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ECONOMETRICS 1
EXERCISES 2

Covariances and covariance matrices

1. Respond by TRUE, FALSE or UNCERTAIN to each one of the following statements and explain your answer (Maximum: 1 page per statement)

   (a) If a random variable X has zero variance, it is unpredictable.

   (b) If the correlation between two random variables X and Y is one, this means that one of them has variance zero.

   (c) If X is a random variable with zero variance, then its covariance with any other random variable Y cannot be negative.

   (d) If X is a random variable with variance equal to zero, then its correlation with any other random variable Y must be positive.

   (e) If X is a random variable with variance one, then its correlation with any other random variable Y must be positive.

2. Let X and Y be two random variables with finite variances $\mathbb{V}(X)$ and $\mathbb{V}(Y)$. Prove the following inequality:
$$\mathsf{C}(X, Y)^2 \le \mathbb{V}(X)\,\mathbb{V}(Y)\,. \tag{1}$$

3. Let $\mathbf{X} = (X_1, \ldots, X_k)'$ a $k \times 1$ random vector, α a scalar, $\mathbf{a}$ and $\mathbf{b}$ fixed $k \times 1$ vectors, and A a fixed $g \times k$ matrix. Then, provided the moments considered are finite, show that the following properties hold:

   (a) $\mathsf{E}(\mathbf{X}+\mathbf{a}) = \mathsf{E}(\mathbf{X}) + \mathbf{a}$;

   (b) $\mathsf{E}(\alpha\mathbf{X}) = \alpha\mathsf{E}(\mathbf{X})$;

   (c) $\mathsf{E}(\mathbf{a}'\mathbf{X}) = \mathbf{a}'\mathsf{E}(\mathbf{X})$, $\mathsf{E}(A\mathbf{X}) = A\mathsf{E}(\mathbf{X})$;

   (d) $\mathsf{V}(\mathbf{X}+\mathbf{a}) = \mathsf{V}(\mathbf{X})$;

   (e) $\mathsf{V}(\alpha\mathbf{X}) = \alpha^2\mathsf{V}(\mathbf{X})$;

   (f) $\mathsf{V}(\mathbf{a}'\mathbf{X}) = \mathbf{a}'\mathsf{V}(\mathbf{X})\,\mathbf{a}$, $\mathsf{V}(A\mathbf{X}) = A\mathsf{V}(\mathbf{X})A'$;

   (g) $\mathsf{C}(\mathbf{a}'\mathbf{X}, \mathbf{b}'\mathbf{X}) = \mathbf{a}'\mathsf{V}(\mathbf{X})\,\mathbf{b} = \mathbf{b}'\mathsf{V}(\mathbf{X})\,\mathbf{a}$.

4. Let $\mathbf{X} = (X_1, \ldots, X_k)'$ be a random vector with finite second moments and let $\Sigma = \mathsf{V}(\mathbf{X})$ be its covariance matrix. Prove the following properties.

(a) $\Sigma' = \Sigma$.

(b) Σ is a positive semidefinite matrix.