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ECON 469 – ECONOMETRICS

MID-TERM EXAM

SOLUTIONS

(20) 1. (a) This is a MA(1) model, so that

$$\gamma(0) = (1+\theta^2)\sigma^2,$$

$$\gamma(1) = -\theta\sigma^2,$$

$$\rho(1) = -\theta/(1+\theta^2).$$

The derivative of $\rho(1)$ with respect to θ is:

$$\frac{\partial\rho(1)}{\partial\theta} = \frac{-(1+\theta^2)+\theta(2\theta)}{(1+\theta^2)^2} = \frac{-1+\theta^2}{(1+\theta^2)^2}$$

hence

$$\frac{\partial\rho(1)}{\partial\theta} \begin{cases} >0 & \text{if } \theta<-1 \\ =0 & \text{if } \theta=-1 \\ <0 & \text{if } |\theta|<1 \\ =0 & \text{if } \theta=1 \\ >0 & \text{if } \theta>1 \end{cases}$$

This entails that $\rho(1)$ has a minimum at $\theta=-1$ and a maximum at $\theta=1$

Since

$\rho(1) = 0.5$ if $\theta = -1$,

$\rho(1) = -0.5$ if $\theta = 1$,

$\rho(1) \longrightarrow 0$ as $\theta \rightarrow -\infty$,

$\rho(1) \longrightarrow 0$ as $\theta \rightarrow +\infty$,

it follows that

$|\rho(1)| \leq 0.5$.

(5) The upper bound of $|\rho(1)|$ is attained at $\theta = -1$ and $\theta = 1$ with

$\rho(1) = -0.5$ if $\theta = -1$

$\rho(1) = 0.5$ if $\theta = +1$.

2. We can consider the model:

$$X_t = 10 + u_t - 0.75\, u_{t-1} + 0.125\, u_{t-1}$$

$$\{u_t : t \in \mathbb{Z}\} \overset{iid}{\sim} N(0,1)$$

(a) Yes. This process is an MA(2) and consequently it is stationary.

(b) Yes.

We have:

$$X_t = 10 + \theta(B)\, u_t$$

where

$$\theta(B) = 1 - 0.75B + 0.125B^2$$

$$= (1-0.5B)(1-0.25B)$$

The roots of the equation

$$\theta(z) = 0$$

are $z_1 = 1/0.5 = 2.0$, $z_2 = 1/0.25 = 4$

Since $|z_1| > 1$ and $|z_2| > 1$, the roots are outside the unit circle.

2.(c) i) $E(X_t) = 10$

ii) $\gamma(0) = Var(X_t) = [1 + (0.75)^2 + (0.125)^2] = 1.578125$

$\gamma(1) = E[(X_t - 10)(X_{t-1} - 10)]$

$= E[(u_t - 0.75u_{t-1} + 0.125u_{t-2})(u_{t-1} - 0.75u_{t-2} + 0.125u_{t-3})]$

$= 0.75 - (0.125)(0.75) = -0.84375$

$\gamma(2) = E[(X_t - 10)(X_{t-2} - 10)]$

$= E[(u_t - 0.75u_{t-1} + 0.125u_{t-2})(u_{t-2} - 0.75u_{t-3} + 0.125u_{t-4})]$

$= 0.125$

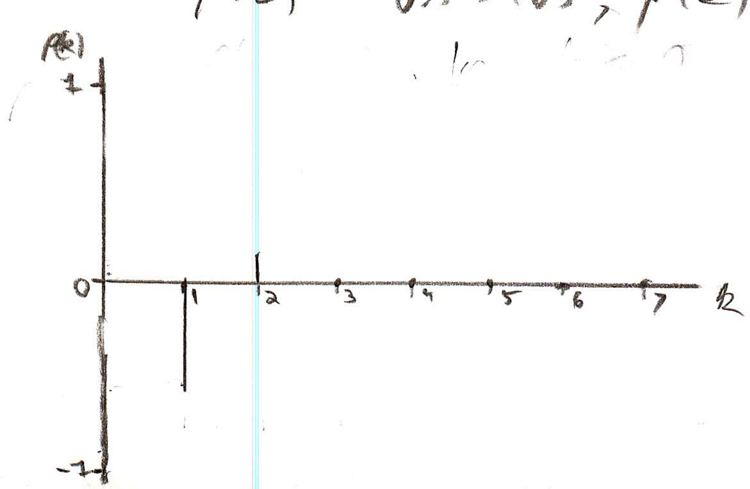
$\gamma(k) = 0$, for $k \geq 3$

Thus: $\gamma(0) = 1.578125$, $\gamma(1) = -0.84375$, $\gamma(2) = 0.125$
$\gamma(k) = 0$, for $k \geq 3$

iii) $\rho(k) = \gamma(k)/\gamma(0)$

$\rho(1) = -0.53465$, $\rho(2) = 0.07921$, $\rho(k) = 0$, for $k \geq 0$

(d)



2. (e) Since the model is already in MA form, the coefficients of the MA representation are the same:

$$X_t = 10 + \sum_{k=0}^{\infty} \psi_k u_{t-k}$$

where

$\psi_0 = 1$, $\psi_1 = -0.75$, $\psi_2 = 0.125$,

$\psi_k = 0$, for $k \geq 3$.

