Distributed lag models *

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1. Models with lagged explanatory variables

It is quite common to explain a variable using current and past values of another variable, as in the following equations:

$$I_t = \alpha + \beta_0 V_t + \beta_1 V_{t-1} + \beta_2 V_{t-2} + \cdots + \beta_k V_{t-k} + u_t \tag{1.1}$$

where I_t represents investment (t time t) and V_t represents sales at time t;

$$\hat{p}_t = \alpha + \beta_0 \hat{M}_t + \beta_1 \hat{M}_{t-1} + \cdots + \beta_k \hat{M}_{t-k} + u_t \tag{1.2}$$

where $\hat{p}_t$ is the rate of inflation and $\hat{M}_t$ is the growth rate of a monetary aggregate. If we suppose that

$$u_t \overset{ind}{\sim} N\left[0, \sigma^2\right], \ t = 1, \ldots, T, \tag{1.3}$$

these equations can be estimated by ordinary least squares.

However, if k is large, one typically has a large number of strongly collinear explanatory variables. Restrictions on the coefficients $\beta_i, i = 0, 1, \ldots, k$, are typically needed to improve estimation precision. Several approaches have been suggested.

Consider a general equation of the form:

$$y_t = \alpha + \beta_0 X_t + \beta_1 X_{t-1} + \ldots + \beta_k X_{t-k} + u_t = \alpha + \sum_{i=0}^{k} \beta_1 X_{t-i} + u_t\,.$$

(1) Arithmetic progression (I. Fisher, 1937):

$$\begin{aligned} \beta_i &= (k+1-i)\,\beta\,, \quad 0 \le i \le k \\ &= 0\,, \qquad\qquad\qquad i > k \end{aligned} \tag{1.4}$$

$$\begin{aligned} y_t &= \alpha + \sum_{i=0}^{k} \beta_i X_{t-i} + u_t \\ &= \alpha + \left[\sum_{i=0}^{k} (k+1-i)\, X_{t-i}\right] \beta + u_t \quad , \quad t = 1, \ldots, T. \end{aligned} \tag{1.5}$$

(2) Inverted V progression (De Leeuw, 1962):

$$\begin{aligned} \beta_i &= i\beta \quad , \quad 0 \le i \le k/2 \\ &= (k-i)\,\beta \quad , \quad k/2 \le i \le k \end{aligned} \tag{1.6}$$

$$y_t = \alpha + \beta \left[\sum_{i=0}^{k/2} X_{t-i} + \sum_{(k/2)+1}^{k} (k-i) X_{t-i} \right] + u_t \, . \tag{1.7}$$

(3) Polynomial progression (Almon, 1965):

$$\beta_i = a_0 + a_1 i + a_2 i^2 + \cdots + a_r i^r \ , \qquad 0 \leq r < k \, . \tag{1.8}$$

For example, if $r = 2$ and $k = 3$,we get:

$$y_t = \beta_0 X_t + \beta_1 X_{t-1} + \beta_2 X_{t-2} + \beta_3 X_{t-3} + u_t \tag{1.9}$$

where

$$\begin{aligned} \beta_t &= a_0 + a_1 t + a_2 t^2 \, , \\ \beta_0 &= a_0 \, , \\ \beta_1 &= a_0 + a_1 + a_2 \, , \\ \beta_2 &= a_0 + 2a_1 + 4a_2 \, , \\ \beta_3 &= a_0 + 3a_1 + 9a_2 \, , \end{aligned}$$

hence

$$y_t = a_0 \left(\sum_{i=0}^{3} X_{t-i} \right) + a_1 \left[\sum_{i=0}^{3} i X_{t-i} \right] + a_2 \left[\sum_{i=0}^{3} i^2 X_{t-i} \right] . \tag{1.10}$$

The number of parameters goes from 4 to 3.

$$\begin{gathered} \beta_{-1} = \beta_{k+1} = 0 \, , \\ \beta_{k+1} = 0 \, , \\ \beta_{-1} = 0 \, . \end{gathered}$$

For example,

$$\beta_{-1} = 0 = a_0 - a_1 + a_2 \Longrightarrow a_2 = a_1 - a_0 \ . \tag{1.11}$$

The Almon method produces very regular patterns for β_i, but we should not forget this is due to the imposed restrictions.

One can test whether a lagged variable is present (given k). If

$$y_t = \alpha + \sum_{i=0}^{k} \beta_i X_{t-i} + \sum_{j=0}^{\ell} \gamma_j Z_{t-j} + u_t \, ,$$

one can assume two different polynomials for β_i and γ_j.

