Notions of asymptotic theory *

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List of Definitions, Propositions and Theorems

1. Stochastic convergence

1.1 Definition *Let* $\{X_n = X_n(\omega) : n = 1, 2, ...\}$ *a sequence of real r.v.'s defined on a probability space* $(\Omega, \mathcal{A}, P)$ *and* $X = X(\omega)$ *another real r.v. defined on the same space.*

(a) X_n *converges* in probability *to* X *as* $n \to \infty$ $($*denoted* $X_n \xrightarrow{p} X)$ *iff*

$$\lim_{n\to\infty} P\left[|X_n - X| > \varepsilon\right] = 0\,,\ \forall\varepsilon > 0\,. \tag{1.1}$$

(b) X_n *converges* almost surely *to* X *as* $n \to \infty$ $($*denoted* $X_n \xrightarrow{a.s.} X)$ *iff*

$$P\left[\lim_{n\to\infty} X_n = X\right] = 1\,. \tag{1.2}$$

(c) X_n *converges* completely *to* X *as* $n \to \infty$ $($*denoted* $X_n \xrightarrow{c} X)$ *iff*

$$\Sigma_{n=1}^{\infty} P\left[|X_n - X| > \varepsilon\right] < \infty\ ,\ \forall\varepsilon > 0\,. \tag{1.3}$$

(d) *Suppose* $E|X_n|^r < \infty\ ,\forall n$, *where* $r > 0$. X_n *converges* in mean of order r *to* X $($*denoted* $X_n \xrightarrow{r} X)$ *iff*

$$\lim_{n\to\infty} E\left[|X_n - X|^r\right] = 0\,. \tag{1.4}$$

In this case, we also say that X_n *converges to* X *in* L_r. *If* $r = 2$, *we say* X_n *converges to* X *in quadratic mean* $(q.m.)$.

(e) *Let* $F_n(x)$ *and* $F(x)$ *be the distribution functions of* X_n *and* X *respectively.* X_n *converges* in law (or in distribution) *to* X *as* $n \to \infty$ $($*denoted* $X_n \xrightarrow{L} X)$ *iff*

$$\lim_{n\to\infty} F_n(x) = F(x) \text{ at all continuity points of } F(x)\,. \tag{1.5}$$

1.2 Proposition UNICITY OF PROBABILITY LIMIT. *Let* $\{X_n : n = 1, 2, ...\}$ *be a sequence of real r.v.'s defined on a probability space* $(\Omega, \mathcal{A}, P)$, *and let* X *and* Y *be two real r.v.'s defined on the same probability space. Then*

$$X_n \xrightarrow{p} X \text{ and } X_n \xrightarrow{p} Y \Rightarrow P[X \neq Y] = 0\,. \tag{1.6}$$

2. Relations between convergence concepts

2.1 Assumption *Let* $\{X_n\} \equiv \{X_n : n = 1, 2, ...\}$ *be a sequence of real r.v.'s defined on a probability space* $(\Omega, \mathcal{A}, P)$ *and* X *another real r.v. defined on the same space.*

Unless stated otherwise, this assumption will hold for all the definitions, propositions and theorems in this section.

2.2 Proposition RELATIONS BETWEEN CONVERGENCE CONCEPTS.

$$(a)\quad X_n \xrightarrow{a.s.} X \Leftrightarrow \sup_{k\geq n}|X_k - X| \xrightarrow{p} 0 \text{ as } n \to \infty$$
$$\Leftrightarrow \lim_{n\to\infty} P\left[\sup_{k\geq n}|X_k - X| > \varepsilon\right] = 0\,,\ \forall\varepsilon > 0\,.$$

$$(b)\quad X_n \xrightarrow{c} X \Rightarrow X_n \xrightarrow{a.s.} X \Rightarrow X_n \xrightarrow{p} X \Rightarrow X_n \xrightarrow{L} X\,.$$

$$(c)\quad X_n \xrightarrow{r} X \Rightarrow X_n \xrightarrow{s} X \text{ for all } s \text{ such that } 0 < s \leq r \Rightarrow X_n \xrightarrow{p} X \Rightarrow X_n \xrightarrow{L} X\,.$$

2.3 Remark In general, the implications in **2.2** (b) and (c) cannot be reversed.

2.4 Proposition MOMENT CONDITION FOR CONVERGENCE IN MEAN OF ORDER r.

$$\sum_{n=1}^{\infty} E\left[|X_n - X|^r\right] < \infty \text{ for some } r > 0$$
$$\Rightarrow X_n \xrightarrow{c} X \text{ and } X_n \xrightarrow{r} X\,.$$

2.5 Proposition SUFFICIENT CONDITION FOR COMPLETE CONVERGENCE. *Let* $g : \mathbb{R} \to \mathbb{R}$ *be a nondecreasing monotonic function for* $x \geq 0$*, such that* $g(x) \geq 0$ *and* $g(x) = g(-x)$*. Then*

$$\sum_{n=1}^{\infty} E\left[g(X_n - X)\right] < \infty \Rightarrow X_n \xrightarrow{c} X\,.$$

2.6 Proposition $X_n \xrightarrow{p} X \Rightarrow$ *there is a subsequence* $\{X_{n_k} : k = 1, 2\ , ...\} \subseteq \{X_n\}$ *such that* $X_{n_k} \underset{k\to\infty}{\overset{a.s.}{\to}} X$ *and*

$$\sum_{k=1}^{\infty} P\left[|X_{n_k} - X| \geq \frac{1}{2^k}\right] < \infty\,.$$

2.7 Proposition $X_n \xrightarrow{p} X \Leftrightarrow$ *each subsequence* $\{X_{n_k}\}$ *of* $\{X_n\}$ *contains a subsequence* $\{X_{m_k}\} \subseteq \{X_{n_k}\}$ *such that* $X_{m_k} \overset{a.s.}{\to} X$.

