Properties of moments of random variables *

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Contents

List of Definitions, Assumptions, Propositions and Theorems

Let X and Y be real random variables, and let r and s be real positive constants ($r > 0,\ s > 0$). The distribution functions of X and Y are denoted $F_X(x) = \mathbb{P}[X \leq x]$ and $F_Y(x) = \mathbb{P}[Y \leq x]$.

1. Existence of moments

1.1 EXISTENCE OF ABSOLUTE AND ORDINARY MOMENTS. $\mathbb{E}(|X|)$ always exists in the extended real numbers $\overline{\mathbb{R}} \equiv \mathbb{R} \cup \{\infty\} \cup \{-\infty\}$ and $\mathbb{E}(|X|) \in [0, \infty]$; *i.e.*, either $\mathbb{E}(|X|)$ is a non-negative real number or $\mathbb{E}(|X|) = \infty$.

1.2 $\mathbb{E}(X)$ exists and is finite $\Leftrightarrow \mathbb{E}(|X|) < \infty$.

1.3 $\mathbb{E}(|X|) < \infty \Rightarrow |\mathbb{E}(X)| \leq \mathbb{E}(|X|) < \infty$.

1.4 If $0 < r \leq s$, then

$$\mathbb{E}(|X|^s) < \infty \Rightarrow \mathbb{E}(|X|^r) < \infty. \tag{1.1}$$

1.5 MONOTONICITY OF L_r. $L_s \subseteq L_r$ for $0 < r \leq s$.

1.6 $\mathbb{E}(|X|^r) < \infty \Rightarrow \mathbb{E}(X^k)$ exists and is finite for all integers k such that $0 < k \leq r$.

2. Moment inequalities

2.1 c_r-INEQUALITY.

$$\mathbb{E}(|X+Y|^r) \leq c_r[\mathbb{E}(|X|^r) + \mathbb{E}(|Y|^r)] \tag{2.1}$$

where

$$\begin{aligned} c_r &= 1, && \text{if } 0 < r \leq 1\,, \\ &= 2^{r-1}, && \text{if } r > 1\,. \end{aligned} \tag{2.2}$$

2.2 MEAN FORM OF c_r-INEQUALITY.

$$\begin{aligned} \mathbb{E}(|\tfrac{1}{2}(X+Y)|^r) &\leq \left(\tfrac{1}{2}\right)^r [\mathbb{E}(|X|^r) + \mathbb{E}(|Y|^r)]\,, && \text{if } 0 < r \leq 1, \\ &\leq \tfrac{1}{2}[\mathbb{E}(|X|^r) + \mathbb{E}(|Y|^r)]\,, && \text{if } r > 1. \end{aligned} \tag{2.3}$$

2.3 CLOSURE OF L_r. Let a and b be real numbers. Then

$$X \in L_r \text{ and } Y \in L_r \Rightarrow aX + bY \in L_r\,. \tag{2.4}$$

2.4 HÖLDER INEQUALITY. If $r > 1$ and $\frac{1}{r} + \frac{1}{s} = 1$, then
$$\mathbb{E}(|XY|) \le [\mathbb{E}(|X|^r)]^{1/r}[\mathbb{E}(|Y|^s)]^{1/s}. \tag{2.5}$$

2.5 CAUCHY-SCHWARZ INEQUALITY.
$$\mathbb{E}(|XY|) \le [\mathbb{E}(X^2)]^{1/2}[\mathbb{E}(Y^2)]^{1/2}. \tag{2.6}$$

2.6 MINKOWSKI INEQUALITY. If $r \ge 1$, then
$$\mathbb{E}(|X+Y|^r)^{1/r} \le [\mathbb{E}(|X|^r)]^{1/r} + [\mathbb{E}(|Y|^r)]^{1/r}. \tag{2.7}$$

2.7 MOMENT MONOTONICITY. $[\mathbb{E}(|X|^r)]^{1/r}$ is a non-decreasing function of r, *i.e.*
$$0 < r \le s \Rightarrow [\mathbb{E}(|X|^r)]^{1/r} \le [\mathbb{E}(|X|^s)]^{1/s}. \tag{2.8}$$

2.8 Theorem LIAPUNOV THEOREM. $\log[\mathbb{E}(|X|^r)]$ *is a convex function of* r, i.e. *for any* $\lambda \in [0,1]$,
$$\log[\mathbb{E}(|X|^{\lambda r+(1-\lambda)s})] \le \lambda \log[\mathbb{E}(|X|^r)] + (1-\lambda)\log[\mathbb{E}(|X|^s)]. \tag{2.9}$$

2.9 LOWER BOUNDS ON THE MOMENTS OF A SUM. If $\mathbb{E}(|X|^r) < \infty$, $\mathbb{E}(|Y|^r) < \infty$ and $\mathbb{E}(Y \mid X) = 0$, then
$$\mathbb{E}(|X+Y|^r) \ge \mathbb{E}(|X|^r), \quad \text{for } r \ge 1. \tag{2.10}$$

2.10 JENSEN INEQUALITY. If $g(x)$ is a convex function on $\mathbb{R}$ and $\mathbb{E}(|X|) < \infty$, then, for any constant $c \in \mathbb{R}$,
$$g(c) \le \mathbb{E}[g(X - EX + c)] \tag{2.11}$$

and, in particular,
$$g(EX) \le \mathbb{E}[g(X)]. \tag{2.12}$$

2.11 CONCAVE JENSEN INEQUALITY. If $g(x)$ is a concave function on $\mathbb{R}$ and $\mathbb{E}(|X|) < \infty$, then, for any constant $c \in \mathbb{R}$,
$$g(c) \ge \mathbb{E}[g(X - EX + c)] \tag{2.13}$$

and, in particular,
$$g(EX) \ge \mathbb{E}[g(X)]. \tag{2.14}$$

3. Moment-based bounds on tail probabilities

3.1. Markov-type inequalities

3.1 Theorem MARKOV INEQUALITY. *Let X be a real random variable such that*

$$\mathbb{P}[X \geq 0] = 1\,. \tag{3.1}$$

If X has a finite expected value $\mathbb{E}[X]$ and $a > 0$, then

$$\mathbb{P}[X \geq a] \leq \frac{\mathbb{E}(X)}{a}\,. \tag{3.2}$$

If we adopt the conventions that $0/0 \equiv 1$ and $x/0 = +\infty$ for $x > 0$, inequality (3.2) remains formally valid for $a = 0$ or $\mathbb{E}(X) = 0$. It is easy to see that (3.2) entails:

$$\mathbb{P}[X > a] \leq \frac{\mathbb{E}(X)}{a}\,, \text{ for } a > 0\,, \tag{3.3}$$

$$\mathbb{P}[X < a] \leq \mathbb{P}[X \leq a] \geq 1 - \frac{\mathbb{E}(X)}{a} = \frac{a - \mathbb{E}(X)}{a}\,, \text{ for } a \geq \mathbb{E}(X)\,, \tag{3.4}$$

$$\mathbb{P}[X = a] \leq \frac{\mathbb{E}(X)}{a}\,. \tag{3.5}$$

Further, if $\mathbb{E}(X) > 0$, we can replace a by $a\mathbb{E}[X]$, hence

$$\mathbb{P}[X > a\mathbb{E}(X)] \leq \mathbb{P}[X \geq a\mathbb{E}(X)] \leq \frac{1}{a}\,, \text{ for any } a > 0\,, \tag{3.6}$$

$$\mathbb{P}[X < a\mathbb{E}(X)] \leq \mathbb{P}[X \leq a\mathbb{E}(X)] \geq 1 - \frac{1}{a} = \frac{a-1}{a}\,, \text{ for } a \geq 1\,, \tag{3.7}$$

$$\mathbb{P}[X = a\mathbb{E}(X)] \leq \frac{1}{a}\,, \text{ for any } a > 0\,. \tag{3.8}$$

The Markov inequality provides useful insights in many statistical contexts: for example,

1. the probability that a person lives more twice life expectancy (at birth) cannot be larger than $1/2$;
2. the probability that the lifetime of a machine exceeds three times the average lifetime cannot be larger than $1/3$.

Consequently, it provides useful information in many areas, such as demography, actuarial mathematics, and reliability theory.

In general, if we define

$$F_X(x) := \mathbb{P}[X \leq x], \tag{3.9}$$

$$F_X^{-1}(q) := \inf\{x : F_X(x) \geq q\}\ ,\ 0 < q < 1\ , \tag{3.10}$$

the Markov inequality entails: for $0 < q < 1$,

$$1 - q \leq \mathbb{P}[X \geq F_X^{-1}(q)] \leq \frac{\mathbb{E}(X)}{F_X^{-1}(q)} \tag{3.11}$$

hence

$$\mathbb{E}(X) \geq (1-q)\, F_X^{-1}(q) \tag{3.12}$$

or, equivalently,

$$\mathbb{E}(X) \geq q\, F_X^{-1}(1-q) \tag{3.13}$$

For example, the mean of a positive random variable is at least as large as half its median:

$$\mathbb{E}(X) \geq \frac{1}{2} F_X^{-1}(1/2)\,. \tag{3.14}$$

3.2. Markov inequality and ecological fallacies

Question – Under the assumptions of Theorem 3.1, can we give lower bounds on $\mathbb{P}[X \geq a]$? Clearly, if $\mathbb{E}[X] > 0$,

$$\mathbb{P}[X \geq \mathbb{E}(X)] > 0\,, \tag{3.15}$$

$$\mathbb{P}[X > a] > 0 \text{ for } 0 \leq a < \mathbb{E}(X)\,. \tag{3.16}$$

This raises the question whether a more informative bound is possible. The answer to this question is no: for any $\mu > 0$ and $0 \leq a < \mu$ and $0 < p < 1$, we can find a nonnegative random variable X such that

$$\mathbb{E}[X] = \mu \text{ and } \mathbb{P}[X \leq a] = p\,. \tag{3.17}$$

3.2 Proposition TIGHTNESS OF MARKOV INEQUALITY. *For any real numbers μ and a satisfying $0 \leq \mu < a$, there exists a nonnegative random variable X such that*

$$\mathbb{E}[X] = \mu \textit{ and } \mathbb{P}[X \geq a] = \frac{\mu}{a}\,. \tag{3.18}$$

3.3 Proposition IRRELEVANCE OF THE MEAN FOR QUANTILES. *Let X be a random variable such that*

$$\mathbb{P}[X \geq 0] = 1\,. \tag{3.19}$$

If X has a finite expected value $\mathbb{E}(X)$ and $\mathbb{E}(X) > 0$, then

$$\mathbb{P}[X \geq \mathbb{E}(X)] > 0\,, \tag{3.20}$$

and

$$\mathbb{P}[X > a] > 0 \text{ for } 0 \leq a < \mathbb{E}(X)\,. \tag{3.21}$$

Further, for any real numbers μ, a and p satisfying $0 < a < \mu$ and $0 < p < 1$, there exists a nonnegative random variable Y such that

$$\mathbb{E}(Y) = \mu \text{ and } \mathbb{P}[Y \leq a] = p\,. \tag{3.22}$$

The above propositions show three things:

1. the Markov inequality is tight: if $0 < \mathbb{E}(X)/a < 1$, there is a nonnegative random variable X such that

$$\mathbb{P}[X \geq a] = \mathbb{E}(X)/a\,; \tag{3.23}$$

   the inequality (3.2) cannot be improved without introducing further restrictions;

2. if $\mathbb{E}(X) > 0$, the probability $\mathbb{P}[X \geq a]$ can be arbitrarily large or small for any a in the interval $0 < a < \mathbb{E}(X)$;

3. the mean of X can be larger than any quantile of X, *i.e.* we can have $\mathbb{E}(X) > F_X^{-1}(q)$ for $0 < q < 1$.