Another possibility consists in assuming that

$$\beta_i = \alpha_i + \gamma_i W_{t-i}$$

hence

$$y_t = \alpha + \sum_{i=0}^{k} \alpha_i X_{t-i} + \sum_{i=0}^{k} \gamma_i W_{t-i} X_{t-i} + u_t$$

with two different on α_i and γ_j.
(4) Geometric progression (Koyck, 1954)

$$y_t = \alpha + \beta \sum_{i=0}^{\infty} W_i X_{t-i} + u_t$$

where

$$W_i = (1-\lambda)\lambda^i \,,\ 0 < \lambda < 1,\ i = 0, 1, \ldots \tag{1.12}$$

This leads to the following model:

$$\begin{aligned} y_t &= \alpha + \beta (1-\lambda) \sum_{i=0}^{\infty} \lambda^i X_{t-i} + u_t \\ &= \alpha + \beta (1-\lambda) \sum_{i=0}^{\infty} \lambda^i L^i X_t + u_t \end{aligned} \tag{1.13}$$

where the lag operator L is defined as follows:

$$\begin{aligned} LX_t &= X_{t-1} \,, \\ L^2 X_t &= L(LX_t) = L(X_{t-1}) = X_{t-2} \,, \\ L^n X_t &= X_{t-n} \,. \end{aligned}$$

The model then becomes:

$$\begin{aligned} y_t &= \alpha + \beta (1-\lambda) \sum_{i=0}^{\infty} (\lambda L)^i X_t + u_t \\ &= \alpha + \beta (1-\lambda) \left[\sum_{i=0}^{\infty} (\lambda L)^i \right] X_t + u_t \\ &= \alpha + \beta (1-\lambda) \frac{1}{1-\lambda L} X_t + u_t \end{aligned} \tag{1.14}$$

hence

$$\begin{aligned}(1-\lambda L)\, y_t &= (1-\lambda L)\, \alpha + \beta\, (1-\lambda)\, X_t + (1-\lambda L)\, u_t \\ y_t - \lambda y_{t-1} &= (1-\lambda)\, \alpha + \beta\, (1-\lambda)\, X_t + u_t - \lambda u_{t-1}\end{aligned}$$

or

$$y_t = (1-\lambda)\, \alpha + \lambda y_{t-1} + \beta\, (1-\lambda)\, X_t \quad + \quad \underset{\substack{\uparrow \\ \text{Moving average}}}{(u_t - \lambda u_{t-1})} \quad t = 2, \ldots, T\,.$$

This provides a considerable reduction of the number of parameters. However there is a lagged dependent variable on the right-hand side with autocorrelated errors:
Y_{t-1} is not independent of $u_t - \lambda u_{t-1}$.

We may also wish not to impose the geometric scheme on the first β_i coefficients. For example, if we do this for the first two coefficients, we get:

$$\begin{aligned}y_t &= \alpha + \beta_0 X_t + \beta_1 X_{t-1} + \beta_2 \sum_{i=2}^{\infty} \lambda^i X_{t-i} \;+\; u_t \\ &= \alpha + \beta_0 X_t + \beta_1 X_{t-1} + \beta_2 \frac{\lambda^2}{1-\lambda L} X_{t-2} \;+\; u_t \qquad (1.15)\end{aligned}$$

or

$$y_t = \alpha\, (1-\lambda) + \lambda y_{t-1} + \beta_0 X_t + (\beta_1 - \lambda\beta_0)\, X_{t-1} + (\beta_2 - \lambda\beta_1)\, X_{t-2} + u_t. \qquad (1.16)$$

Another possibility consists in considering geometric progressions on two explanatory variables:

$$\begin{aligned}y_t &= \alpha + \frac{\beta\, (1-\lambda)}{1-\lambda L} X_t + \frac{\gamma\, (1-\lambda)}{1-\lambda L} Z_t + u_t\,, \\ y_t &= \alpha\, (1-\lambda) + \beta\, (1-\lambda)\, X_t + \gamma\, (1-\lambda)\, Z_t + (u_t - \lambda u_{t-1})\ .\end{aligned}$$

Other proposed schemes include: Pascal, rational (Jorgenson), Gamma.