2.8 Proposition *Let* $\{X_n\}$ *a sequence of non-negative r.v.'s such that* $X_n \leq X_{n+1}\ , \ \forall n$, *or* $X_n \geq X_{n+1}\ , \ \forall n$. *Then*

$$X_n \xrightarrow{p} X \Rightarrow X_n \overset{a.s.}{\to} X\ .$$

2.9 Definition SEQUENCE OF UNIFORMLY CONTINUOUS INTEGRALS. *The integrals of the r.v.'s* $\{X_n : n = 1, 2, ...\}$ *are uniformly continuous iff*

$$\sup_{n\geq 1} \int_A X_n dP \to 0 \text{ as } P(A) \to 0\,.$$

2.10 Definition UNIFORMLY INTEGRABLE SEQUENCE. *Let* $B_n = \{\omega \in \Omega : |X_n| \geq a\}$. *The sequence of r.v.'s* $\{X_n : n = 1, 2, ...\}$ *is uniformly integrable iff*

$$\sup_{n\geq 1} \int_{B_n} X_n dP \to 0 \text{ as } a \to \infty\,.$$

2.11 Proposition $X_n \xrightarrow{r} X \Leftrightarrow X_n \xrightarrow{p} X$ *and at least one of the three following conditions holds:*

(a) $E\left(|X_n|^r\right) \to E\left[|X|^r\right]$;

(b) *the sequence of r.v.'s* $\{|X_n|^r : n = 1, 2, ...\}$ *is uniformly integrable;*

(c) *the integrals of the r.v.'s* $\{|X_n|^r : n = 1,\ 2,\ ...\}$ *or those of* $\{|X_n - X|^r : n = 1,\ 2,\ ...\}$ *are uniformly continuous.*

2.12 Proposition *If* $E|X_n|^r \leq c < \infty\ ,\ \forall n$, *for* $r > 0$, *then*

$$X_n \xrightarrow{p} X \Rightarrow X_n \xrightarrow{r'} X \text{ for all } 0 < r' < r\,.$$

2.13 Proposition *If* $|X_n| \leq Y\ ,\ \forall n$, *and* $E\left(|Y|^r\right) < \infty$, *then*

$$X_n \xrightarrow{p} X \Rightarrow X_n \xrightarrow{r} X \text{ and } E\left(|X|^r\right) < \infty\,.$$

2.14 Proposition *If* $|X_n| \leq c < \infty\ ,\ \forall n$, *then*

$$X_n \xrightarrow{p} X \Rightarrow X_n \xrightarrow{r} X \text{ and } E\left(|X|^r\right) < \infty \text{ for all } r > 0\,.$$

2.15 Proposition *If the r.v.'s* X_n *are independent, then*

$$X_n \xrightarrow{a.s.} X \Rightarrow X_n \xrightarrow{c} X\,.$$

2.16 Proposition *If* $P[X = d] = 1$, *then*

$$X_n \xrightarrow{p} d \Leftrightarrow X_n \xrightarrow{L} d\,.$$

2.17 Proposition CAUCHY CRITERION.

(a) *There is a r.v.* X *such that* $X_n \xrightarrow{p} X$

$$\begin{aligned} &\Leftrightarrow \lim_{m,n\to\infty} P[|X_m - X_n| > \varepsilon] = 0\ ,\ \forall\varepsilon > 0 \\ &\Leftrightarrow \lim_{n\to\infty} \left\{\sup_{k\geq 1} P[|X_{n+k} - X_n| > \varepsilon]\right\}\ ,\ \forall\varepsilon > 0\,. \end{aligned}$$

(b) *There is a r.v.* X *such that* $X_n \stackrel{a.s.}{\to} X$

$$
\begin{aligned}
&\Leftrightarrow \quad \lim_{n\to\infty} P\left[\sup_{k,\ell\geq n}|X_k - X_\ell| > \varepsilon\right] = 0\,,\ \forall\varepsilon > 0 \\
&\Leftrightarrow \quad \lim_{n\to\infty} P\left[\sup_{k\geq 1}|X_{n+k} - X_n| > \varepsilon\right] = 0\,,\ \forall\varepsilon > 0 \\
&\Leftrightarrow \quad \sup_{k\geq 1}|X_{n+k} - X_n| \underset{n\to\infty}{\stackrel{p}{\to}} 0 \\
&\Leftrightarrow \quad \sup_{k\geq 1}|X_{n+k} - X_n| \underset{n\to\infty}{\stackrel{a.s.}{\to}} 0\,.
\end{aligned}
$$

(c) *There is a r.v.* $X \in L_r$ *such that* $X_n \stackrel{r}{\to} X$

$$
\begin{aligned}
&\Leftrightarrow \quad \lim_{m,n\to\infty} E[|X_m - X_n|^r] = 0 \\
&\Leftrightarrow \quad \sup_{k\geq 1} E|X_{n+k} - X_n|^r \underset{n\to\infty}{\to} 0\,.
\end{aligned}
$$

2.18 Proposition *Let* $\{\varepsilon_n\}_{n=1}^{\infty}$ *be a sequence of positive constants such that* $\lim_{n\to\infty}\varepsilon_n = 0$. *Then*

$$\sum_{n=1}^{\infty} P[|X_n - X| > \varepsilon_n] < \infty \Leftrightarrow X_n \stackrel{a.s.}{\to} X\,.$$

2.19 Proposition $X_n \stackrel{a.s.}{\to} X \Leftrightarrow \exists$ *a sequence* ε_n *such that* $\lim_{n\to\infty}\varepsilon_n = 0$ *and*

$$\lim_{n\to\infty} P\left[|X_k - X| \geq \varepsilon_k,\ \text{for at least one } k \geq n\right] = 0\,.$$

3. Convergence of expectations and functions of random variables

3.1 Assumption *Let* $\{X_n : n = 1, 2, ...\}$ *be a sequence of real r.v.'s,* X *a real r.v. and* $g : \mathbb{R} \to \mathbb{R}$ *a function such that* $g(X)$ *and* $g(X_n)$, $n = 1, 2, ...$, *are real r.v.'s.*

Unless stated otherwise, this assumption will hold for all the definitions, propositions and theorems in this section.