The observations provide a way of illustrating the "ecological fallacy".
Proposition 3.3 entails we can find, for any $0 < q < 1$, a random variable Y such that

$$\mathbb{E}(X) > F_X^{-1}(q)\,. \tag{3.24}$$

Question – If X and Y are random variables, does the value of $\mathbb{E}(XY)$ impose restrictions on $\mathbb{P}[XY > a]$ or $\mathbb{P}[XY < a]$? For example, if $\mathbb{E}(XY) = \gamma > 0$, does this constrain $\mathbb{P}[XY > 0]$ or $\mathbb{P}[XY < 0]$?

3.3. Markov-type inequalities for bounded variables

When the random variable X has bounded support, both lower and upper bounds can be given for $\mathbb{P}[X \geq a]$ as follows.

3.4 Theorem TWO-SIDED MARKOV-TYPE INEQUALITIES FOR BOUNDED VARIABLES. *Let X be a real random variable such that*

$$\mathbb{P}[M_0 \leq X \leq M_1] = 1 \tag{3.25}$$

where $-\infty \le M_0 \le M_1 \le +\infty$. *If* X *has finite expected value* $\mathbb{E}(X)$, $a \in \mathbb{R}$ *and* $M_0 < a < M_1$, *then:*

$$\max\left\{0, \frac{\mathbb{E}(X)-a}{M_1-a}\right\} \le \mathbb{P}[X > a] \le \mathbb{P}[X \ge a] \le \min\left\{1, \frac{\mathbb{E}(X)-M_0}{a-M_0}\right\}, \tag{3.26}$$

$$\max\left\{0, \frac{a-\mathbb{E}(X)}{a-M_0}\right\} \le \mathbb{P}[X < a] \le \mathbb{P}[X \le a] \le \min\left\{1, \frac{M_1-\mathbb{E}(X)}{M_1-a}\right\}. \tag{3.27}$$

In Theorem 3.4, the positivity assumption is replaced by the more general support assumption (3.25), so X can take negative values. If $\mathbb{E}(X) = 0$ and $M_0 \le 0 \le M_1$, we have:

$$\frac{-a}{M_1-a} \le \mathbb{P}[X > a] \le \mathbb{P}[X \ge a] \le \frac{-M_0}{a-M_0}. \tag{3.28}$$

For example, for $M_0 = -1$ and $M_1 = 1$, we get:

$$\mathbb{P}[X > 0.5] \le \mathbb{P}[X \ge 0.5] \le \frac{1}{0.5+1} = \frac{1}{3}, \tag{3.29}$$

$$\mathbb{P}[X \ge -0.5] \ge \mathbb{P}[X > -0.5] \ge \frac{0.5}{1+0.5} = \frac{1}{3}. \tag{3.30}$$

The Markov inequality [Theorem 3.1] corresponds to the special case where $M_0 = 0$ and $M_1 = +\infty$. If $-\infty < M_0 = M_1 < +\infty$, the random variable X is degenerate at M_1, and $\mathbb{P}[X = M_1] = 1$.

For $M_0 \ge 0$ and $M_0 < \mathbb{E}(X) \le a < M_1$, we also have

$$\frac{\mathbb{E}(X)-a}{M_1-a} \ge \frac{\mathbb{E}(X)-a}{M_1}, \quad \frac{\mathbb{E}(X)-M_0}{a-M_0} \le \frac{\mathbb{E}(X)}{a} \tag{3.31}$$

so that

$$\frac{\mathbb{E}(X)-a}{M_1} \le \mathbb{P}[X > a] \le \mathbb{P}[X \ge a] \le \frac{\mathbb{E}(X)}{a}. \tag{3.32}$$

The latter result follows from the *Basic inequality* of Loève (1977, Volume I, Section 9, p. 159) on taking $g(x) = x$. The bounds in (3.32) are however less tight than those in (3.26).

As in (3.3) - (3.7), we can see that (3.26) implies: for $M_0 < a < M_1$,

$$\frac{a-\mathbb{E}(X)}{a-M_0} \le \mathbb{P}[X < a] \le \mathbb{P}[X \le a] \le \frac{M_1-\mathbb{E}(X)}{M_1-a}, \tag{3.33}$$

$$\mathbb{P}[X = a] \le \min\left\{\frac{\mathbb{E}(X)-M_0}{a-M_0}, \frac{M_1-\mathbb{E}(X)}{M_1-a}\right\}. \tag{3.34}$$

For example, for $a=0$, we get:

$$\frac{\mathbb{E}(X)}{M_0} \leq \mathbb{P}[X<0] \leq \mathbb{P}[X \leq 0] \leq \frac{M_1-\mathbb{E}(X)}{M_1}\,. \tag{3.35}$$

On taking $a=F_X^{-1}(q)$, we get for $0<q<1$:

$$\frac{F_X^{-1}(q)-\mathbb{E}(X)}{F_X^{-1}(q)-M_0} \leq \mathbb{P}[X<F_X^{-1}(q)] \leq \mathbb{P}[X \leq F_X^{-1}(q)] \leq \frac{M_1-\mathbb{E}(X)}{M_1-F_X^{-1}(q)} \tag{3.36}$$

hence

$$q \leq \mathbb{P}[X \leq F_X^{-1}(q)] \leq \frac{M_1-\mathbb{E}(X)}{M_1-F_X^{-1}(q)}\,, \tag{3.37}$$

$$\frac{F_X^{-1}(q)-\mathbb{E}(X)}{F_X^{-1}(q)-M_0} \leq \mathbb{P}[X<F_X^{-1}(q)] \leq q\,, \tag{3.38}$$

and

$$F_X^{-1}(q)-q[F_X^{-1}(q)-M_0] \leq \mathbb{E}(X) \leq M_1-q[M_1-F_X^{-1}(q)]\,, \tag{3.39}$$

$$M_0q+(1-q)F_X^{-1}(q) \leq \mathbb{E}(X) \leq qF_X^{-1}(q)+(1-q)M_1\,. \tag{3.40}$$

Correspondingly,

$$\frac{\mathbb{E}(X)-(1-q)M_1}{q} \leq F_X^{-1}(q) \leq \frac{\mathbb{E}(X)-M_0q}{(1-q)}\,. \tag{3.41}$$

If $M_0=0$, we have:

$$(1-q)F_X^{-1}(q) \leq \mathbb{E}(X) \leq qF_X^{-1}(q)+(1-q)M_1\,, \tag{3.42}$$

$$\frac{\mathbb{E}(X)-(1-q)M_1}{q} \leq F_X^{-1}(q) \leq \frac{\mathbb{E}(X)}{(1-q)}\,. \tag{3.43}$$

If $M_1=0$, we have:

$$M_0q+(1-q)F_X^{-1}(q) \leq \mathbb{E}(X) \leq qF_X^{-1}(q)\,, \tag{3.44}$$

$$\frac{\mathbb{E}(X)}{q} \leq F_X^{-1}(q) \leq \frac{\mathbb{E}(X)-M_0q}{(1-q)}\,. \tag{3.45}$$

Another advantage of (3.26) is the possibility of working in terms of deviation from the mean, for $X-\mathbb{E}(X)$ typically can take negative and positive values:

$$\frac{-a}{M_1-a} \leq \mathbb{P}[X-\mathbb{E}(X)>a] \leq \mathbb{P}[X-\mathbb{E}(X) \geq a] \leq \frac{-M_0}{a-M_0}\,, \tag{3.46}$$

$$\frac{a}{a-M_0} \leq \mathbb{P}[X-\mathbb{E}(X)<a] \leq \mathbb{P}[X-\mathbb{E}(X) \leq a] \leq \frac{M_1}{M_1-a}, \tag{3.47}$$

$$\mathbb{P}[X-\mathbb{E}(X)=a] \leq \min\left\{\frac{-M_0}{a-M_0}, \frac{M_1}{M_1-a}\right\}. \tag{3.48}$$

3.4. Chebyshev-type inequalities

3.5 Theorem TWO-SIDED MONOTONIC MARKOV-TYPE INEQUALITIES. *Let $g:\mathbb{R}\to\mathbb{R}$ be a function such that $g(X)$ is a real random variable, $\mathsf{E}(|g(X)|)<\infty$ and*

$$\mathbb{P}[0 \leq g(X) \leq M] = 1 \tag{3.49}$$

where $M\in[0,\infty]$. If $g(x)$ is a non-decreasing function on $\mathbb{R}$, then, for all $a\in\mathbb{R}$ such that $g(a)>0$,

$$\frac{\mathsf{E}[g(X)]-g(a)}{M} \leq \mathbb{P}[X\geq a] \leq \frac{\mathsf{E}[g(X)]}{g(a)}. \tag{3.50}$$

If $g(x)$ is a non-decreasing function on $[0,\infty)$ and $g(x)=g(-x)$ for any x, then, for all $a\geq 0$ such that $g(a)>0$,

$$\frac{\mathsf{E}[g(X)]-g(a)}{M} \leq \mathbb{P}[|X|\geq a] \leq \frac{\mathsf{E}[g(X)]}{g(a)} \tag{3.51}$$

where $0/0\equiv 1$.

3.6 Corollary CHEBYSHEV INEQUALITY. *If $\mathbb{E}(|X|^r)<\infty$, for $r>0$, and $\mathbb{P}[m\leq|X|\leq M]=1$, where $0\leq m\leq M\in$, $m<\infty$ and $M\leq\infty$, then, for all $a\in(m,M)$*

$$\mathbb{P}[[|X|\geq a] \leq \frac{\mathbb{E}(|X|^r)}{a^r}. \tag{3.52}$$

Note the above result does not require that X be positive. It is easy to see that (3.50) entails

$$\mathbb{P}[X\leq a] \geq \mathbb{P}[X<a] \geq 1-\frac{\mathbb{E}[g(X)]}{g(a)} \tag{3.53}$$

while (3.51) entails

$$\mathbb{P}[|X|\leq a] \geq \mathbb{P}[|X|<a] \geq 1-\frac{\mathbb{E}[g(X)]}{g(a)}. \tag{3.54}$$

On taking $g(x) = x^r$, $r > 0$, we see that the distribution function of X is dominated by the Pareto distribution

$$\begin{aligned} F_r(x) &= 1 - \frac{\mathbb{E}[X^r]}{x^r} \quad \text{for } x \geq \mathbb{E}[X^r] \\ &= 0 \qquad\qquad\quad \text{for } x < \mathbb{E}[X^r]\,. \end{aligned} \tag{3.55}$$