2. Models with lagged dependent variables: economic examples

2.1. Partial adjustment model (Nerlove, 1958)

$$
\begin{aligned}
Y_t^* &= \alpha + \beta X_t && \text{Equilibrium equation} \quad (2.1)\\
Y_t - Y_{t-1} &= \gamma \left(Y_t^* - Y_{t-1}\right) + u_t\,, && \text{Adjustment equation} \quad (2.2)\\
& 0 < \gamma \leq 1 \\
& u_t \overset{ind}{\sim} N\left[0, \sigma_u^2\right]
\end{aligned}
$$

$$
\begin{aligned}
Y_t &= Y_{t-1} + \gamma\left[\alpha + \beta X_t - Y_{t-1}\right] + u_t \\
Y_t &= \alpha\gamma + (1-\gamma)\,Y_{t-1} + \gamma\beta X_t + u_t \qquad (2.3)
\end{aligned}
$$

Equation (2.2) can be obtained as the solution a cost minimization problem:

$$
C_t = \underset{\text{Disequilibrium cost}}{\underset{\uparrow}{a\left(Y_t - Y_t^*\right)}} + \underset{\text{Adjustment cost}}{\underset{\uparrow}{b\left(Y_t - Y_{t-1}\right)^2}} \quad , \quad a \geq 0,\ b \geq 0 \quad , \quad a + b \neq 0\,. \qquad (2.4)
$$

Differentiating (2.4) with respect to Y_t, we get:

$$
\begin{aligned}
\tfrac{\partial C_t}{\partial Y_t} = \;& 2a\left(Y_t - Y_t^*\right) + 2b\left(Y_t - Y_{t-1}\right) = 0 \\
\Longrightarrow \;& (a+b)\,Y_t - aY_t^* - bY_{t-1} = 0 \\
& (a+b)\,Y_t - (a+b)\,Y_{t-1} + aY_{t-1} - aY_t^* = 0 \\
& Y_t - Y_{t-1} = \tfrac{a}{a+b}\left(Y_t^* - Y_{t-1}\right) \\
& \qquad\qquad = \gamma\left(Y_t^* - Y_{t-1}\right)
\end{aligned}
$$

$$
0 \leq \frac{a}{a+b} \leq 1.
$$

2.2. Adaptive expectations

$$
\begin{aligned}
Y_t &= \alpha + \beta X_t^* + u_t,\ t = 1, \ldots, T \qquad (2.5)\\
& u_t \overset{ind}{\sim} N\left[0, \sigma_u^2\right]
\end{aligned}
$$

$$
\begin{aligned}
X_t^* - X_{t-1}^* &= \delta\left(X_{t-1} - X_{t-1}^*\right) && \text{Expectation equation} \qquad (2.6) \\
0 &< \delta < 1 \\
X_t^* &= \text{expected level of } X_t
\end{aligned}
$$

$$
\begin{aligned}
X_t^* - (1-\delta) X_{t-1}^* &= \delta X_{t-1} && B = L \\
\left[1-(1-\delta) B\right] X_t^* &= \delta X_{t-1} \\
X_t^* &= \frac{\delta}{1-(1-\delta) B} X_{t-1} \\
Y_t &= \alpha + \beta \frac{\delta}{1-(1-\delta) B} X_{t-1} + u_t \\
\left[1-(1-\delta) B\right] Y_t &= \left[1-(1-\delta) B\right] \alpha + \beta\delta X_{t-1} + \left[1-(1-\delta) B\right] u_t \\
Y_t - (1-\delta) Y_{t-1} &= \alpha\delta + \beta\delta X_{t-1} + u_t - (1-\delta) u_{t-1}
\end{aligned}
$$

$$
\begin{aligned}
Y_t &= \alpha\delta + \lambda Y_{t-1} + \beta\delta X_{t-1} + (u_t - \lambda u_{t-1}) \qquad (2.7) \\
\lambda &= 1-\delta.
\end{aligned}
$$

3. Estimation of models with lagged dependent variables

Consider an equation of the type:

$$
Y_t = \beta_0 + \beta_1 Y_{t-1} + \underset{\substack{\downarrow \\ \text{Non-stochastic}}}{\beta_2 X_t} + v_t \,, \; t = 0, 1, \ldots, T,
$$

with three alternative hypotheses on the errors:

$$
\begin{aligned}
&\text{(I)} \quad v_t \overset{ind}{\sim} N\left[0, \sigma_v^2\right] \\
&\text{(II)} \quad v_t = u_t - \lambda u_{t-1} \\
&\qquad (a) \quad u_t \overset{ind}{\sim} N\left[0, \sigma_u^2\right] \qquad (3.1) \\
&\qquad (b) \quad u_t = \rho u_{t-1} + \varepsilon_t \\
&\text{(III)} \quad v_t = \rho v_{t-1} + \varepsilon_t
\end{aligned}
$$