3.2 Theorem HELLY-BRAY. $X_n \stackrel{L}{\to} X \Rightarrow E[g(X_n)] \to E[g(X)]$ *for any bounded continuous function* $g : \mathbb{R} \to \mathbb{R}$.

3.3 Definition UNIFORMLY INTEGRABLE FUNCTION. $|g|$ *is uniformly integrable with respect to the distributions* F_n *iff*

$$\forall \varepsilon > 0\,,\ \exists\, a(\varepsilon) \text{ and } b(\varepsilon) \text{ which do not depend on } n \text{ such that}$$
$$a \leq a(\varepsilon) \text{ and } b \geq b(\varepsilon) \text{ implies}$$
$$\int |g(x)| dF_n(x) - \int_a^b |g(x)| d\, F_n(x) < \varepsilon\,,\ \forall n\,.$$

3.4 Proposition *If* $g(x)$ *is a continuous function and* $X_n \xrightarrow{L} X$, *then*

(*a*) $\liminf_{n\to\infty} E\left[|g(X_n)|\right] \geq E[|g(X)|]$;

(*b*) $|g(x)|$ *is uniformly integrable with respect to* $F_n \Rightarrow E[g(X_n)] \to E[g(X)]$;

(*c*) $E\left[|g(X_n)|\right] \to E[|g(X)|] < \infty \Leftrightarrow |g(x)|$ *is uniformly integrable with respect to* F_n .

3.5 Proposition *If* $E\left[|X_n|^{r_0}\right] \leq c < \infty\ ,\ \forall n$, *for* $r_0 > 0$, *then*

(*a*) $X_n \xrightarrow{L} X \Rightarrow E\left[|X_n|^r\right] \to E\left[|X|^r\right] < \infty$ *for all* $0 < r < r_0$,

and

(*b*) $E\left[X_n^k\right] \to E\left[X^k\right] \neq \pm\infty$ *for any integer* $0 < k < r_0$, $k \in \mathbb{N}$.

3.6 Proposition *If* $E\left(X_n^k\right)$ *exists and is finite for all* k $(k = 1,\ 2,\ \ldots)$ *and* n, *then*

(*a*) $X_n \xrightarrow{L} X$ *and* $E\left(X_n^k\right) \to \mu_k \neq \pm\infty \Rightarrow E(X^k) = \mu_k$;

(*b*) $E\left(X_n^k\right) \to \mu_k \neq \pm\infty$, *where the sequence* $\{\mu_k : k = 1,\ 2,\ \ldots\}$ *defines a unique distribution function* $F(x) \Rightarrow X_n \xrightarrow{L} X$, *where* X *is a r.v. whose distribution function is* $F(x)$.

3.7 Proposition *If* $E|X_n|^r < \infty, \forall n$ *, for* $r > 0$ *, then*

$$\begin{aligned} X_n \xrightarrow{r} X \quad &\Rightarrow \quad E|X_n|^r \to E|X|^r \text{ and } E|X|^r < \infty \\ &\Rightarrow \quad E\left(X_n^k\right) \to E(X^k) \text{ for } 0 < k < r\,. \end{aligned}$$

3.8 Proposition *If* $E\left(X_n^2\right) < \infty$ *,* $\forall n$*, then* $X_n \xrightarrow{2} X$ *implies*

$$E\left(X_n^2\right) \to E(X^2) < \infty \text{ and } E(X_n) \to E(X) \neq \pm\infty\,.$$

3.9 Proposition *If* $E|X_n|^r < \infty$ *,* $\forall n$ *, and* $E|X|^r < \infty$ *, for* $r > 0$ *, then*

$$X_n \xrightarrow{r} X \Rightarrow E\left(X_n^k\right) \to E(X^k)\,, \text{ for any integer } 0 < k \le r\,.$$

3.10 Proposition *Let* $g : \mathbb{R} \to \mathbb{R}$ *a continuous function everywhere on* $\mathbb{R}$*, except possibly in a set* $A \subseteq \mathbb{R}$*, and let* X *a r.v. such that* $P[X \in A] = 0$ *. Then*

(*a*) $X_n \xrightarrow{p} X \Rightarrow g(X_n) \xrightarrow{p} g(X)$;

(*b*) $X_n \xrightarrow{a.s.} X \Rightarrow g(X_n) \xrightarrow{a.s.} g(X)$;

(*c*) $X_n \xrightarrow{L} X \Rightarrow g(X_n) \xrightarrow{L} g(X)$.

3.11 Proposition *Let* $\{X_n\}$ *and* $\{Y_n\}$ *two sequences of random variables. Then*

(*a*) $X_n \xrightarrow{p} X$ *and* $Y_n \xrightarrow{p} Y \Rightarrow X_n + Y_n \xrightarrow{p} X + Y$;

(*b*) $X_n \xrightarrow{a.s.} X$ *and* $Y_n \xrightarrow{a.s.} Y \Rightarrow X_n + Y_n \xrightarrow{a.s.} X + Y$;

(*c*) $X_n \xrightarrow{p} X$ *and* $Y_n \xrightarrow{p} Y \Rightarrow g(X_n, Y_n) \xrightarrow{p} g(X, Y)$ *for any continuous function* $g(x, y)$.