Take $M = \infty$, and suppose the function $g(x)$ is homogeneous of degree k, *i.e.*

$$g(\lambda x) = \lambda^k g(x) \text{ for all } x\,. \tag{3.56}$$

Then, (3.50) implies:

$$\mathbb{P}\big[X \geq a\,(\mathbb{E}[g(X)])\big] \leq \frac{\mathbb{E}[g(X)]}{\mathbb{E}[g(X)]^k\, g(a)} = \frac{\mathbb{E}[g(X)]^{1-k}}{g(a)}\,, \tag{3.57}$$

$$\mathbb{P}\big[X \geq a\,(\mathbb{E}[g(X)])^{1/k}\big] \leq \frac{\mathbb{E}[g(X)]}{\mathbb{E}[g(X)]\, g(a)} = \frac{1}{g(a)}\,. \tag{3.58}$$

hence for $g(x) = x^k$ and $a > 0$,

$$\mathbb{P}\big[X \geq a(\mathbb{E}\big[X^k\big])\big] \leq \frac{1}{a^k}\,, \tag{3.59}$$

$$\mathbb{P}\big[X \geq a(\mathbb{E}\big[X^k\big])^{1/k}\big] \leq \frac{1}{a^k}\,. \tag{3.60}$$

In particular,

$$\mathbb{P}\big[X \geq a\,\mathbb{E}[X]\big] \leq \frac{1}{a}\,, \tag{3.61}$$

$$\mathbb{P}\big[X \geq a\,(\mathbb{E}\big[X^2\big])^{1/2}\big] \leq \frac{1}{a^2}\,. \tag{3.62}$$

3.7 Theorem TWO-SIDED BOUNDED MONOTONIC MARKOV-TYPE INEQUALITIES. *Let X be a real random variable, and $g : \mathbb{R} \to \mathbb{R}$ a function such that $g(X)$ is a real random variable which satisfies $\mathbb{E}(|g(X)|) < \infty$ and*

$$\mathbb{P}[m \leq g(X) \leq M] = 1 \tag{3.63}$$

where $-\infty \leq m \leq M \leq +\infty$. If $g(x)$ is a non-decreasing function on $\mathbb{R}$, then, for all $a \in \mathbb{R}$ such that $m < g(a) < M$,

$$\frac{\mathbb{E}[g(X)] - g(a)}{M - g(a)} \leq \mathbb{P}[|X| > a] \leq \mathbb{P}[|X| \geq a] \leq \frac{\mathbb{E}[g(X)] - m}{g(a) - m}\,. \tag{3.64}$$

If $g(x)$ *is a non-decreasing function on* $[0,\infty)$ *and* $g(x) = g(-x)$ *for any* x*, then, for all* $a \geq 0$ *such that* $g(a) > 0$*,*

$$\frac{\mathbb{E}\,[g\,(X)] - g\,(a)}{M - g\,(a)} \leq \mathbb{P}[|X| > a] \leq \mathbb{P}[|X| \geq a] \leq \frac{\mathbb{E}\,[g\,(X)] - m}{g\,(a) - m} \tag{3.65}$$

where $0/0 \equiv 1$.

3.8 Corollary CHEBYSHEV INEQUALITIES. *If* $\mathbb{E}\,(|X|^r) < \infty$*, for* $r > 0$*, and* $\mathbb{P}[m \leq |X| \leq M] = 1$*, where* $0 \leq m \leq M \in$*,* $m < \infty$ *and* $M \leq \infty$*, then, for all* $a \in (m, M)$*,*

$$\frac{\mathbb{E}\,(|X|^r) - a^r}{M^r - a^r} \leq \mathbb{P}[|X| \geq a] \leq \frac{\mathbb{E}\,(|X|^r) - m^r}{a^r - m^r}\,. \tag{3.66}$$

3.5. Cantelli inequality

The Markov inequality is fundamentally two-sided. It can extended and improved by considering a one-sided version due to Cantelli (1928).

3.9 Theorem CANTELLI INEQUALITY. *Let* X *be a real random variable such that* $\mathbb{E}(X^2) < \infty$*, and* $\sigma^2(X) := \mathbb{E}\{[X - \mathbb{E}(X)]^2\}$*.Then, for any* $a \geq 0$*,*

$$\mathbb{P}[X - \mathbb{E}(X) \geq a] \leq \frac{\sigma^2(X)}{\sigma^2(X) + a^2}\,, \tag{3.67}$$

$$\mathbb{P}[X - \mathbb{E}(X) \leq -a] \leq \frac{\sigma^2(X)}{\sigma^2(X) + a^2}\,. \tag{3.68}$$

From (3.67) - (3.68), it follows that

$$\mathbb{P}[X \geq \mathbb{E}(X) + a\,\sigma(X)] = \mathbb{P}[X - \mathbb{E}(X) \geq a\,\sigma(X)] \leq \frac{1}{1 + a^2}\,, \tag{3.69}$$

$$\mathbb{P}[X \leq \mathbb{E}(X) - a\,\sigma(X)] = \mathbb{P}[X - \mathbb{E}(X) \leq a\,\sigma(X)] \leq \frac{1}{1 + a^2}\,. \tag{3.70}$$

For example, if $\mathbb{E}(X) = 5$ and $\sigma(X) = 1$, we have:

$$\mathbb{P}[X \leq 0] = \mathbb{P}[X \leq 5 - 5\,\sigma(X)] \leq \frac{1}{1 + 25} = 0.038\,. \tag{3.71}$$

The inequalities (3.67) - (3.68) can also be written as follows:

$$\mathbb{P}[X \geq a] \leq \begin{cases} \frac{\sigma^2(X)}{\sigma^2(X) + [a - \mathbb{E}(X)]^2} \quad = \frac{\mathbb{E}\{[X - \mathbb{E}(X)]^2\}}{\mathbb{E}[(X-a)^2]} & \text{if } a \geq \mathbb{E}(X)\,, \\ 1 & \text{if } a < \mathbb{E}(X)\,, \end{cases} \tag{3.72}$$

$$\mathbb{P}[X \le a] \le \begin{cases} \frac{\sigma^2(X)}{\sigma^2(X)+[a-\mathbb{E}(X)]^2} \quad = \frac{\mathbb{E}\{[X-\mathbb{E}(X)]^2\}}{\mathbb{E}[(X-a)^2]} & \text{if } a \le \mathbb{E}(X)\,, \\ 1 & \text{if } a > \mathbb{E}(X)\,. \end{cases} \tag{3.73}$$

3.6. Generalized Markov-type inequalities

3.10 Theorem GENERIC MARKOV-TYPE INEQUALITIES. *Let Y be a real random variable and an event A such that*

$$\mathbb{P}[A \cap \{m(A) \le Y \le M(A)\}] = \mathbb{P}(A)\,, \tag{3.74}$$

$$\mathbb{P}[A^c \cap \{m(A^c) \le Y \le M(A^c)\}] = \mathbb{P}(A^c)\,, \tag{3.75}$$

where $-\infty \le m(A) \le M(A) \le +\infty$ and $-\infty \le m(A^c) \le M(A^c) \le +\infty$. If Y has finite expected value $\mathbb{E}(Y)$, then

$$[m(A) - m(A^c)]\,\mathbb{P}(A) \le \mathbb{E}(Y) - m(A^c)\,, \tag{3.76}$$

$$[M(A^c) - M(A)]\,\mathbb{P}(A) \le M(A^c) - \mathbb{E}(Y)\,, \tag{3.77}$$

In particular,

$$\mathbb{P}(A) \le \frac{\mathbb{E}(Y) - m(A^c)}{m(A) - m(A^c)}\,, \quad \textit{if } m(A) > m(A^c)\,, \tag{3.78}$$

$$\mathbb{P}(A) \ge \frac{\mathbb{E}(Y) - m(A^c)}{m(A) - m(A^c)}\,, \quad \textit{if } m(A) < m(A^c)\,, \tag{3.79}$$

$$\mathbb{P}(A) \le \frac{M(A^c) - \mathbb{E}(Y)}{M(A^c) - M(A)}\,, \quad \textit{if } M(A) < M(A^c)\,, \tag{3.80}$$

$$\mathbb{P}(A) \ge \frac{M(A^c) - \mathbb{E}(Y)}{M(A^c) - M(A)}\,, \quad \textit{if } M(A) > M(A^c)\,. \tag{3.81}$$

Two-sided inequalities easily follow from Theorem 3.10. If $m(A) > m(A^c)$ and $M(A) > M(A^c)$, we have

$$\frac{M(A^c) - \mathbb{E}(Y)}{M(A^c) - M(A)} \le \mathbb{P}(A) \le \frac{\mathbb{E}(Y) - m(A^c)}{m(A) - m(A^c)}\,. \tag{3.82}$$

Similarly, if $m(A) < m(A^c)$ and $M(A) < M(A^c)$, we get:

$$\frac{\mathbb{E}(Y) - m(A^c)}{m(A) - m(A^c)} \le \mathbb{P}(A) \le \frac{M(A^c) - \mathbb{E}(Y)}{M(A^c) - M(A)}\,. \tag{3.83}$$

3.11 Theorem MARKOV-TYPE INEQUALITIES BASED ON TRUNCATED MOMENTS. *Let $g : \mathbb{R} \to \mathbb{R}$ be a function such that $g(X)$ is a real random variable, $\mathbb{E}(|g(X)|) < \infty$ and*

$$0 \leq g(x) \leq M_U \text{ for } x \geq A_U\,, \tag{3.84}$$
$$0 \leq g(x) \leq M_L \text{ for } x \leq A_L\,, \tag{3.85}$$

where $0 \leq M_U \leq \infty$, $0 \leq M_L \leq \infty$, $0 \leq A_U \leq \infty$ and $0 \leq A_L \leq \infty$. Let also

$$C_U(g, a) = \int_{[a,\infty)} g(x)\, dF_X(x)\,, \quad C_L(g, a) = \int_{(-\infty,a]} g(x)\, dF_X(x)\,. \tag{3.86}$$

(a) *If $g(x)$ is nondecreasing on $[A_U, \infty)$, then, for $a \geq A_U$,*

$$\frac{C_U(g, a)}{M_U} \leq \mathbb{P}[X \geq a] \leq \frac{C_U(g, a)}{g(a)}\,. \tag{3.87}$$

(b) *If $g(x)$ is nonincreasing on $(-\infty, A_L]$, then, for $a \leq A_L$,*

$$\frac{C_L(g, a)}{M_L} \leq \mathbb{P}[X \leq a] \leq \frac{C_L(g, a)}{g(a)}\,. \tag{3.88}$$

(c) *If $g(x)$ is nondecreasing on $[A_U, \infty)$ and nonincreasing on $(-\infty, A_L]$, then, for $a \geq \max\{|A_U|, |A_L|\}$,*

$$\begin{aligned} \mathbb{P}[|X| \geq a] &\leq \frac{C_U(g, a)}{g(a)} + \frac{C_L(g, a)}{g(-a)} \\ &\leq \frac{C_U(g, a) + C_L(g, a)}{\min\{g(a), g(-a)\}}\,, \end{aligned} \tag{3.89}$$

$$\begin{aligned} \mathbb{P}[|X| \geq a] &\geq \frac{C_U(g, a)}{M_U} + \frac{C_L(g, a)}{M_L} \\ &\geq \frac{C_U(g, a) + C_L(g, a)}{\max\{M_U, M_L\}}\,. \end{aligned} \tag{3.90}$$

