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**TIME SERIES ANALYSIS**
**EXERCISES**
**STOCHASTIC PROCESSES 2**

1. Discuss the convergence conditions for the series

$$\sum_{j=-\infty}^{\infty} \psi_j u_{t-j}$$

where $\{u_t : t \in \mathbb{Z}\} \sim WN(0, \sigma^2)$. In particular, give sufficient conditions under which:

   (a) $\sum_{j=-\infty}^{\infty} \psi_j u_{t-j}$ converges in mean of order 2;

   (b) $\sum_{j=-\infty}^{\infty} \psi_j u_{t-j}$ converges in mean of order $r > 0$;

   (c) $\sum_{j=-\infty}^{\infty} \psi_j u_{t-j}$ converges almost surely;

   (d) $\sum_{j=-\infty}^{\infty} \psi_j u_{t-j}$ converges in probability.

2. Consider a $MA(1)$ model:

$$X_t = \overline{\mu} + u_t - \theta u_{t-1}\,,\ t \in \mathbb{Z}$$

where $u_t \sim WN(0, \sigma^2)$ and $\sigma^2 > 0$.

   (a) Prove that the first autocorrelation of this model cannot be greater than $0.5$ in absolute value.

   (b) Find the values of the model parameters for which this upper bound is attained.

3. Let $\{x_t : t \in \mathbb{Z}\}$ an MA$(q)$ process. For $q = 3, 4, 5, 6,$ check whether the following inequalities are correct:

(a) $|\rho(1)| \leq 0.10$ ;

(b) $|\rho(2)| \leq 0.50$ ;

(c) $|\rho(3)| \leq 0.50$ ;

(d) $|\rho(4)| \leq 0.50$ ;

(e) $|\rho(5)| \leq 0.50$ ;

(f) $|\rho(6)| \leq 0.50$ .

4. Consider the following models:

$$
\begin{array}{ll}
(1) & X_t = 0.9\, X_{t-1} + u_t, \\
(2) & X_t = 10 - 0.5\, X_{t-1} + u_t, \\
(3) & X_t = 10 + 0.9\, X_{t-1} - 0.2\, X_{t-2} + u_t, \\
(4) & X_t = 10 + u_t - 0.5\, u_{t-1} + 0.125\, u_{t-2}, \\
(5) & X_t = 0.5\, X_{t-1} + u_t - 0.95\, u_{t-1}, \\
(6) & X_t = 0.9\, X_{t-1} + u_t - 0.9\, u_{t-1},
\end{array}
$$

where $\{u_t : t \in \mathbb{Z}\}$ is an $i.i.d.$ $N(0, 1)$ sequence. For each one of these models, answer the following questions.

(a) Is this model stationary? Why?

(b) Is this model invertible? Why?

(c) Compute:

i. $E(X_t)$ ;

ii. $\gamma(k)$ , $k = 1, \ldots , 8$;

iii. $\rho(k)$ , $k = 1, 2, \ldots , 8$.

(d) Graph $\rho(k)$ , $k = 1, 2, \ldots , 8$.

(e) Find the coefficients of $u_t$, $u_{t-1}$, $u_{t-2}$, $u_{t-3}$ and $u_{t-4}$ in the moving average representation of $X_t$.

(f) Find the autocovariance generating function of $X_t$.

(g) Find and graph the spectral density of $X_t$.

(h) Compute the first four partial autocorrelations of $X_t$.