3.12 Proposition *Let* $\{X_n\}$ *and* $\{Y_n\}$ *two sequences of r.v.'s such that* $X_n \xrightarrow{L} X$ *and* $Y_n \xrightarrow{p} c$ *, where* X *is a r.v. and* c *is a real constant* $(-\infty < c < +\infty)$*. Then*

(*a*) $X_n + Y_n \xrightarrow{L} X + c$;

(b) $X_n Y_n \xrightarrow{L} Xc$;

(c) $X_n / Y_n \xrightarrow{L} X/c$ if $c \neq 0$;

(d) $(X_n,\, Y_n) \xrightarrow{L} (X,\, c)$.

3.13 Proposition *Let $\{X_n\}$ and $\{Y_n\}$ two sequences of r.v.'s such that $X_n - Y_n \xrightarrow{p} 0$ and $Y_n \xrightarrow{L} Y$, and let $g : \mathbb{R} \to \mathbb{R}$ be a continuous function. Then*
(a) $X_n \xrightarrow{L} Y$;

(b) $g(X_n) - g(Y_n) \xrightarrow{p} 0$;

(c) $g(X_n) \xrightarrow{L} g(Y)$.

3.14 Proposition $X_n \xrightarrow{2} X \Leftrightarrow \lim\limits_{\substack{m\to\infty \\ n\to\infty}} E(X_m X_n)$ *exists and is finite irrespective of the way* m *and* $n \to \infty$.

3.15 Proposition $X_n \xrightarrow{2} X$ *and* $Y_n \xrightarrow{2} Y \Rightarrow E(X_n + Y_n) \to E(X+Y)$ *and* $E(X_n Y_n) \to E(XY)$, *where* $E(X+Y)$ *and* $E(XY)$ *are finite real numbers.*

4. Random series

4.1. Definitions

4.1.1 Definition *Let $\{X_t \,:\, t \in \mathbb{N}\}$ be a real-valued stochastic process and consider the series* $\sum\limits_{t=1}^{\infty} X_t$.

4.1.2 Definition *We say* $\sum\limits_{t=1}^{\infty} X_t$ *converges (according to given mode of convergence) iff there exists a real r.v. Y such that*

$$\sum_{t=1}^{N} X_t \underset{N\to\infty}{\longrightarrow} Y \ (\textit{according to the same mode of convergence})\ .$$

4.1.3 Remark Mode of convergence : a.s., in probability or in mean of order r .

4.1.4 QUESTIONS :

(1) Under which conditions does the series $\sum_{t=1}^{\infty} X_t$ converge?

(2) Under which conditions can one write

$$E\left(\sum_{t=1}^{\infty} X_t\right) = \sum_{t=1}^{\infty} E(X_t)\,?$$

4.1.5 Definition *We say $\sum_{t=1}^{\infty} X_t$ converges absolutely (according to a given convergence mode) iff $\sum_{t=1}^{\infty} |X_t|$ converges (according to the same mode of convergence). If $\sum_{t=1}^{\infty} |X_t|$ converges with probability one (a.s.), we write $\sum_{t=1}^{\infty} |X_t| < \infty\ a.s.$*

4.2. General convergence criteria

4.2.1 Proposition *$\sum_{t=1}^{\infty} |X_t| < \infty$ a.s. $\Rightarrow \sum_{t=1}^{\infty} X_t$ converges a.s.*

4.2.2 Proposition *$\sum_{t=1}^{\infty} X_t$ converges absolutely in probability $\Leftrightarrow$ $\sum_{t=1}^{\infty} X_t$ converges absolutely a.s.*

4.2.3 Proposition *If there is a sequence of positive constants $\{\varepsilon_t\}_{t=1}^{\infty}$ such that*

$$\begin{aligned} &(a) \quad \sum_{t=1}^{\infty} \varepsilon_t < \infty\,, \\ &(b) \quad \sum^{\infty} P\left(|X_t| \geq \varepsilon_t\right) < \infty\,, \end{aligned}$$

then $\sum_{t=1}^{\infty} X_t$ *converges a.s.* (*i.e. there exists a r.v.* X *such that* $\sum_{t=1}^{N} X_t \xrightarrow[N\to\infty]{a.s.} X$) .

4.2.4 Proposition

$$\sum_{t=1}^{\infty} |X_t| < \infty \; a.s. \Leftrightarrow P\left(\sum_{t=1}^{n} |X_t| < x\right) \underset{n\to\infty}{\longrightarrow} F(x) \,,\ \forall x \,, \tag{4.1}$$

where $F(x)$ *is the distribution function of a r.v.*

4.2.5 Proposition $\sum_{t=1}^{\infty} E|X_t| < \infty \Rightarrow \sum_{t=1}^{\infty} |X_t| < \infty$ *a.s. and* $E\left[\sum_{t=1}^{\infty} |X_t|\right] = \sum_{t=1}^{\infty} E|X_t|$.

4.2.6 Proposition $\sum_{t=1}^{\infty} (E|X_t|^r)^{1/r} < \infty$, *where* $r \geq 1 \Rightarrow \sum_{t=1}^{\infty} |X_t|$ *and* $\sum_{t=1}^{\infty} X_t$ *converge a.s. and in mean of order* r .