4. Moments as integrals of distribution and quantile functions

4.1 Proposition DISTRIBUTION DECOMPOSITION OF THE FIRST MOMENT. *Let X be a random variable with distribution function $F_X(x)$. If $\mathbb{E}|X| < \infty$, then*

$$\begin{aligned} \int_0^\infty x\,dF_X(x) &= \int_0^\infty \mathbb{P}[X \geq x]dx = \int_0^\infty \mathbb{P}[X > x]dx \\ &= \int_0^\infty [1 - F_X(x)]dx, \end{aligned} \tag{4.1}$$

$$\begin{aligned} \int_{-\infty}^0 x\,dF_X(x) &= -\int_{-\infty}^0 \mathbb{P}[X \leq x]dx = -\int_{-\infty}^0 \mathbb{P}[X < x]dx \\ &= -\int_{-\infty}^0 F_X(x)dx, \end{aligned} \tag{4.2}$$

$$\begin{aligned} \mathbb{E}(X) &= \int_0^\infty \mathbb{P}[X \geq x]dx - \int_{-\infty}^0 \mathbb{P}[X \leq x]dx \\ &= \int_0^\infty [1 - F_X(x)]dx - \int_{-\infty}^0 F_X(x)dx \\ &= \int_0^\infty [1 - F_X(x) - F_X(-x)]dx. \end{aligned} \tag{4.3}$$

4.2 Corollary DISTRIBUTION DECOMPOSITION OF THE FIRST ABSOLUTE MOMENT. *Let X be a random variable with distribution function $F_X(x)$. If $\mathbb{E}|X| < \infty$, then*

$$\begin{aligned} \mathbb{E}(|X|) &= \int_0^\infty [1 - F_X(x)]dx + \int_{-\infty}^0 F_X(x)dx \\ &= \int_0^\infty \mathbb{P}[X \geq x]dx + \int_{-\infty}^0 \mathbb{P}[X \leq x]dx \\ &= \int_0^\infty \mathbb{P}[|X| \geq x]dx. \end{aligned} \tag{4.4}$$

4.3 Corollary DISTRIBUTION DECOMPOSITION OF THE MEAN OF A POSITIVE VARIABLE. *Let X be a random variable with distribution function $F_X(x)$. If $\mathbb{E}|X| < \infty$ and $\mathbb{P}[X \geq 0] = 1$, then*

$$\mathbb{E}(X) = \int_0^\infty [1 - F_X(x)]dx$$

$$= \int_0^\infty \mathbb{P}[X \geq x] dx. \tag{4.5}$$

4.4 Theorem DISTRIBUTION DECOMPOSITION OF THE EXPECTED VALUE OF $g(X)$. *Let X be a real random variable, and $g : \mathbb{R} \to \mathbb{R}$ a function such that $g(X)$ is a real random variable which satisfies $\mathbb{E}(|g(X)|) < \infty$. If $g(x)$ is differentiable a.e. on $\mathbb{R}$, then, for any $a \in \mathbb{R}$,*

$$\begin{aligned}
\int_a^\infty g(x)\,dF_X(x) &= g(a)[1-F_X(a)] + \int_a^\infty g'(x)\,[1-F_X(x)]\,dx \\
&= g(a)\mathbb{P}[X > a] + \int_a^\infty g'(x)\,\mathbb{P}[X > x]\,dx \\
&= g(a)\mathbb{P}[X \geq a] + \int_a^\infty g'(x)\,\mathbb{P}[X \geq x]\,dx \\
&= \mathbb{P}[X \geq a]\{g(a) + \int_{-\infty}^\infty g'(x)\,\mathbb{P}[X \geq x\,|\,X \geq a]\,dx\}
\end{aligned} \tag{4.6}$$

$$\begin{aligned}
\int_{-\infty}^a g(x)\,dF_X(x) &= g(a)F_X(a) - \int_{-\infty}^a g'(x)\,F_X(x)\,dx \\
&= g(a)\mathbb{P}[X \leq a] - \int_{-\infty}^a g'(x)\,\mathbb{P}[X \leq x]\,dx \\
&= \mathbb{P}[X \leq a]\{g(a) - \int_{-\infty}^\infty g'(x)\,\mathbb{P}[X \leq x\,|\,X \leq a]\,dx\},
\end{aligned} \tag{4.7}$$

$$\begin{aligned}
\mathbb{E}[g(X)] &= g(a) + \int_a^\infty g'(x)[1-F_X(x)]dx - \int_{-\infty}^a g'(x)F_X(x)dx \\
&= g(a) + \int_a^\infty g'(x)\mathbb{P}[X > x]\,dx - \int_{-\infty}^a g'(x)\,\mathbb{P}[X \leq x]\,dx \\
&= g(a) + \mathbb{P}[X \geq a]\int_{-\infty}^\infty g'(x)\,\{\mathbb{P}[X \geq x\,|\,X \geq a]\,dx \\
&\quad -\mathbb{P}[X \leq a]\int_{-\infty}^\infty g'(x)\,\mathbb{P}[X \leq x\,|\,X \leq a]\,dx.
\end{aligned} \tag{4.8}$$

It is of interest to spell out a number of special cases of (4.8):

$$\mathbb{E}[g(X)] = g(0) + \int_0^\infty g'(x)[1-F_X(x)]dx - \int_{-\infty}^0 g'(x)F_X(x)dx \tag{4.9}$$

$$= g(0)+\int_0^\infty g'(x)[1-F_X(x)]dx+\int_0^\infty g'(-x)F_X(-x)dx \tag{4.10}$$

$$= g(0)+\int_0^\infty \{g'(x)[1-F_X(x)]+g'(-x)F_X(-x)\}dx \tag{4.11}$$

In particular, if $g(-x)=g(x)$, we have $g'(-x)=-g'(x)$, hence

$$\mathbb{E}[g(X)] = g(0)+\int_0^\infty g'(x)[1-F_X(x)]dx-\int_0^\infty g'(x)F_X(-x)dx \tag{4.12}$$

$$= g(0)+\int_0^\infty g'(x)[1-F_X(x)-F_X(-x)]dx \tag{4.13}$$

and, if X has a distribution symmetric about zero [$F_X(-x)=1-F_X(x)$, for all x],

$$\mathbb{E}[g(X)]=g(0)\,. \tag{4.14}$$

$$\mathbb{E}[g(X)]-g(\mathbb{E}[X])=\int_{\mathbb{E}[X]}^\infty g'(x)[1-F_X(x)]dx-\int_{-\infty}^{\mathbb{E}[X]} g'(x)F_X(x)dx\,. \tag{4.15}$$

When $g(x)$ is convex, we see that

$$\mathbb{E}[g(X)]-g(\mathbb{E}[X])=\int_{\mathbb{E}[X]}^\infty g'(x)[1-F_X(x)]dx-\int_{-\infty}^{\mathbb{E}[X]} g'(x)F_X(x)dx\geq 0\,. \tag{4.16}$$

This yields a closed-form expression for the difference $\mathbb{E}[g(X)]-g(\mathbb{E}[X])$.

4.5 Corollary DISTRIBUTION DECOMPOSITION OF r-MOMENTS. *Let X a random variable with distribution function $F_X(x)$ and $r>0$. If $\mathbb{E}(|X|^r)<\infty$, then*

$$\int_0^\infty x^r dF_X(x)=r\int_0^\infty x^{r-1}[1-F_X(x)]dx\,, \tag{4.17}$$

$$\begin{aligned}\mathbb{E}(|X|^r) &= r\int_0^\infty x^{r-1}\mathbb{P}(|X|\geq x)dx\\ &= r\int_0^\infty x^{r-1}[1-F_X(x)+F_X(-x)]dx\,.\end{aligned} \tag{4.18}$$

4.6 Proposition MOMENT-TAIL AREA INEQUALITIES. *Let $g(x)$ be a nonnegative strictly increasing function on $[0,\infty)$ and let $g^{-1}(x)$ be the inverse function of g. Then,*

$$\sum_{n=1}^{\infty}\mathbb{P}\left[|X|\geq g^{-1}(n)\right]\leq\mathbb{E}[g(X)]\leq\sum_{n=0}^{\infty}\mathbb{P}\left[|X|>g^{-1}(n)\right]. \tag{4.19}$$

In particular, for any $r>0$,

$$\begin{aligned}\sum_{n=1}^{\infty}\mathbb{P}(|X|\geq n^{1/r}) &\leq \mathbb{E}(|X|^r)\leq\sum_{n=0}^{\infty}\mathbb{P}(|X|>n^{1/r})\\ &\leq 1+\sum_{n=1}^{\infty}\mathbb{P}(X>n^{1/r}).\end{aligned} \tag{4.20}$$

4.7 Corollary MEAN-TAIL AREA INEQUALITIES. *If X is a positive random variable,*

$$\sum_{n=1}^{\infty}\mathbb{P}(X\geq n)\leq\mathbb{E}(X)\leq 1+\sum_{n=1}^{\infty}\mathbb{P}(X>n). \tag{4.21}$$

4.8 Proposition INTEGRAL OF THE PROBABILITY OF AN INTERVAL. *Let X a random variable with distribution function $F_X(x)$ and $0\leq a\leq b$. If $\mathbb{E}|X|<\infty$, then*

$$\int\mathbb{P}[a<X\leq x+b]dx=\int[F_X(x+b)-F_X(x)]dx=b, \tag{4.22}$$

$$\int\mathbb{P}[x-a<X\leq x+b]dx=\int[F_X(x+b)-F_X(x-a)]dx=a+b. \tag{4.23}$$

4.9 Corollary INTEGRAL OF THE PROBABILITY OF A GENERAL INTERVAL. *Let X a random variable with distribution function $F_X(x)$ and $a\leq b$. If $\mathbb{E}|X|<\infty$, then*

$$\int\mathbb{P}[x+a<X\leq x+b]dx=\int[F_X(x+b)-F_X(x+a)]dx=b-a. \tag{4.24}$$

4.10 Proposition QUANTILE REPRESENTATION OF THE MEAN. *Let X a random variable with distribution function $F_X(x)$ and quantile function*

$$F_X^{-1}(q)=\inf\{x:F_X(x)\geq q\},\quad 0<q<1. \tag{4.25}$$

If $\mathbb{E}|X|<\infty$, then

$$\mathbb{E}(X)=\int_0^1 F_X^{-1}(q)\,dq. \tag{4.26}$$

5. Integration by parts

In order to study conditions for the existence of moments, it will be useful to recall some results on integration by parts.

5.1. Standard results

5.1 Theorem EXISTENCE OF RIEMANN-STIELTJES INTEGRAL. *Each one of the following conditions is sufficient for the existence of the Riemann-Stieltjes integral* $\int_a^b g(x)\,df(x)$.

(a) $g(x)$ *is continuous on* $[a,b]$ *and* $f(x)$ *is of bounded variation on* $[a,b]$.

5.2 Theorem REPRESENTATION OF RIEMANN-STIELTJES INTEGRAL FOR CONTINUOUS-BV FUNCTIONS. *If* $g(x)$ *is continuous on* $[a,b]$ *and* $f(x)$ *is of bounded variation on* $[a,b]$, *then*

$$\int_a^b g(x)\,df(x) = \sum_{x\in J} g(x)[f(x+)-f(x-)] + \int_a^b g(x)df_c(x) \tag{5.1}$$

where J *is the set of discontinuities of* g *on* $[a,b]$,

$$f_c(x) = f(x) - f_s(x), \tag{5.2}$$

and

$$\begin{aligned} f_s(x) &= f(x) - f(x-) + \sum_{y\in J\cap[a,x)} [f(y+)-f(y-)] \quad \text{for } x\in(a,b] \\ &= 0 \quad \text{for } x = a \end{aligned} . \tag{5.3}$$

$g_s(x)$ is called the *saltus function* of g on $[a,b]$.