3.1. Hypothesis I

$$\begin{aligned}
L \quad &= \quad P\left(Y_1,\,\dots\,,\,Y_t \mid Y_0, X\right) = \frac{1}{\left(2\Pi\sigma_v^2\right)^{T/2}} \exp\left\{-\frac{1}{2\sigma_v^2}\sum_{t=1}^{T}\left(Y_t-\beta_0-\beta_1 Y_{t-1}-\beta_2 X_t\right)^2\right\} \\
Max\,L \quad &\Longrightarrow \quad Min\sum_{t=1}^{T}\left(Y_t-\beta_0-\beta_1 Y_{t-1}-\beta_2 X_t\right)^2 \\
&\rightarrow \quad MCO \text{ are valid } \left\langle \begin{array}{l}\text{Consistent}\\ \text{Asymptotically normal, efficient}\end{array}\right.
\end{aligned}$$

(Tests and confidence intervals are only approximate).

The problem gets more complicated when v_t and Y_{t-1} are correlated. Consider:

$$Y_t = \beta Y_{t-1} + v_t$$

$$v_t = \rho v_{t-1} + \varepsilon_t\,, \qquad \varepsilon_t \overset{ind}{\sim} N\left[0, \sigma_\varepsilon^2\right]\ , \quad |\rho| < 1$$

$$\hat{\beta} = \frac{\sum_{t=1}^{T} Y_t Y_{t-1}}{\sum_{t=2}^{T} Y_{t-1}^2}$$

$$Y_t = \beta Y_{t-1} + \rho v_{t-1} + \varepsilon_t$$

$$v_{t-1} = Y_{t-1} - \beta Y_{t-2}$$

$$\begin{aligned}
Y_t &= \beta Y_{t-1} + \rho\left(Y_{t-1} - \beta Y_{t-2}\right) + \varepsilon_t \\
&= \left(\beta+\rho\right) Y_{t-1} - \beta\rho Y_{t-2} + \varepsilon_t
\end{aligned}$$

$$\sum_{t=2}^{T} Y_t Y_{t-1} = \left(\beta+\rho\right)\sum_{t=2}^{T} Y_{t-1}^2 - \beta\rho\sum_{t=2}^{T} Y_{t-1}Y_{t-2} + \sum_{t=1}^{T}\varepsilon_t Y_{t-1}$$

$$\hat{\beta} = \left(\beta+\rho\right) - \beta\rho\,\frac{\sum_{t=2}^{T} Y_{t-1}Y_{t-2}}{\sum_{t=2}^{T} Y_{t-1}^2} + \frac{\sum_{t=1}^{T}\varepsilon_t Y_{t-1}}{\sum_{t=2}^{T} Y_{t-1}^2}$$

$$p\lim\frac{\sum_{t=1}^{T}\varepsilon_t Y_{t-1}}{\sum_{t=2}^{T} Y_{t-1}^2} = 0$$

$$p\lim\hat{\beta} = \left(\beta+\rho\right) - \beta\rho\; p\lim\hat{\beta}$$

$$p\lim\hat{\beta} = \frac{\beta+\rho}{1+\beta\rho}$$

$$p\lim\hat{\beta} - \beta = \rho\left(1-\beta^2\right)/\left(1+\beta\rho\right)$$

The bias $p\lim\hat{\beta}-\beta$ can be large when $\beta \simeq 0.5$ or $\rho \simeq 0.5$. Similarly, the Durbin-Watson test is not valid:

$$\begin{aligned}
d &= \sum_{t=2}^{T} (\hat{v}_t - \hat{v}_{t-1})^2 / \sum_{t=1}^{T} \hat{v}_t^2 \\
&= \frac{1}{\sum_{t=1}^{T} \hat{v}_t^2} \left[\sum_{t=2}^{T} \hat{v}_t^2 + \sum_{t=2}^{T} \hat{v}_{t-1}^2 - 2 \sum_{t=1}^{T} \hat{v}_t \hat{v}_{t-1} \right] \\
p \lim d &= 2 - 2\, p \lim \frac{\sum_{t=2}^{T} \hat{v}_t \hat{v}_{t-1}}{\sum_{t=1}^{T} \hat{v}_t^2}
\end{aligned}$$