4.2.7 Proposition *If* $X_t \in L_2$, $\mu_t \equiv E(X_t)$ *and* $\sigma_t^2 \equiv Var(X_t)$, $\forall t$, *then* $\sum_{t=1}^{\infty} [\sigma_t^2 + \mu_t^2]^{\frac{1}{2}} < \infty$ *implies*

$$\sum_{t=1}^{\infty} |X_t| \text{ and } \sum_{t=1}^{\infty} X_t \text{ converge a.s. and q.m.}$$

In particular, if $E(X_t) = 0$, $\forall t$,

$$\sum_{t=1}^{\infty} \sigma_t < \infty \Rightarrow \sum_{t=1}^{\infty} |X_t| \text{ and } \sum_{t=1}^{\infty} X_t \text{ converge a.s. and q.m.}$$

4.2.8 Proposition

$$\sum_{t=1}^{\infty} E|X_t| < \infty \Rightarrow E\left[\sum_{t=1}^{\infty} X_t\right] = \sum_{t=1}^{\infty} E(X_t) \,.$$

4.2.9 Proposition *If $X_t \in L_2, \forall t$, and if there are non-negative constants $\{\bar{\rho}_t : t = 1, 2, ...\}$ such that $0 \leq \bar{\rho}_t \leq 1$,*

$$E(X_s X_t) \leq \bar{\rho}_{t-s} \left[E\left(X_s^2\right) E\left(X_t^2\right)\right]^{\frac{1}{2}} \text{ , for } 0 < s \leq t \text{ ,}$$

and

$$\sum_{t=1}^{\infty} \bar{\rho}_t < \infty \text{ ,}$$

then

$$\sum_{t=1}^{\infty} (\log\ t)^2 E\left(X_t^2\right) < \infty \Rightarrow \sum_{t=1}^{\infty} X_t \text{ converges a.s.}$$

4.2.10 Remark When $E(X_t) = 0$, we have

$$Corr(X_s,\, X_t) \leq \bar{\rho}_{t-s} \,.$$

4.3. Series of orthogonal variables

4.3.1 Proposition *Let $\{X_t\}_{t=1}^{\infty}$ a sequence of r.v.'s such that $X_t \in L_2$, $E(X_t) = 0$ and $E(X_s X_t) = 0$ for $s \neq t$. Then*

(*a*) $\sum\limits_{t=1}^{\infty} X_t$ *converges in quadratic mean* $\Leftrightarrow \sum\limits_{t=1}^{\infty} E|X_t|^2 < \infty$;

(*b*) $\sum\limits_{t=1}^{\infty} E|X_t|^2 < \infty \Rightarrow E|X|^2 = \sum\limits_{t=1}^{\infty} E|X_t|^2 < \infty$, *where* $\sum\limits_{t=1}^{N} X_t \xrightarrow[N\to\infty]{2} X$;

(*c*) $\sum\limits_{t=1}^{\infty} \frac{E|X_t|^2}{b_t^2} < \infty$ *and* $b_n \uparrow \infty \Rightarrow \frac{1}{b_n} \Sigma_{t=1}^{n} X_t \xrightarrow{2} 0$; (*Kolmogorov*)

(*d*) $\sum\limits_{t=1}^{\infty} (\log\ t)^2 E|X_t|^2 < \infty \Rightarrow \sum\limits_{t=1}^{\infty} X_t$ *converges a.s. and in quadratic mean;*

(*e*) $\sum\limits_{t=1}^{\infty} \left(\frac{\log\ t}{b_t}\right)^2 E|X_t|^2 < \infty$ *and* $b_n \uparrow \infty \Rightarrow \frac{1}{b_n} \sum\limits_{t=1}^{n} X_t \xrightarrow{a.s.} 0$.

4.3.2 Proposition *Let $\{X_t\}_{t=1}^{\infty}$ a sequence of r.v.'s such that $X_t \in L_2$ and $Cov(X_s,\, X_t) = 0$ for $s \neq t$, and let $\mu_t = E(X_t)$. Then*

(a) $\sum\limits_{t=1}^{\infty}(X_t-\mu_t)$ *converges in quadratic mean to a r.v.* $Y \Leftrightarrow \sum\limits_{t=1}^{\infty}Var(X_t)<\infty$;

(b) $\sum\limits_{t=1}^{\infty}Var(X_t)<\infty \Rightarrow Var(Y)=\sum\limits_{t=1}^{\infty}Var(X_t)$, *where* $\sum\limits_{t=1}^{N}(X_t-\mu_t) \xrightarrow[N\to\infty]{2} Y$ *;*

(c) $\sum\limits_{t=1}^{\infty}\frac{Var(X_t)}{b_t^2}<\infty$ *and* $b_n\uparrow\infty \Rightarrow \frac{1}{b_n}\sum\limits_{t=1}^{n}(X_t-\mu_t)\xrightarrow{2}0$ *;*

(d) $\sum\limits_{t=1}^{\infty}(\log\ t)^2Var(X_t)<\infty \Rightarrow \sum\limits_{t=1}^{\infty}(X_t-\mu_t)$ *converges in quadratic mean and a.s. ;*

(e) $\sum\limits_{t=1}^{\infty}\left(\frac{\log\ t}{b_t}\right)^2Var(X_t)<\infty$ *and* $b_n\uparrow\infty \Rightarrow \frac{1}{b_n}\sum\limits_{t=1}^{n}(X_t-\mu_t)\xrightarrow{a.s.}0$.