5.3 Theorem INTEGRATION BY PARTS. *The Riemann-Stieltjes integral* $\int_a^b g(x)\,df(x)$ *exists if and only the Riemann-Stieltjes integral* $\int_a^b f(x)\,dg(x)$ *exists. Further, when* $\int_a^b g(x)\,df(x)$ *exists, we have:*

$$\int_a^b f(x)\,dg(x) + \int_a^b g(x)\,df(x) = f(b)g(b) - f(a)g(a)\,. \tag{5.4}$$

5.2. Extensions

5.4 Lemma RIEMANN-STIELTJES INTEGRATION BY PARTS. *Let* $f:\mathbb{R}\to\mathbb{R}$ *and* $g:\mathbb{R}\to\mathbb{R}$ *two real-valued functions and* $-\infty < a \le b < +\infty$. *If the (Riemann-Stieltjes) integral* $\int_a^b g(x)\,df(x)$ *exists, then the integrals* $\int_a^b f(x)\,dg(x)$ *and* $\int_a^b [A - f(x)]\,dg(x)$ *also exist and*

$$\int_a^b g(x)\,df(x) \quad = \quad g(b)f(b) - g(a)f(a) - \int_a^b f(x)\,dg(x)$$

$$= \quad [A-f(a)]g(a)-g(b)[A-f(b)]+\int_a^b [A-f(x)]\,dg(x)\,, \tag{5.5}$$

for any real constant A, with

$$\int_a^b f(x)\,dg(x) = \int_a^b f(x)\,g'(x)dx \tag{5.6}$$

and

$$\int_a^b [A-f(x)]\,dg(x) = \int_a^b [A-f(x)]\;g'(x)dx \tag{5.7}$$

if g is continuous on $[a,\,b]$ as well as differentiable on (a,b) and the Riemann integral $\int_a^b f(x)\,g'(x)dx$ exists (where g' can take arbitrary real values at a and b).

5.5 Lemma CENTERED RIEMANN-STIELTJES INTEGRATION BY PARTS. *Let $f:\mathbb{R}\to\mathbb{R}$ and $g:\mathbb{R}\to\mathbb{R}$ two real-valued functions and $-\infty<a\le c\le b<+\infty$. If the integrals $\int_a^b g(x)\,df(x)$, $\int_a^c g(x)\,df(x)$ and $\int_c^b g(x)\,df(x)$ exist, then the integrals $\int_a^c f(x)\,dg(x)$ and $\int_c^b f(x)\;dg(x)$ also exist, and*

$$\begin{aligned}\int_a^b g(x)\,df(x) \quad &= \quad Ag(c)-\{g(b)[A-f(b)]+g(a)f(a)\} \\ &\quad +\int_c^b [A-f(x)]\;dg(x)-\int_a^c f(x)\;dg(x)\end{aligned} \tag{5.8}$$

for any real constant A, with

$$\int_a^c f(x)\,dg(x) = \int_a^c f(x)\,g'(x)\,dx \tag{5.9}$$

if g is continuous on $[a,\,c]$ as well as differentiable on (a,c) and the Riemann integral $\int_a^c f(x)\,g'(x)dx$ exists (where g' can take arbitrary real values at a and c), and

$$\int_c^b [A-f(x)]\,dg(x) = \int_c^b [A-f(x)]\,g'(x)\,dx \tag{5.10}$$

if g is continuous on $[c,\,b]$ as well as differentiable on (c,b) and the Riemann integral $\int_c^b [A-f(x)]\,g'(x)dx$ exists (where g' can take arbitrary real values at c and b).

5.6 Lemma IMPROPER RIEMANN-STIELTJES INTEGRATION BY PARTS. *Let $f:\mathbb{R}\to\mathbb{R}$ and $g:\mathbb{R}\to\mathbb{R}$ two real-valued functions, and $-\infty<a\le c\le b<+\infty$. Let us also denote*

$$\nu_{-\infty}[gf] = \lim_{a\to-\infty} g(a)f(a)\,,\; \nu_{\infty}[gf] = \lim_{b\to\infty} g(b)f(b)\;, \tag{5.11}$$

$$\nu_{-\infty}[g(1-f)] = \lim_{a\to-\infty}[1-f(a)]g(a)\,,\; \nu_{\infty}[g(1-f)] = \lim_{b\to\infty} g(b)[1-f(b)]\;, \tag{5.12}$$

whenever the limit considered exists.

(a) *If the (improper Riemann-Stieltjes) integral* $\int_a^{+\infty} g(x)\,df(x)$ *exists, then*

(i) $\int_a^{+\infty} f(x)\,dg(x)$ *exists and*

$$\int_a^{+\infty} g(x)\,df(x) = \nu_\infty[gf] - g(a)f(a) - \int_a^{+\infty} f(x)\,dg(x) \tag{5.13}$$

provided the limit $\nu_\infty[gf]$ *exists;*

(ii) $\int_a^{+\infty}[1-f(x)]\,dg(x)$ *exists and*

$$\int_a^{+\infty} g(x)\,df(x) = [1-f(a)]g(a) - \nu_\infty[g(1-f)] + \int_a^{+\infty}[1-f(x)]\,dg(x) \tag{5.14}$$

provided the limit $\nu_\infty[g(1-f)]$ *exists.*

(b) *If the integral* $\int_{-\infty}^{b} g(x)\,df(x)$ *exists, then*

(i) $\int_{-\infty}^{b} f(x)\,dg(x)$ *exists and*

$$\int_{-\infty}^{b} g(x)\,df(x) = g(b)f(b) - \nu_{-\infty}[gf] - \int_{-\infty}^{b} f(x)\,dg(x) \tag{5.15}$$

provided the limit $\nu_{-\infty}[gf]$ *exists;*

(ii) $\int_{-\infty}^{b}[1-f(x)]\,dg(x)$ *exists and*

$$\int_{-\infty}^{b} g(x)\,df(x) = \nu_{-\infty}[g(1-f)] - g(b)[1-f(b)] + \int_{-\infty}^{b}[1-f(x)]\,dg(x) \tag{5.16}$$

provided the limit $\nu_{-\infty}[g(1-f)]$ *exists.*

(c) *If the integral* $\int_{-\infty}^{+\infty} g(x)\,df(x)$ *exists, then*

(i) $\int_{-\infty}^{+\infty} f(x)\,dg(x)$ *exists and*

$$\int_{-\infty}^{+\infty} g(x)\,df(x) = \nu_\infty[gf] - \nu_{-\infty}[gf] - \int_{-\infty}^{+\infty} f(x)\,dg(x)\,, \tag{5.17}$$

provided the limits $\nu_\infty[gf]$ *and* $\nu_{-\infty}[gf]$ *exist;*

(ii) $\int_{-\infty}^{+\infty}[1-f(x)]\,dg(x)$ *exists and*

$$\int_{-\infty}^{+\infty} g(x)\,df(x) = \nu_{-\infty}[g(1-f)] - \nu_\infty[g(1-f)] + \int_{-\infty}^{+\infty}[1-f(x)]\,dg(x)\,, \tag{5.18}$$

provided the limits $\nu_{-\infty}[g(1-f)]$ *and* $\nu_\infty[g(1-f)]$ *exist.*

Furthermore, $dg(x)$ may be replaced by $g'(x)dx$ whenever g is differentiable everywhere in interior of the interval of integration considered and the corresponding Riemann integral exists.

5.7 Lemma IMPROPER CENTERED RIEMANN-STIELTJES INTEGRATION BY PARTS. *Let $f:\mathbb{R}\to\mathbb{R}$ and $g:\mathbb{R}\to\mathbb{R}$ two real-valued functions and $-\infty<a\le c\le b<+\infty$.*

(a) *If the integrals $\int_a^{+\infty} g(x)\,df(x)$, $\int_a^c g(x)\,df(x)$ and $\int_c^{+\infty} g(x)\,df(x)$ exist, then the integrals $\int_a^c f(x)\,dg(x)$ and $\int_c^{+\infty}[1-f(x)]\,dg(x)$ also exist and*

$$\begin{aligned}\int_a^{+\infty} g(x)\,df(x) &= g(c)-\{\nu_\infty[g(1-f)]+g(a)f(a)\}\\ &\quad+\int_c^{+\infty}[1-f(x)]\,dg(x)-\int_a^c f(x)\,dg(x),\end{aligned}\tag{5.19}$$

provided the limit $\nu_\infty[g(1-f)]$ exists.

(b) *If the integrals $\int_{-\infty}^b g(x)\,df(x)$, $\int_{-\infty}^c g(x)\,df(x)$ and $\int_c^b g(x)\,df(x)$ exist, then the integrals $\int_{-\infty}^c f(x)\,dg(x)$ and $\int_c^b[1-f(x)]\,dg(x)$ also exist and*

$$\begin{aligned}\int_{-\infty}^b g(x)\,df(x) &= g(c)-\{g(b)[1-f(b)]+\nu_{-\infty}[gf]\}\\ &\quad+\int_c^b[1-f(x)]\,dg(x)-\int_{-\infty}^c f(x)\,dg(x),\end{aligned}\tag{5.20}$$

provided the limit $\nu_{-\infty}[gf]$ exists.

(c) *If the integrals $\int_{-\infty}^{+\infty} g(x)\,df(x)$, $\int_{-\infty}^c g(x)\,df(x)$ and $\int_c^{+\infty} g(x)\,df(x)$ exist, then the integrals $\int_{-\infty}^c f(x)\,dg(x)$ and $\int_c^{+\infty}[1-f(x)]\,dg(x)$ also exist and*

$$\begin{aligned}\int_{-\infty}^{+\infty} g(x)\,df(x) &= g(c)-\{\nu_\infty[g(1-f)]+\nu_{-\infty}[gf]\}\\ &\quad+\int_c^{+\infty}[1-f(x)]\,dg(x)-\int_{-\infty}^c f(x)\,dg(x),\end{aligned}\tag{5.21}$$

provided the limits $\nu_\infty[g(1-f)]$and $\nu_{-\infty}[gf]$ exist.

Furthermore, $dg(x)$ may be replaced by $g'(x)dx$ whenever g is differentiable everywhere in interior of the interval of integration considered and the corresponding Riemann integral exists.