$$\begin{aligned}
r_1 &= \sum_{t=2}^{T} \hat{v}_t \hat{v}_{t-1} / \sum_{t=1}^{T} \hat{v}_t^2 \\
\hat{v}_t &= Y_t - \hat{\beta} Y_{t-1} \qquad (\text{if } Y_t = \beta Y_{t-1} + v_t) \\
\hat{v}_{t-1} &= Y_{t-1} - \hat{\beta} Y_{t-2}.
\end{aligned}$$

$$\begin{aligned}
p \lim\, r_1 &= \frac{\beta\rho\,(\beta + \rho)}{1 + \beta\rho} = \rho - \frac{\rho\left(1 - \beta^2\right)}{1 + \beta\rho} \\
&= \frac{\rho\,(1 + \beta\rho) - \rho\left(1 - \beta^2\right)}{1 + \beta\rho} = \frac{\beta\rho^2 - \rho\beta^2}{1 + \beta\rho} \\
p \lim\, d &= 2 \left[1 - \frac{\beta\rho\,(\beta + \rho)}{1 + \beta\rho} \right] .
\end{aligned}$$

If we knew the true errors v_t, we could compute:

$$\begin{aligned}
d^* &= \sum_{t=2}^{T} (v_t - v_{t-1})^2 / \sum_{t=1}^{T} v_t^2 \\
&= \left[\sum_{t=2}^{T} v_t^2 + \sum_{t=2}^{T} v_{t-1}^2 - 2 \sum_{t=2}^{T} v_t v_{t-1} \right] / \sum_{t=1}^{T} v_t^2 \\
p \lim d^* &= 2\,(1 - \rho) \\
p \lim\,(d - d^*) &= \frac{2\rho\left(1 - \beta^2\right)}{1 + \beta\rho} = 2 p \lim \left(\hat{\beta} - \beta\right) \\
& \quad d \text{ is biased upward } (NS)\,.
\end{aligned}$$

A possible solution consists in using Durbin's test:

$$\begin{aligned} r &= \sum_{t=2}^{T} \hat{v}_t \hat{v}_{t-1} / \sum_{t=1}^{T-1} \hat{v}_t^2 \simeq 1 - \frac{1}{2} d \\ h &= r \sqrt{\frac{n}{1 - n\hat{V}(b_1)}} \end{aligned}$$

where $n = T - 1$ and $\hat{V}(b_1)$ is the estimator of the variance of b_1, and β_1 is the coefficient de Y_{t-1}.Then

$$h \sim N[0,1] \text{ under } H_0.$$

This test is applicable even if $Y_{t-2}, Y_{t-3}, \ldots$ also appear in the equation. The test is not applicable if $n\hat{V}(b_1) \geq 1$.

In view of then latter fact, an alternative test is based on considering a regression of the type:

$$\hat{v}_t = \gamma \hat{v}_{t-1} + \text{ other regressors, } t = 2, \ldots, T.$$

We then test $\gamma = 0$ using a standard t standard, e.g.

$$\hat{v}_t = \gamma \hat{v}_{t-1} + \gamma_0 + \gamma_1 Y_{t-1} + \gamma_2 X_t + e_t, \ t = 2, ..., T \, .$$

3.2. Hypothesis IIa

$$\begin{aligned} v_t = u_t - \lambda u_{t-1} \qquad & \text{Koyck scheme} \\ u_t \overset{ind}{\sim} N[0, \sigma_n^2] \qquad & \text{Adaptive expectations} \end{aligned}$$

$$\begin{aligned} E(v_t) &= 0, \forall t \\ E(v_t^2) &= \sigma_u^2 \left(1 + \lambda^2\right), \forall t \\ E[v_t v_{t+s}] &= -\lambda \sigma_u^2 \,, \, s = \pm 1, \forall t \\ &= 0 \text{ , pour } |s| \geq 2, \forall t \end{aligned}$$

$$V = E[v\, v'] = \sigma_u^2 \begin{bmatrix} 1+\lambda^2 & -\lambda & 0 & \ldots & 0 \\ -\lambda & 1+\lambda^2 & -\lambda & \ldots & 0 \\ \vdots & \vdots & \vdots & & \\ 0 & 0 & 0 & \ldots & 1+\lambda^2 \end{bmatrix} .$$