4.3.3 Proposition *Let* $\{X_t\}_{t=1}^{\infty}$ *a sequence of r.v.'s such that* $X_t\in L_2$ *and* $E(X_sX_t)=0$ *for* $s\neq t$ *. Then*

(a) $\sum\limits_{t=1}^{\infty}E\left(X_t^2\right)<\infty \Rightarrow \sum\limits_{t=1}^{\infty}X_t$ *converges in q.m.;*

(b) $\sum\limits_{t=1}^{\infty}(\log\ t)^2E\left(X_t^2\right)<\infty \Rightarrow \sum\limits_{t=1}^{\infty}X_t$ *converges a.s.*

4.4. Series of independent variables

4.4.1 Proposition *Let* $\{X_t\}_{t=1}^{\infty}$ *a sequence of independent r.v.'s such that* $X_t\in L_2$ *.Then*

(a) $\sum\limits_{t=1}^{\infty}$ Var$(X_t)<\infty \Rightarrow \sum\limits_{t=1}^{\infty}(X_t-EX_t)$ *converges a.s. ;*

(b) *if* $|X_t|\leq c<\infty, \forall t$,

$$\sum_{t=1}^{\infty}(X_t-EX_t) \text{ converges a.s. } \Leftrightarrow \sum_{t=1}^{\infty}Var(X_t)<\infty\ ;$$

(c) $\sum\limits_{t=1}^{\infty}E(X_t)$ *converges to a finite number and* $\sum\limits_{t=1}^{\infty}$ Var$(X_t)<\infty \Rightarrow \sum\limits_{t=1}^{\infty}X_t$ *converges a.s. and in quadratic mean ;*

(d) $|X_t|\leq c<\infty$, $\forall t$, *and* $\sum\limits_{t=1}^{\infty}X_t$ *converges a.s.* $\Rightarrow \sum\limits_{t=1}^{\infty}X_t$ *and* $\sum\limits_{t=1}^{\infty}$ Var(X_t) *converge.*

4.4.2 Proposition THREE SERIES THEOREM (KOLMOGOROV). *Let $\{X_t\}_{t=1}^{\infty}$ a sequence of independent r.v.'s and let*

$$\begin{aligned} X_t^c &= X_t\,, \text{ if } |X_t| < c \\ &= 0, \text{ if } |X_t| \geq c \end{aligned}$$

where $c > 0$. The series $\sum_{t=1}^{\infty} X_t$ converges a.s. $\Leftrightarrow$ there exists a constant $c > 0$ such that the three following series converge in $\mathbb{R}$:

$$\begin{array}{ll} (a) & \sum_{t=1}^{\infty} P\left[|X_t| \geq c\right] \; ; \\ (b) & \sum_{t=1}^{\infty} Var\left(X_t^c\right) \; ; \\ (c) & \sum_{t=1}^{\infty} E\left(X_t^c\right) \; . \end{array}$$

4.4.3 Proposition *Let $\{X_t\}_{t=1}^{\infty}$ a sequence of independent r.v.'s. If the two series $\sum_{t=1}^{\infty} E(X_t)$ and $\sum_{t=1}^{\infty} E\left(|X_t|^p\right)$ where $1 < p \leq 2$ converge, then the series $\sum_{t=1}^{\infty} X_t$ converges a.s.*

4.4.4 Proposition *Let $\{X_t\}_{t=1}^{\infty}$ a sequence of independent r.v.'s. Then*

(a) $\sum_{t=1}^{\infty} X_t$ converges $a.s.$ $\Leftrightarrow$ $\sum_{t=1}^{\infty} X_t$ converges in probability $\Leftrightarrow$ $\sum_{t=1}^{\infty} X_t$ converges in law;

(b) if $|X_t| \leq c < \infty$ and $E(X_t) = 0$, $\forall t$, then $\sum_{t=1}^{\infty} X_t$ converges $a.s.$ iff

$$\sum_{t=1}^{\infty} X_t \text{ converges in quadratic mean.}$$

5. Laws of large numbers

5.1 Proposition *Let $\{X_t\}_{t=1}^{\infty}$ a sequence of r.v.'s such that $X_t \in L_2$ and $Cov(X_s,\ X_t) = 0$ for $s \neq t$, and let $\mu_t = E(X_t)$. Then* (*a*) $\sum\limits_{n=1}^{\infty} \frac{1}{n^2} Var(X_n) < \infty \Rightarrow \bar{X}_n - \bar{\mu}_n \xrightarrow[n\to\infty]{2} 0$ (*Chebychev law*)

where $\bar{X}_n = \sum\limits_{t=1}^{n} X_t/n$ *and* $\bar{\mu}_n = \sum\limits_{t=1}^{n} \mu_t/n$, *and*

(*b*) $\sum\limits_{n=1}^{\infty} \left(\frac{\log\, n}{n}\right)^2 Var(X_n) < \infty \Rightarrow \bar{X}_n - \bar{\mu}_n \xrightarrow[n\to\infty]{a.s.} 0\,.$

In particular, if $Var(X_t) = \sigma^2 < \infty$ *and* $E(X_t) = \mu$ *for all* t, *then*

$$\frac{1}{n}\sum_{t=1}^{n} X_t \xrightarrow[n\to\infty]{a.s.} \mu \text{ and } \frac{1}{n}\sum_{t=1}^{n} X_t \xrightarrow[n\to\infty]{2} \mu\,.$$

5.2 Theorem KHINTCHINE WEAK LAW OF LARGE NUMBERS. *Let* $\{X_t\}_{t=1}^{\infty}$ *a sequence of independent and identically distributed r.v.'s whose mean* $E(X_t)$ *exists. Then*

$$E(X_t) = \mu \Rightarrow \bar{X}_n \xrightarrow[n\to\infty]{p} \mu\,.$$

5.3 Theorem FIRST KOLMOGOROV'S STRONG LAW OF LARGE NUMBERS. *Let* $\{X_t\}_{t=1}^{\infty}$ *a sequence of independent r.v.'s such that* $E(X_t) = \mu_t$ *and* $Var(X_t) = \sigma_t^2$ *exist for all* t. *Then*

$$\sum_{n=1}^{\infty} (\sigma_n/\, n)^2 < \infty \Rightarrow \bar{X}_n - \bar{\mu}_n \xrightarrow[n\to\infty]{a.s.} 0\,.$$