5.8 Lemma BOUNDED MONOTONICITY CONDITION FOR TAIL CONVERGENCE OF AN INTEGRABLE FUNCTION. *Let $f:\mathbb{R}\to\mathbb{R}$ and $g:\mathbb{R}\to\mathbb{R}$ two real-valued functions, and let m, M be two real constants.*

(a) *If $f(x)$ is monotonic nondecreasing on the interval $(-\infty, m)$ with finite limit as $x \to -\infty$, and if g satisfies the inequality*

$$|g(a)| \le B_L(x)\,, \text{ for } x \le a < m \tag{5.22}$$

where $B_L(x)$ is a real-valued function such that $\int_{-\infty}^{m} B_L(x)\,df(x)$ exists, then

$$0 \le |g(a)|\,[f(a) - f(-\infty)] \le \int_{-\infty}^{a} B_L(x)\,df(x)\,, \text{ for } a < m, \tag{5.23}$$

where $f(-\infty) = \lim_{x\to-\infty} f(x) > -\infty$, and

$$\lim_{a\to\infty} g(a)[f(a) - f(-\infty)] = 0\,. \tag{5.24}$$

(b) *If $f(x)$ is monotonic nondecreasing on the interval $[M, \infty)$ with finite limit as $x \to \infty$, and if g satisfies the inequality*

$$|g(b)| \le |g(x)| + B_U(x)\,, \text{ for } x \ge b > M \tag{5.25}$$

where $B_U(x)$ is a real-valued function such that $\int_{M}^{\infty} B_U(x)\,df(x)$ exists, then

$$0 \le |g(b)|\;[f(\infty) - f(b)] \le \int_{b}^{\infty} B_U(x)\,df(x)\,, \text{ for } b > M\,, \tag{5.26}$$

where $f(\infty) = \lim_{x\to\infty} f(x) < \infty$, and

$$\lim_{b\to\infty} g(b)\,[f(\infty) - f(b)] = 0\,. \tag{5.27}$$

It is easy to see that (5.22) holds whenever $\int_{-\infty}^{m} |g(x)|\,df(x)$ exists and one of the following conditions holds: for some real constant B,

$$|g(a)| \le |g(x)| + B\,, \text{ for } x \le a < m\,; \tag{5.28}$$

$$|g(x)| \text{ is nondecreasing on the interval } (M, \infty)\,; \tag{5.29}$$

$$g(x) \text{ is bounded on the interval } (M, \infty)\,. \tag{5.30}$$

Further, in case (5.28), we have:

$$0 \le |g(a)|\,[f(a) - f(-\infty)] \le \int_{-\infty}^{a} |g(x)|\,df(x) + B\,[f(a) - f(-\infty)]\,. \tag{5.31}$$

Similarly, (5.25) holds whenever one of the following conditions holds: for some real constant B,

$$|g(a)| \leq |g(x)| + B\,, \text{ for } x > b > M\,; \tag{5.32}$$

$$|g(x)| \text{ is nonincreasing on the interval } (-\infty, m)\,; \tag{5.33}$$

$$g(x) \text{ is bounded on the interval } (-\infty, m)\,. \tag{5.34}$$

Further, in case (5.32), we have:

$$0 \leq |g(b)|\,[f(\infty) - f(b)] \leq \int_b^\infty |g(x)|\,df(x) + B\,[f(\infty) - f(b)] \tag{5.35}$$

6. Tail decay rates and the existence of moments

It follows from Markov-type inequalities in Theorem 3.7 that tail probabilities decay to zero at least as fast as $g(x)$ or x^k increases. More precisely, on taking $m = 0$, we see from Theorem 3.7, that

$$g\,(a)\;\mathbb{P}[|X| \geq a] \leq \mathbb{E}\,[g\,(X)] \text{ for all } a > 0 \tag{6.1}$$

hence

$$\limsup_{a\to\infty}\{g\,(a)\;\mathbb{P}[|X| \geq a]\} = \limsup_{a\to\infty}\left\{\frac{\mathbb{P}[|X| \geq a]}{1/g\,(a)}\right\} \leq \mathbb{E}\,[g\,(X)]\,; \tag{6.2}$$

the rate of convergence of $\mathbb{P}[|X| \geq a]$ to zero must be at least as fast as the one of $1/g\,(a)$. In particular, if $\mathbb{E}\,(|X|^r) < \infty$ and $m = 0$,

$$a^r\,\mathbb{P}[|X| \geq a] \leq \mathbb{E}\,(|X|^r) \text{ for all } a > 0 \tag{6.3}$$

hence

$$\limsup_{a\to\infty}\{a^r\,\mathbb{P}[|X| \geq a]\} = \limsup_{a\to\infty}\left\{\frac{\mathbb{P}[|X| \geq a]}{1/a^r}\right\} \leq \mathbb{E}\,(|X|^r)\,; \tag{6.4}$$

the rate of convergence of $\mathbb{P}[|X| \geq a]$ to zero must be at least as fast as the one of $1/a^r$.

However, it is possible to make stronger statements on these rates of convergence by considering further generalizations of the Markov inequality.

6.1 Proposition TAIL DECAY RATES BASED ON TRUNCATED MOMENTS. *Under the assumptions of Theorem 3.11, the following limits hold:*

(a) *if $g(x)$ is nondecreasing on $[A_U, \infty)$,and if $g(a) > 0$ for $a \geq A_U$, then*

$$\limsup_{a\to\infty}\{g\,(a)\;\mathbb{P}[X \geq a]\} = \limsup_{a\to\infty}\left\{\frac{\mathbb{P}[X \geq a]}{1/g\,(a)}\right\} = 0\,; \tag{6.5}$$

(b) *if $g(x)$ is nonincreasing on $(-\infty, A_L]$, and if $g(a) > 0$ for $a \le A_L$, then,*

$$\limsup_{a\to-\infty}\{g(a)\,\mathbb{P}[X \le a]\} = \limsup_{a\to-\infty}\left\{\frac{\mathbb{P}[X \le a]}{1/g(a)}\right\} = 0\,; \tag{6.6}$$

(c) *if $g(x)$ is nondecreasing on $[A_U, \infty)$ and nonincreasing on $(-\infty, A_L]$, and if $g(a) > 0$ for $|a| \ge \max\{|A_U|, |A_L|\}$, then*

$$\limsup_{a\to\infty}\{g(a)\,\mathbb{P}[|X| \ge a]\} = \limsup_{a\to\infty}\left\{\frac{\mathbb{P}[|X| \ge a]}{1/g(a)}\right\} = 0\,. \tag{6.7}$$

Note that the limits (6.5) - (6.7), can be rewritten using the "little oh" notation $o(\cdot)$ [see Serfling (1980, Section 1.1.2)]:

$$\mathbb{P}[X \ge a] = o\big(g(a)^{-1}\big)\,,\ a \to \infty\,, \tag{6.8}$$

$$\mathbb{P}[X \le a] = o\big(g(a)^{-1}\big)\,,\ a \to -\infty\,, \tag{6.9}$$

$$\mathbb{P}[|X| \ge a] = o\big(g(a)^{-1}\big)\,,\ a \to \infty\,. \tag{6.10}$$

In particular, if $g(x) = |x|^r$, we have:

$$\mathbb{P}[X \ge a] = o\big(|a|^{-r}\big)\,,\ a \to \infty\,, \tag{6.11}$$

$$\mathbb{P}[X \le a] = o\big(|a|^{-r}\big)\,,\ a \to -\infty\,, \tag{6.12}$$

$$\mathbb{P}[|X| \ge a] = o\big(|a|^{-r}\big)\,,\ a \to \infty\,. \tag{6.13}$$

6.2 Corollary MOMENT EXISTENCE AND TAIL AREA DECAY. *Let $r > 0$. If $\mathbb{E}(|X|^r) < \infty$, then*

$$\begin{aligned}\lim_{x\to\infty}\{x^r\,\mathbb{P}[X \ge x]\} &= \lim_{x\to-\infty}\{|x|^r\,\mathbb{P}[X \le x]\}\\ &= \lim_{x\to\infty}\{x^r\,\mathbb{P}[|X| \ge x]\} = 0\,.\end{aligned} \tag{6.14}$$

In particular, if $\mathbb{E}(|X|) < \infty$, then

$$\begin{aligned}\lim_{x\to\infty}\{x\,\mathbb{P}[X \ge x]\} &= \lim_{x\to-\infty}\{|x|\,\mathbb{P}[X \le x]\}\\ &= \lim_{x\to\infty}\{x\,\mathbb{P}[|X| \ge x]\} = 0\,.\end{aligned} \tag{6.15}$$

The following proposition provides a general sufficient condition for the existence of a finite mean in terms of tail areas. It is a direct consequence of Proposition 4.1.

6.3 Proposition NECESSARY AND SUFFICIENT CONDITION FOR THE EXISTENCE OF THE MEAN. *Let X be a random variable with distribution function $F_X(x)$. Then*

$$\begin{aligned}\mathbb{E}(|X|) < \infty &\Leftrightarrow \mathbb{P}(|X| \geq x) \text{ is integrable on } (0, +\infty)\\ &\Leftrightarrow [1 - F_X(x) - F_X(-x)] \text{ is integrable on } (0, +\infty)\\ &\Leftrightarrow \int_0^\infty [1 - F_X(x)]dx < \infty \text{ and } \int_{-\infty}^0 F_X(x)dx < \infty .\end{aligned} \tag{6.16}$$

6.4 Proposition NECESSARY AND SUFFICIENT CONDITION FOR THE EXISTENCE OF r-MOMENTS. *Let X a random variable with distribution function $F_X(x)$ and $r > 0$. If $\mathbb{E}(|X|^r) < \infty$, then*

$$\begin{aligned}\mathbb{E}(|X|^r) < \infty &\Leftrightarrow x^{r-1}\mathbb{P}(|X| \geq x) \text{ is integrable on } (0, +\infty)\\ &\Leftrightarrow |x|^{r-1}[1 - F_X(x) + F_X(-x)] \text{ is integrable on } (0, +\infty)\\ &\Leftrightarrow \int_0^\infty x^{r-1}[1 - F_X(x)]dx < \infty \text{ and } \int_{-\infty}^0 |x|^{r-1} F_X(x)dx < \infty .\end{aligned} \tag{6.17}$$

7. Moments of sums of random variables

In this section, we consider a sequence $X_1, \ldots, X_n$ of random variables, and study the moments of the corresponding sum and average:

$$S_n = \sum_{i=1}^{n} X_i \,, \quad \bar{X}_n = S_n/n \,. \tag{7.1}$$

7.1 Proposition BOUNDS ON THE ABSOLUTE MOMENTS OF A SUM OF RANDOM VARIABLES.

$$\begin{aligned}\mathbb{E}(|S_n|^r) &\leq \sum_{i=1}^{n} \mathbb{E}(|X_i|^r) , && \text{if } 0 < r \leq 1 ,\\ &\leq n^{r-1} \sum_{i=1}^{n} \mathbb{E}(|X_i|^r) , && \text{if } r > 1 ,\end{aligned} \tag{7.2}$$

and

$$\begin{aligned}\mathbb{E}(|\bar{X}_n|^r) &\leq \left(\tfrac{1}{n}\right)^r \sum_{i=1}^{n} \mathbb{E}(|X_i|^r) , && \text{if } 0 < r \leq 1 ,\\ &\leq \tfrac{1}{n} \sum_{i=1}^{n} \mathbb{E}(|X_i|^r) , && \text{if } r > 1 .\end{aligned} \tag{7.3}$$

7.2 Proposition MINKOWSKI INEQUALITY FOR n VARIABLES. *If* $r \geq 1$, *then*

$$[\mathbb{E}(|S_n|^r)]^{1/r} \leq \sum_{i=1}^{n} [\mathbb{E}(|X_i|^r)]^{1/r} \tag{7.4}$$

and

$$\begin{aligned} [\mathbb{E}(|\bar{X}_n|^r)]^{1/r} &\leq \frac{1}{n}\sum_{i=1}^{n} [\mathbb{E}(|X_i|^r)]^{1/r} \\ &\leq \left\{\frac{1}{n}\sum_{i=1}^{n} \mathbb{E}(|X_i|^r)\right\}^{1/r}. \end{aligned} \tag{7.5}$$

7.3 Proposition BOUNDS ON THE ABSOLUTE MOMENTS OF A SUM OF RANDOM VARIABLES UNDER CONDITIONAL SYMMETRY. *If the distribution of* X_{k+1} *given* S_i *is symmetric about zero for* $k = 1, \dots, n-1$, *and* $\mathbb{E}(|X_i|^r) < \infty$, $i = 1, \dots, n$, *then*

$$\mathbb{E}(|S_n|^r) \leq \sum_{i=1}^{n} \mathbb{E}(|X_i|^r) \quad \textit{for } 1 \leq r \leq 2, \tag{7.6}$$

and

$$\mathbb{E}(|\bar{X}_n|^r) \leq \left(\frac{1}{n}\right)^r \sum_{i=1}^{n} \mathbb{E}(|X_i|^r) \quad \textit{for } 1 \leq r \leq 2, \tag{7.7}$$

with equality holding when $r = 2$.