In Koyck models or with adaptive expectations, we have:

$$\begin{aligned} \beta_1 &= \lambda \\ Y_t &= \beta_0 + \lambda Y_{t-1} + \beta_2 X_t + v_t \end{aligned}$$

$$
\begin{aligned}
y_t &= Y_t - \lambda Y_{t-1} = \beta_0 + \beta_2 X_t + v_t \\
\hat{\beta} &= \left(X'V^{-1}X\right)^{-1} X'V^{-1}y \qquad y = \begin{pmatrix} y_1 \\ \vdots \\ y_t \end{pmatrix}.
\end{aligned}
$$

From a more practical viewpoint,

$$
\begin{aligned}
W_t &= Y_t - u_t \\
W_t &= \lambda W_{t-1} + \beta_0 + \beta_2 X_t \\
W_t &= \lambda \left[\lambda W_{t-2} + \beta_0 + \beta_2 X_{t-1}\right] + \beta_0 + \beta_2 X_t \\
&= \lambda^2 W_{t-2} + \beta_0 (1 + \lambda) + \beta_2 (X_t + \lambda X_{t-1})
\end{aligned}
$$

$$
\begin{aligned}
Y_t &= \lambda^t W_0 + \beta_0 \left(1 + \lambda + \lambda^2 + \ldots + \lambda^{t-1}\right) \\
&\quad + \beta_2 (X_t + \lambda X_{t-1} + \lambda^2 X_{t-2} \\
&\quad + \ldots + \lambda^{t-1} X_1) + u_t. \\
&\quad W_0 \text{ is a parameter.}
\end{aligned}
$$

We choose a grid of values of λ over the admissible interval $0 \leq \lambda < 1$.Then we compute a linear regression and select the value of λ which minimizes the residual sum of squares [Zellner and Geisel (1968)]. This is asymptotically equivalent to applying maximum likelihood.

3.3. Hypothesis IIb

$$
\begin{aligned}
v_t &= u_t - \lambda u_{t-1} \\
u_t &= \rho u_{t-1} + \varepsilon_t \qquad |\rho| < 1, \qquad \varepsilon_t \overset{ind}{\sim} N\left[0, \sigma^2\right] \\
Y_t &= \beta_0 + \lambda Y_{t-1} + \beta_2 X_t + u_t - \lambda u_{t-1} \\
\rho Y_{t-1} &= \rho \beta_0 + \lambda \rho Y_{t-2} + \beta_2 \rho X_{t-1} + \rho u_{t-1} - \lambda \rho u_{t-2} \\
Y_t - \rho Y_{t-1} &= \beta_0 (1 - \rho) + \lambda \left[Y_{t-1} - \rho Y_{t-2}\right] + \beta_2 \left[X_t - \rho X_{t-1}\right] + \varepsilon_t - \lambda \varepsilon_{t-1} \\
Y_t(\rho) &= \beta_0 (1 - \rho) + \lambda Y_{t-1}(\rho) + \beta_2 X_t(\rho) + \varepsilon_t - \lambda \varepsilon_{t-1}.
\end{aligned}
$$

Proceeding as for IIa, we define:

$$
\begin{aligned}
Y_t(\rho) &= Y_t - \rho Y_{t-1} \\
X_t(\rho) &= X_t - \rho X_{t-1}
\end{aligned}
$$

$$
W_t(\rho) = W_t - \rho W_{t-1}
$$

$$
\begin{aligned}
Y_t(\beta) &= \lambda^t W_0(\rho) + \beta_0 (1-\rho) \left[1 + \lambda + \ldots + \lambda^{t-1}\right] \\
&\quad + \beta_2 \left[X_t(\rho) + \lambda X_{t-1}(\rho) + \ldots + \lambda^{t-1} X_1(\rho)\right] + \varepsilon_t.
\end{aligned}
$$

We select a grid over

$$
\begin{aligned}
-1 &\leq \rho \leq 1 \\
0 &\leq \lambda < 1.
\end{aligned}
$$

3.4. Hypothesis III

$$
\begin{aligned}
v_t &= \rho v_{t-1} + \varepsilon_t \quad |\rho| < 1 \quad , \quad \varepsilon_t \sim N\left[0, \sigma^2\right] \\
Y_t &= \beta_0 + \beta_1 Y_{t-1} + \beta_2 X_t + v_t \\
Y_t - \rho Y_{t-1} &= \beta_0 (1-\rho) + \beta_1 (Y_{t-1} - \rho Y_{t-2}) + \beta_2 (X_t - \rho X_{t-1}) + \varepsilon_t.
\end{aligned}
$$

This can be estimated by applying Cochrane-Orcutt or the Hildreth-Lu algorithm. Without such transformations, least squares estimators would be inconsistent.