5.4 Theorem SECOND KOLMOGOROV'S STRONG LAW OF LARGE NUMBERS. *Let* $\{X_t\}_{t=1}^{\infty}$ *a sequence of independent and identically distributed r.v.'s. Then*

$$E(X_t) \text{ exists and is equal to } \mu \Leftrightarrow \bar{X}_n - \bar{\mu}_n \xrightarrow[n\to\infty]{a.s.} 0\,.$$

5.5 Theorem LAW OF LARGE NUMBERS FOR CORRELATED R.V.'S. *Let* $\{X_t\}_{t=1}^{\infty}$ *a sequence of r.v.'s with means* $E(X_t) = \mu_t$ *and satisfying the following condition: there exists a sequence of real numbers* $\{r_j\}_{j=0}^{\infty}$ *such that*

(*a*) $0 \le r_j \le 1$, *for all* j ,

(*b*) $Cov(X_s, X_t) \le r_{t-s}[Var(X_s)Var(X_t)]^{1/2}$, *for* $t \ge s$, (*c*) $\sum_{j=1}^{\infty} r_j < \infty$,

(*d*) $\sum_{n=1}^{\infty} \left(\frac{\log(n)}{n}\right)^2 Var(X_n) < \infty$.

Then

$$\bar{X}_n - \bar{\mu}_n \xrightarrow[n\to\infty]{a.s.} 0 .$$

6. Central limit theorems

6.1 Theorem LINDEBERG-LÉVY CENTRAL LIMIT THEOREM. *Let* $\{X_t\}_{t=1}^{\infty}$ *a sequence of independent and identically distributed r.v.'s in* L_2 *such that* $E(X_t) = \mu$ *and* $Var(X_t) = \sigma^2 > 0$. *Then*

$$\frac{1}{\sqrt{n}}\sum_{t=1}^{n}(X_t - \mu)/\ \sigma = \sqrt{n}(\bar{X}_t - \mu)/\ \sigma \xrightarrow[n\to\infty]{L} Z$$

where $Z \sim N(0,1)$.

6.2 Theorem LIAPUNOV CENTRAL LIMIT THEOREM. *Let* $\{X_t\}_{t=1}^{\infty}$ *a sequence of independent r.v.'s in* L_3 *such that* $E(X_t) = \mu_t$, $Var(X_t) = \sigma_t^2 \neq 0$, $E[|X_t - \mu_t|^3] = \beta_t$ *for all* t. *Moreover,*

$$B_n = \left(\sum_{t=1}^{n}\beta_t\right)^{1/3}, C_n = \left(\sum_{t=1}^{n}\sigma_t^2\right)^{1/2} .$$

If $\lim_{n\to\infty}(B_n/\ C_n) = 0$, *then*

$$\Sigma_{t=1}^{n}(X_t - \mu_t)/\ C_n = \sqrt{n}(\bar{X}_t - \bar{\mu}_n)/(C_n/\sqrt{n})\sigma \xrightarrow[n\to\infty]{L} Z$$

where $Z \sim N(0,1)$.

6.3 Theorem LINDEBERG-FELLER CENTRAL LIMIT THEOREM. *Let* $\{X_t\}_{t=1}^{\infty}$ *a sequence of independent r.v.'s in* L_2 *such that*

$$P[X_t \leq x] = G_t(x)\ ,\ E(X_t) = \mu_t\ ,\ Var(X_t) = \sigma_t^2 \neq 0\ ,$$

for all t. Then

$$\sum_{t=1}^{n}(X_t-\mu_t)/\,C_n \xrightarrow[n\to\infty]{L} Z \text{ and } \lim_{n\to\infty}\max_{1\le t\le n}(\sigma_t/\,C_n)=0$$

iff

$$\lim_{n\to\infty}\frac{1}{C_n^2}\sum_{t=1}^{n}\int_{|x-\mu_t|>\varepsilon C_n}(x-\mu_t)^2 dG_t(x)=0, \forall\varepsilon>0\,.$$

7. Extension to random vectors

7.1 Definition STOCHASTIC CONVERGENCE FOR VECTORS. *Let $\{X_n\}_{n=1}^{\infty}$ a sequence of vectors of dimension k,*

$$X_n=(X_{1n},X_{2n},...,X_{kn})',n=1,2,...$$

whose components are real random variables all defined on the same probability space (Ω, Q, P), and

$$X=(X_1,X_2,...,X_k)'$$

another random vector of dimension k whose components are defined on the same space.