7.4 Proposition BOUNDS ON THE ABSOLUTE MOMENTS OF A SUM OF RANDOM VARIABLES UNDER MARTINGALE CONDITION. *If*

$$\mathbb{E}(X_{k+1} \,|\, S_k) = 0 \quad \textit{a.s.}, \quad k = 1, \dots, n-1, \tag{7.8}$$

and $\mathbb{E}(|X_i|^r) < \infty$, $i = 1, \dots, n$, *then*

$$\mathbb{E}(|S_n|^r) \leq 2\sum_{i=1}^{n} \mathbb{E}(|X_i|^r), \quad \textit{for } 1 \leq r \leq 2, \tag{7.9}$$

and

$$\mathbb{E}(|\bar{X}_n|^r) \leq 2\left(\frac{1}{n}\right)^r \sum_{i=1}^{n} \mathbb{E}(|X_i|^r), \quad \textit{for } 1 \leq r \leq 2. \tag{7.10}$$

Furthermore, for $r = 2$,

$$\mathbb{E}({S_n}^2) = \sum_{i=1}^{n} \mathbb{E}(X_i^2)\,. \tag{7.11}$$

7.5 Proposition BOUNDS ON THE ABSOLUTE MOMENTS OF A SUM OF RANDOM VARIABLES UNDER TWO-SIDED MARTINGALE CONDITION. *Let*

$$S_{m(k)} = \sum_{i=1,\ i\neq k}^{m+1} X_i\,, \quad 1 \le k \le m+1 \le n. \tag{7.12}$$

If

$$\mathbb{E}(X_k \,|\, S_{m(k)}) = 0 \quad \text{a.s.,} \quad \text{for } 1 \le k \le m+1 \le n, \tag{7.13}$$

and $\mathbb{E}(|X_i|^r) < \infty$, $i = 1, \ldots, n$, *then*

$$\mathbb{E}(|S_n|^r) \le \left(2 - \frac{1}{n}\right) \sum_{i=1}^{n} \mathbb{E}(|X_i|^r)\,, \quad \text{for } 1 \le r \le 2\,, \tag{7.14}$$

and

$$\mathbb{E}(|\bar{X}_n|^r) \le \left(\frac{1}{n}\right)^r \left(2 - \frac{1}{n}\right) \sum_{i=1}^{n} \mathbb{E}(|X_i|^r)\,, \quad \text{for } 1 \le r \le 2\,. \tag{7.15}$$

7.6 Proposition BOUNDS ON THE ABSOLUTE MOMENTS OF A SUM OF INDEPENDENT RANDOM VARIABLES. *Let the random variables* $X_1, \ldots, X_n$ *be independent with* $\mathbb{E}(X_i) = 0$ *and* $\mathbb{E}(|X_i|^r) < \infty$, $i = 1, \ldots, n$, *and let*

$$D(r) = [13.52/(2.6\pi)^r]\,\Gamma(r)\sin(r\pi/2)\,. \tag{7.16}$$

If $D(r) < 1$ *and* $1 \le r \le 2$, *then*

$$\mathbb{E}(|S_n|^r) \le [1 - D(r)]^{-1} \sum_{i=1}^{n} \mathbb{E}(|X_i|^r)\,, \tag{7.17}$$

and

$$\mathbb{E}(|\bar{X}_n|^r) \le \left(\frac{1}{n}\right)^r [1 - D(r)]^{-1} \sum_{i=1}^{n} \mathbb{E}(|X_i|^r)\,, \quad \text{for } 1 \le r \le 2\,. \tag{7.18}$$

8. Proofs and references

1.1 to 3.8. See Loève (1977, Volume I, Sections 9.1 and 9.3, pp. 151-162). For Jensen inequality, see also Chow and Teicher (1988, Section 4.3, pp. 103-106). Hannan (1985), Lehmann and Shaffer (1988), Piegorsch and Casella (1988) and Khuri and Casella (2002) discussed conditions for the existence of the moments of $1/X$.
3. Reviews of moment-based bounds on tail probabilities are available in Godwin (1964).
2.9. See von Bahr and Esseen (1965, Lemma 3).
3.3. Discussions and extensions of the Markov inequality are given by Ghosh (2002), Cohen (2015).
3.11
PROOF OF THEOREM 3.11 (a) For $x \geq a \geq A_U$, we have $g(x) \geq g(a)$ and $g(x) \leq M_U$, hence

$$C_U(g,a) = \int_{[a,\infty)} g(x)\, dF_X(x) \geq g(a) \int_{[a,\infty)} dF_X(x) = g(a)\,\mathbb{P}[X \geq a]$$

and

$$\int_{[a,\infty)} g(x)\, dF_X(x) \leq M_U \mathbb{P}[X \geq a]\,,$$

from which we get the inequality

$$\frac{C_U(g,a)}{M_U} \leq \mathbb{P}[X \geq a] \leq \frac{C_U(g,a)}{g(a)}\,.$$

(b) For $x \leq a \leq A_L$, we have $g(x) \geq g(a)$ and $g(x) \leq M_L$, hence

$$C_L(g,a) = \int_{[-\infty,a)} g(x)\, dF_X(x) \geq g(a) \int_{[-\infty,a)} dF_X(x) = g(a)\,\mathbb{P}[X \leq a]$$

and

$$\int_{[-\infty,a)} g(x)\, dF_X(x) \leq M_L \mathbb{P}[X \leq a]$$

from which we get the inequality

$$\frac{C_L(g,a)}{M_L} \leq \mathbb{P}[X \leq a] \leq \frac{C_L(g,a)}{g(a)}\,.$$

(c) For $a \geq \max(|A_U|, |A_L|)$, we have $a \geq A_U$ and $-a \leq A_L$, hence

$$\mathbb{P}\mathbb{P}[|X| \geq a] \quad = \quad \mathbb{P}[X \geq a] + \mathbb{P}[X \leq -a]$$

$$
\begin{aligned}
&\leq \frac{C_U(g,a)}{g(a)} + \frac{C_L(g,a)}{g(a)} \\
&\leq \frac{C_U(g,a) + C_L(g,a)}{\min\{g(a), g(-a)\}}
\end{aligned}
$$

and

$$
\begin{aligned}
\mathbb{P}[|X| \geq a] &\geq \frac{C_U(g,a)}{M_U} + \frac{C_L(g,a)}{M_L} \\
&\geq \frac{C_U(g,a) + C_L(g,a)}{\max(M_U, M_L)}.
\end{aligned}
$$

□

PROOF OF PROPOSITION 4.1 We have:

$$
\mathbb{E}(X) = \int_0^\infty x\, dF_X(x) + \int_{-\infty}^0 x\, dF_X(x). \tag{8.1}
$$

For any statement p, let us set

$$
\begin{aligned}
I(p) &= 1 \quad \text{if } p \text{ is true} \\
&= 0 \quad \text{if } p \text{ is false}
\end{aligned}
. \tag{8.2}
$$

Using this notation, we can write:

$$
\begin{aligned}
x &= \int I(0 < y < x) dy \quad && \text{for } x \geq 0 \\
&= -\int I(x \leq y \leq 0) dy \quad && \text{for } x < 0
\end{aligned}
. \tag{8.3}
$$

Then,

$$
\begin{aligned}
\int_0^\infty x F_X(x) &= \int I(x>0)\, x\, dF_X(x) \\
&= \int I(x>0) \left[\int I(0<y<x) dy\right] dF_X(x) \\
&= \int\int I(x>0) I(0<y<x)\, dF_X(x)\, dy \\
&= \int\int I(y<x) I(y>0)\, dF_X(x)\, dy \\
&= \int \left[\int I(y<x)\, dF_X(x)\right] I(y>0)\, dy
\end{aligned}
$$

$$
\begin{aligned}
&= \int \Big[\int_{x>y} dF_X(x) \Big] I(y>0)\, dy \\
&= \int [1-F_X(y)] I(y>0)\, dy = \int_0^\infty [1-F_X(y)]\, dy \\
&= \int_0^\infty [1-F_X(x)]\, dx = \int_0^\infty \mathbb{P}[X \geq x] dx\,. \qquad (8.4)
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\int_{-\infty}^0 xF_X(x)dx &= \int I(x \leq 0)\, x\, dF_X(x) \\
&= -\int I(x \leq 0) \Big[\int I(x \leq y \leq 0)\, dy \Big]\, dF_X(x) \\
&= -\int\int I(x \leq y \leq 0)\, I(x \leq 0)\, dF_X(x)\, dy \\
&= -\int\int I(x \leq y)\, I(y \leq 0)\, dF_X(x)\, dy \\
&= -\int \Big[\int I(x \leq y)\, dF_X(x) \Big]\, I(y \leq 0)\, dy \\
&= -\int F_X(y)\, I(y \leq 0)\, dy = -\int_{-\infty}^0 F_X(y)\, dy \\
&= -\int_{-\infty}^0 F_X(x)\, dx = -\int_{-\infty}^0 \mathbb{P}[X \leq x] dx\,. \qquad (8.5)
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\mathbb{E}(X) &= \int_0^\infty [1-F_X(x)]dx - \int_{-\infty}^0 F_X(x)dx \\
&= \int_0^\infty \mathbb{P}[X \geq x]dx - \int_{-\infty}^0 \mathbb{P}[X \leq x]dx\,. \qquad (8.6)
\end{aligned}
$$

Finally, (4.4) follows on observing that

$$
\int_{-\infty}^0 |x|\, F_X(x)dx = -\int_{-\infty}^0 xF_X(x)dx = \int_{-\infty}^0 F_X(x)\, dx \int_{-\infty}^0 \mathbb{P}[X \leq x] dx\,. \qquad (8.7)
$$

□

4.2 to 6.2. See Feller (1966, Section V.6, Lemma 1), Chung (1974, Section 3.2, Exercises 17-18), Serfling (1980, Section 1.14, pp. 46-47) and Chow and Teicher (1988, Section 4.3, pp. 103-106). For other inequalities involving absolute moments, the reader may consult Beesack (1984). Further discussion of Proposition 4.1 is available in Hong (2012, 2015) and Lo (2019).