(f) The first partial autocorrelation is

$$r_{11} = \rho(1) = -0.53465$$

The second partial autocorrelation is obtained by solving the equations:

$$\rho(1) = \rho(0) r_{21} + \rho(1) r_{22} = r_{21} + \rho(1) r_{22}$$

$$\rho(2) = \rho(1) r_{21} + \rho(0) r_{22} = r_{21} \rho(1) + r_{22}$$

or

$$\begin{bmatrix} \rho(1) \\ \rho(2) \end{bmatrix} = \begin{bmatrix} 1 & \rho(1) \\ \rho(1) & 1 \end{bmatrix} \begin{bmatrix} r_{21} \\ r_{22} \end{bmatrix}$$

hence

$$r_{22} = \begin{vmatrix} 1 & \rho(1) \\ \rho(1) & \rho(2) \end{vmatrix} \Bigg/ \begin{vmatrix} 1 & \rho(1) \\ \rho(1) & 1 \end{vmatrix} = \frac{\rho(2) - \rho(1)^2}{1 - \rho(1)^2}$$

$$= \frac{-0.20664}{0.71415} = -0.28935$$

2.(g) $X_8 = 11$ is the only information available to make the forecast.

i) We need to use the formula

$$P_L(X_{t+1} \mid X_t) = \beta_0 + \beta_1 X_t$$

$$\beta_1 = \frac{Cov(X_{t+1}, X_t)}{V(X_t)} = \frac{\gamma(1)}{\gamma(0)} = \rho(1) = -0.53465$$

$$\beta_0 = 10 - \beta_1(10) = 10(1-\beta_1) = 15.3465$$

$$P_L(X_9 \mid X_8) = 15.3465 + (-0.53465)(11)$$

$$= 10 - 0.53465 = 9.46535$$

ii) Since the process is Gaussian,

$E(X_{t+1} \mid X_t) = P_L(X_{t+1} \mid X_t)$

hence

$$E(X_9 \mid X_8) = 9.46535$$

(h) i) X_{15} is uncorrelated with all X_t, $t \leq 10$,

thus

$$P_L(X_{15} \mid X_t, t \leq 10) = E(X_t) = 10.$$

ii) By the Gaussian assumption,

X_{15} is independent of $\{X_t, t \leq 10\}$.

thus

$$E(X_{15} \mid X_t, t \leq 10) = E(X_{15}) = 10.$$

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3. (a) i) The sample autocorrelations are

$$r_k = \sum_{t=1}^{T-k} (X_{t+k} - \bar{X})(X_t - \bar{X}) \Big/ \sum_{t=1}^{T} (X_t - \bar{X})^2$$

where $\bar{X} = \frac{1}{T} \sum_{t=1}^{T} X_t$

ii) The sample partial autocorrelations are obtained by solving the equation

$$r_j = \sum_{j=1}^{k} a_{kj} \, r_{j-k} \, , \quad j = 1, 2, \dots, k$$

to obtain a_{kk} $(k = 1, 2, \dots)$

(b) i) For any $m \geq 1$,

$$\sqrt{T}\left[(r_1 - \rho_1), \dots, (r_m - \rho_m)\right]' \xrightarrow[T\to\infty]{L} N[0, I_m]$$

when $\rho_1 = \dots = \rho_m = 0$,

so that

$$\sqrt{T}\left[r_1, \dots, r_m\right]' \xrightarrow[T\to\infty]{L} N[0, I_m]$$

Similarly

$$\sqrt{T}\,(a_{11}, \dots, a_{mm})' \xrightarrow[T\to\infty]{L} N[0, I_m]$$

(b) ii) For any $m \geq 1$

$$\sqrt{T}\left[(r_1-\rho_1), \ldots, (r_m-\rho_m)\right] \to N\left[0, \Sigma_m\right]$$

where the elements of Σ_m are given by the Bartlett formula.

Further, if $\Sigma_m = \left[\sigma_{ij}\right]_{i,j=1,\ldots,m}$

where $\sigma_{ii} = 1+2\sum_{j=1}^{q} \rho_j^2$ for $i \geq q+1$

The sample partial autocorrelations are asymptotically normal.

(c) In question 2, we have an MA(2) process so that $\rho_1 \neq 0$, $\rho_2 \neq 0$ and

$\rho_k = 0$, for $k \geq 3$

The order of the process can be identified by looking at the correlogram

We decide whether

$$t_k = \left|\frac{\sqrt{T}\, r_k}{\hat{\sigma}_k}\right| > t_c(\alpha/2) \quad , k = 1, 2, \ldots$$

$$\hat{\sigma}_k^2 = 1+2\sum_{j=1}^{k-1} r_j^2$$

where $c(\alpha/2)$ is the normal critical value for a test with level α.

For an MA(2) process, we expect that the first two of these tests be non-significant, while the other ones (for $k \geq 3$) will not be significant.