(*a*) *We say X_n converges to X in probability (almost surely, in mean of order r) as $n\to\infty$ if each component of X_n converges to the corresponding component of X in probability (almost surely, in mean of order r) as $n\to\infty$. Depending on the case considered, we then write $X_n \xrightarrow{p} X$, $X_n \xrightarrow{a.s.} X$ or $X_n \xrightarrow{r} X$.*

(*b*) *We say X_n converges in law to X ($X_n \xrightarrow{L} X$) iff*

$$\lim_{n\to\infty}F_{X_n}(x)=F_X(x) \text{ at all continuity points of } F_X(x)\,.$$

where $x=(x_1,x_2,\ldots,x_k)'\in\mathbb{R}^k$,

$$F_{X_n}(x)=P[X_{1n}\le x_1,...,X_{kn}\le x_k]\,,\ n=1,2,...$$

and

$$F_X(x)=P[X_1\le x_1,...,X_k\le x_k]\,.$$

7.2 Theorem UNIVARIATE CHARACTERIZATION OF CONVERGENCE IN LAW FOR A SEQUENCE OF VECTORS.. *Let* $\{X_n\}_{n=1}^{\infty}$ *be a sequence of random vectors of dimension* $k \times 1$ *and let* X *be another random vector of dimension* $k \times 1$. *Then*

$$X_n \xrightarrow[n\to\infty]{L} X \Leftrightarrow \lambda' X_n \xrightarrow[n\to\infty]{L} \lambda' X \, , \, \forall \lambda \in \mathbb{R}^k \, .$$

In particular, if $X \sim N[\mu, \Sigma]$,

$$X_n \xrightarrow[n\to\infty]{L} N[\mu, \Sigma] \Leftrightarrow \lambda' X_n \xrightarrow[n\to\infty]{L} N[\lambda'\mu, \lambda'\Sigma\lambda], \forall \lambda \in \mathbb{R}^k \, .$$

8. Proofs and additional references

Proofs and further discussions of the results presented above may be found the following references.

1.2 Lukacs (1975), Theorem 2.2.6, p. 39.
2.6 Loève (1977), p. 153.
2.7 Lukacs (1975), p. 49.
2.8 Lukacs (1975), pp. 37 and 49.
2.11 Loève (1977), vol. I, p. 165.
2.12 Loève (1977), vol. I, p. 166.
2.13 Loève (1977), vol. I, p. 166.
2.14 Lukacs (1975), p. 38.
2.15 Loève (1977), vol. I, p. 240.
2.17 (a) Loève (1977), vol. I, pp. 114, 118, 153, and Lukacs (1975), p. 48.
(b) Loève (1977), vol. I, p. 153, and Lukacs (1975), p. 45.
(c) Loève (1977), p. 163, and Lukacs (1975), p. 49.
2.18 Stout (1974), Theorem 2.1.2, p. 11.
2.19 Loève (1977), vol. I, p. 175.
3.2 Loève (1977), vol. I, p. 184.
3.4 Loève (1977), vol. I, p. 185.
3.5 Loève (1977), vol. I, p. 186.
3.6 Rao (1973), p. 121.
3.7 Loève (1977), vol. I, p. 165, and Prakasa Rao (1987), p. 10.
3.10 Prakasa Rao (1987), p. 12.
3.11 Prakasa Rao (1987), pp. 43 and 57-58.
3.12 Rao (1973), p. 122) and Prakasa Rao (1987), p. 10.
3.13 Rao (1973), p. 124.
3.14 Lukacs (1975), p. 50.
4.2.2 Implication of Proposition **2.8**.
4.2.3 Lukacs (1975), Corollary 4.2.2, p. 82.
4.2.4 Loève (1977), vol. I, p. 175, Problem 10.
4.2.5 Loève (1977), vol. I, p. 175, Problem 10, Lukacs (1975), Theorem 4.2.1, p. 80, and the Monotone convergence theorem [Loève (1977), vol. I, p. 125].
4.2.6 Loève (1977), vol. I, p. 175, Problem 10, and Proposition **2.13** above.
4.2.8 Loève (1977), p. 111. This result is an implication of the Dominated convergence theorem: Loève (1977), vol. I, p. 126, and Royden (1968), pp. 88-89.
4.2.9 Stout (1974), Corollary 2.4.1, p. 28.
4.3.1 Loève (1977), vol. II, Theorem 30.1, p. 122, and Theorem 30.1B, p. 124.
4.3.3 (a) Loève (1977), Theorem 4.2.4A, p. 85, and Stout (1974), Lemma 2.2.2, p. 15.

(b) Stout (1974), Theorem 2.3.2, p. 20.

4.4.1 (a) Lukacs (1975), Theorem 4.2.4, p. 84, and Loève (1977), vol. I, Theorem 17.3.Ia, p. 248.

(b) Loève (1977), vol. I, Theorem 17.3.Ia, p. 248, and Lukacs (1975), Theorem 4.2.4, Corollary 3, p. 87.

(c) Lukacs (1975), Theorem 4.2.4, Corollary 1, p. 84.

(d) Loève (1977), Theorem 17.3.Ib, p. 248.

4.4.3 Lukacs (1975), Theorem 4.2.6, Corollary, p. 89.

4.4.4 (a) Lukacs (1975), Theorem 4.2.8, p. 92.

5.1 This result can be deduced on considering the case where $b_t = t$ in **4.3.2** (c) and (d).

5.2 Rao (1973), Section 2c.3, p. 112.

5.3 Rao (1973), Section 2c.3, p. 114.

5.4 Rao (1973), Section 2c.3, p. 115.

5.5 Gouriéroux and Monfort (1995), vol. 2, Appendix.

6.1 Rao (1973), Section 2c.5, p. 127.

6.2 Rao (1973), Section 2c.5, p. 127.

6.3 Rao (1973), Section 2c.5, p. 128.

7.2 Rao (1973), Section 2c.5, p. 128.

Several counterexamples relevant to the above results are presented in Stoyanov (1997). About laws of large of numbers, the literature on ergodic theorems is also relevant; for a review, see Petersen (1983). For general presentations of asymptotic theory in view of statistical applications, the reader may consult: Rao (1973, Chapter 2), Serfling (1980), White (1984), Prakasa Rao (1987), McCabe and Tremayne (1993), Davidson (1994), van der Vaart (1998), Spanos (1999, Chapter 9) and Lehmann (1999).

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