PROOF OF THEOREM 4.4 For any $a \in \mathbb{R}$, we have:

$$\mathbb{E}[g(X)] = \int_a^\infty g(x)\, dF_X(x) + \int_{-\infty}^a g(x)\, dF_X(x)\,. \tag{8.8}$$

For any statement p, let us set

$$\begin{aligned} I(p) \quad &= 1 \quad \text{if } p \text{ is true} \\ &= 0 \quad \text{if } p \text{ is false}\,. \end{aligned} \tag{8.9}$$

By the fundamental theorem of calculus, we can write: for any $a \in \mathbb{R}$,

$$\begin{aligned} g(x) \quad &= g(a) + \textstyle\int_a^x g'(y)\, dy \quad \text{if } x \geq a \\ &= g(a) - \textstyle\int_a^x g'(y)\, dy \quad \text{if } x < a \end{aligned}\,. \tag{8.10}$$

Then,

$$\begin{aligned}
\int_a^\infty g(x)\, dF_X(x) \quad &= \quad \int I(a < x)\, g(x)\, dF_X(x) \\
&= \quad \int I(a < x) \left[g(a) + \int_a^x g'(y)\, dy \right] dF_X(x) \\
&= \quad \int I(a < x)\, g(a)\, dF_X(x) + \int I(a < x) \left[\int_a^x g'(y)\, dy \right] dF_X(x) \\
&= \quad g(a)\mathbb{P}[X > a] + \int I(a < x) \left[\int I(a < y \leq x)\, g'(y)\, dy \right] dF_X(x) \\
&= \quad g(a)\mathbb{P}[X > a] + \int \int I(a < x) I(a < y \leq x)\, g'(y)\, dy\, dF_X(x) \\
&= \quad g(a)\mathbb{P}[X > a] + \int \int I(y \leq x) I(a < y)\, g'(y)\, dy\, dF_X(x) \\
&= \quad g(a)\mathbb{P}[X > a] + \int \int I(a < y)\, g'(y)\, I(y \leq x)\, dF_X(x)\, dy \\
&= \quad g(a)\mathbb{P}[X > a] + \int I(a < y)\, g'(y) \left[\int I(y \leq x)\, dF_X(x) \right] dy \\
&= \quad g(a)\mathbb{P}[X > a] + \int I(a < y)\, g'(y)\, \mathbb{P}\left[X \geq y\right] dy
\end{aligned}$$

$$
\begin{aligned}
&= g(a)\mathbb{P}[X > a] + \int I(a < y)\, g'(y)\, \mathbb{P}\,[X > y]\, dy \\
&= g(a)\mathbb{P}[X > a] + \int_a^\infty g'(y)\, \mathbb{P}\,[X > y]\, dy \\
&= g(a)\mathbb{P}[X > a] + \int_a^\infty g'(y)\, [1 - F_X(y)]\, dy\,. \qquad (8.11)
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\int_{-\infty}^a g(x)\, dF_X(x) &= \int I(x \le a)\, g(x)\, dF_X(x) \\
&= \int I(x \le a) \big[g(a) - \int_a^x g'(y)\, dy\big]\, dF_X(x) \\
&= g(a) \int I(x \le a)\, dF_X(x) - \int I(x \le a) \big[\int_a^x g'(y)\, dy\big]\, dF_X(x) \\
&= g(a)\mathbb{P}[X \le a] - \int I(x \le a) \big[\int I(x < y \le a) g'(y)\, dy\big]\, dF_X(x) \\
&= g(a)\mathbb{P}[X \le a] - \int\int I(x \le a) I(x < y \le a) g'(y)\, dy\, dF_X(x) \\
&= g(a)\mathbb{P}[X \le a] - \int\int I(x < y) I(y \le a) g'(y)\, dy\, dF_X(x) \\
&= g(a)\mathbb{P}[X \le a] - \int\int I(y \le a) g'(y)\, I(x < y)\, dF_X(x)\, dy \\
&= g(a)\mathbb{P}[X \le a] - \int I(y \le a) g'(y) \big[\int I(x < y)\, dF_X(x)\big]\, dy \\
&= g(a)\mathbb{P}[X \le a] - \int I(y \le a) g'(y) \mathbb{P}[X < y]\, dy \\
&= g(a)\mathbb{P}[X \le a] - \int I(y \le a) g'(y) \mathbb{P}[X \le y]\, dy \\
&= g(a)\mathbb{P}[X \le a] - \int_{-\infty}^a g'(y) \mathbb{P}[X \le y]\, dy \\
&= g(a) F_X(a) - \int_{-\infty}^a g'(y) F_X(y)\, dy\,. \qquad (8.12)
\end{aligned}
$$

Therefore,

$$\begin{aligned}
\mathbb{E}[g(X)] &= \int_a^\infty g(x)\,dF_X(x) + \int_{-\infty}^a g(x)\,dF_X(x) \\
&= g(a) + \int_a^\infty g'(y)\,[1-F_X(y)]\,dy - \int_{-\infty}^a g'(y)F_X(y)\,dy \\
&= g(a) + \int_a^\infty g'(y)\,\mathbb{P}\,[X > y]\;dy - \int_{-\infty}^a g'(y)\mathbb{P}\,[X \le y]\;dy \\
&= g(a) + \int_a^\infty g'(y)\,\mathbb{P}\,[X \ge y]\;dy - \int_{-\infty}^a g'(y)\mathbb{P}\,[X \le y]\;dy \qquad (8.13)
\end{aligned}$$

□

4.8 The identity (4.22) is stated by Chung (1974, Section 3.2, Exercise 16). We give below a simple proof along with a slight extension.
PROOF OF PROPOSITION 4.8 We can establish (4.22) as follows, through the use of the indicator function $I(p)$ defined in (8.2):

$$\begin{aligned}
\int [F_X(x+b) - F_X(x)]dx &= \int \mathbb{P}[a < X \le x+b]dx \\
&= \int \Big[\int I(x < y \le x+b)dF_X(y)\Big]dx \\
&= \int \Big[\int I(x < y \le x+b)dx\Big]dF_X(y) \\
&= \int b\,dF_X(y) = b\,. \qquad (8.14)
\end{aligned}$$

(4.23) then follows by repeated application of the above identity:

$$\begin{aligned}
\int [F_X(x+b) - F_X(x-a)]dx &= \int \mathbb{P}[x-a < X \le x+b]dx \\
&= \int \mathbb{P}[x-a < X \le x]dx + \int \mathbb{P}[x < X \le x+b]dx \\
&= \int \mathbb{P}[x < X \le x+a]dx + \int \mathbb{P}[x < X \le x+b]dx \\
&= a+b\,. \qquad (8.15)
\end{aligned}$$

□

4.6. See Chow and Teicher (1988, Section 4.1, Corollary 3, p. 90).
4.7. The inequality (4.21) is given by Chung (1974, Theorem 3.2.1) and Serfling (1980, Section 1.3, p. 12).
Theorem 5.1 (a) See Riesz and Sz.-Nagy (1955/1990, Section 54) and Devinatz (1968, Theorem 5.5.2, page 219).
Theorem 5.2. See Devinatz (1968, Theorem 5.5.7, page 225).
Theorem 5.3 See Riesz and Sz.-Nagy (1955/1990, Section 54) and Haaser and Sullivan (1991, Theorem 2.8, page 254).
Theorem 5.4 See Riesz and Sz.-Nagy (1955/1990, Section 54) for $A = 0$.
PROOF OF LEMMA 5.4 The first identity in (5.5) part is given by Devinatz (1968, Theorem 5.4.8, page 213) and Protter and Morrey (1991, Theorem 12.12, page 320), while the second follows from the latter on observing that

$$\begin{aligned} -\int_a^b f(x)\,dg(x) &= \int_a^b [A - f(x) - A]\,dg(x) = \int_a^b [A - f(x)]\,dg(x) - A\int_a^b dg(x) \\ &= \int_a^b [A - f(x)]\,dg(x) - A\,[g(b) - g(a)] \end{aligned} \tag{8.16}$$

and rearranging the terms of the sum. Equation (8.16) also entails the existence of the integral $\int_a^b [1 - f(x)]\,dg(x)$. The identities (5.6)-(5.7) follow on observing that we can write $dg(x) = g'(x)dx$ when g is differentiable [see Devinatz (1968, Theorem 5.4.7, page 213)]. □

5.5
PROOF OF LEMMA 5.5 Using Lemma 5.5, we get:

$$\begin{aligned} \int_a^b g(x)\,df(x) &= \int_a^c g(x)\,df(x) + \int_c^b g(x)\,df(x) \\ &= g(c)f(c) - g(a)f(a) - \int_a^c f(x)\,dg(x) \\ &\quad + [A - f(c)]g(c) - g(b)[A - f(b)] + \int_c^b [A - f(x)]\,dg(x) \\ &= Ag(c) - \{g(b)[A - f(b)] + g(a)f(a)\} \\ &\quad + \int_c^b [A - f(x)]\,dg(x) - \int_a^c f(x)\,dg(x)\,. \end{aligned} \tag{8.17}$$

□

5.8
PROOF OF LEMMA 5.8 (a) The existence of the limit $\lim_{x\to-\infty} f(x)$ entails that the integral $\int_{-\infty}^a df(x) = f(a) - f(-\infty)$ also exists. Since $f(x)$ is monotonic nondecreasing

on the interval $(-\infty, m)$ and $\int_{-\infty}^{m} B_L(x)\, df(x)$ exists, we get from (5.22): for $a < m$,

$$0 \le \int_{-\infty}^{a} |g(a)|\, df(x) \le \int_{-\infty}^{a} B_L(x)\, df(x) \tag{8.18}$$

hence

$$0 \le |g(a)|\, [f(a) - f(-\infty)] \le \int_{-\infty}^{a} B_L(x)\, df(x). \tag{8.19}$$

Letting $a \to -\infty$, this yields

$$0 \le \lim_{a \to -\infty} |g(a)|\, [f(a) - f(-\infty)] \le \lim_{a \to -\infty} \int_{-\infty}^{a} B_L(x)\, df(x) = 0 \tag{8.20}$$

and

$$\lim_{a \to \infty} g(a)[f(a) - f(-\infty)] = 0\,. \tag{8.21}$$

(b) The existence of the limit $\lim_{x \to \infty} f(x)$ entails that the integral $\int_{b}^{\infty} df(x) = f(\infty) - f(b)$ also exists. Since $f(x)$ is monotonic nondecreasing on the interval (M, ∞) and $\int_{M}^{\infty} B_U(x)\, df(x)$ exists, we get from (5.25): for $b > M$,

$$0 \le \int_{b}^{\infty} |g(b)|\, df(x) \le \int_{b}^{\infty} B_U(x)\, df(x) \tag{8.22}$$

hence

$$0 \le |g(b)|\, [f(\infty) - f(b)] \le \int_{b}^{\infty} B_U(x)\, df(x)\,. \tag{8.23}$$

Letting $b \to \infty$, this yields

$$0 \le \lim_{b \to \infty} |g(b)|\, [f(\infty) - f(b)] \le \lim_{b \to \infty} \int_{b}^{\infty} B_U(x)\, df(x) = 0 \tag{8.24}$$

and

$$\lim_{b \to \infty} g(b)\, [f(\infty) - f(b)] = 0\,. \tag{8.25}$$

□

7.1. See von Bahr and Esseen (1965), Chung (1974, p. 48) and Chow and Teicher (1988, p. 108).

7.2. See Chung (1974, p. 48).

PROOF OF PROPOSITION 7.2 The first inequality follows by recursion on applying the Minkowski inequality for two variables. The first part of the second inequality is obtained by multiplying both sides of the first one by $(1/n)$. The second part follows on observing that the function $x^{1/r}$ is concave in x for $x > 0$ when $r > 1$. □

7.3. See von Bahr and Esseen (1965, Theorem 1).

7.4. See von Bahr and Esseen (1965, Theorem 2).
7.5. See von Bahr and Esseen (1965, Theorem 3).
7.6. See von Bahr and Esseen (1965, Theorem 4).

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