Seemingly unrelated regressions *

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1. Model

Consider m linear regressions of the form:

$$y_i = X_i \beta_i + u_i \ , \quad i = 1, \ldots , m \tag{1.1}$$

where

$$y_i \text{ and } u_i \text{ are } T \times 1 \text{ vectors,} \tag{1.2}$$
$$X_i \text{ is a } T \times k_i \text{ matrix,} \tag{1.3}$$
$$1 \leq \operatorname{rank}(X_i) = k_i < T \, , \tag{1.4}$$
$$E\left[u_i u_j'\right] = \sigma_{ij} I_T \, . \tag{1.5}$$

These m relations can be written:

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix} = \begin{bmatrix} X_1 & 0 & \cdots & 0 \\ 0 & X_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & X_m \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_m \end{bmatrix} + \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix} \tag{1.6}$$

or

$$y = X\beta + u$$

where $y,\ X,\ \beta$ and u are vectors (or matrices) of dimensions $(Tm) \times 1$, $(Tm) \times k$, $k \times 1$ and $(Tm) \times 1$ respectively,

$$y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix} \ , \quad X = \begin{bmatrix} X_1 & 0 & \cdots & 0 \\ 0 & X_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & & X_m \end{bmatrix} \ , \quad \beta = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_m \end{bmatrix} \ , \quad u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix} \ ,$$

$$k = k_1 + k_2 + \cdots + k_m \ ,$$

$$V(u) \equiv \Sigma = \begin{bmatrix} \sigma_{11} I_T & \sigma_{12} I_T & \cdots & \sigma_{1m} I_T \\ \sigma_{21} I_T & \sigma_{22} I_T & \cdots & \sigma_{2m} I_T \\ \vdots & \vdots & & \vdots \\ \sigma_{m1} I_T & \sigma_{m2} I_T & \cdots & \sigma_{mm} I_T \end{bmatrix} = \Sigma_c \otimes I_T \ ,$$

$$\Sigma_c = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1m} \\ \sigma_{21} & \sigma_{22} & \cdots & \sigma_{2m} \\ \vdots & \vdots & & \vdots \\ \sigma_{m1} & \sigma_{m2} & \cdots & \sigma_{mm} \end{bmatrix} .$$

Note _ $\otimes$ represents Kronecker's product. If

$$A = [a_{ij}]_{\substack{i=1,\ldots,m \\ j=1,\ldots,n}} \ , \ B = [b_{ij}]_{\substack{i=1,\ldots,p \\ j=1,\ldots,q}} \ ,$$

then

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1m}B \\ a_{21}B & a_{22}B & \cdots & a_{2m}B \\ \vdots & \vdots & & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mm}B \end{bmatrix} .$$

Properties of Kronecker's product:

$$\begin{aligned} (A \otimes B)(C \otimes D) &= (AC) \otimes (BD) \ , && (1.7) \\ (A \otimes B)' &= A' \otimes B', && (1.8) \\ A \otimes (B + C) &= A \otimes B + A \otimes C \, , && (1.9) \\ (A \otimes B)^{-1} &= A^{-1} \otimes B^{-1}. && (1.10) \end{aligned}$$

2. Estimation

2.1. Generalized least squares

The generalized least squares estimator of β in model (1.6) is given by:

$$\hat{\beta}_Z = \left(X'\Sigma^{-1}X\right)^{-1} X'\Sigma^{-1}y \, , \tag{2.1}$$

$$V\left(\hat{\beta}_Z\right) = \left(X'\Sigma^{-1}X\right)^{-1} \ . \tag{2.2}$$

The idea of using generalized least squares to jointly estimate several regression was suggested by Zellner ("Seemingly unrelated regressions"). In general, $\hat{\beta}_Z$ is an estimator of β more efficient than OLS applied to each equation in (1.1).

2.2. Equivalence conditions with ordinary least squares

It is possible to identify cases where the two methods are equivalent.

1. $\sigma_{ij} = 0,\ \forall i \neq j$ (errors uncorrelated between equations).
   In this case,

$$\Sigma_c = \begin{bmatrix} \sigma_{11} & 0 & \cdots & 0 \\ 0 & \sigma_{22} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \sigma_{mm} \end{bmatrix},$$

   hence

$$\Sigma^{-1} = \Sigma_c^{-1} \otimes I_T = \begin{bmatrix} \frac{1}{\sigma_{11}} & 0 & \cdots & 0 \\ 0 & \frac{1}{\sigma_{22}} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \frac{1}{\sigma_{mm}} \end{bmatrix} \otimes I_T = \begin{bmatrix} \frac{1}{\sigma_{11}} I_T & 0 & \cdots & 0 \\ 0 & \frac{1}{\sigma_{22}} I_T & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \frac{1}{\sigma_{mm}} I_T \end{bmatrix},$$

$$X'\Sigma^{-1}X = \begin{bmatrix} \frac{1}{\sigma_{11}}(X_1'X_1) & 0 & \cdots & 0 \\ 0 & \frac{1}{\sigma_{22}}(X_2'X_2) & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \frac{1}{\sigma_{mm}}(X_m'X_m) \end{bmatrix},$$

$$X'\Sigma^{-1}y = \begin{bmatrix} \frac{1}{\sigma_{11}} X_1'y_1 \\ \frac{1}{\sigma_{22}} X_2'y_2 \\ \vdots \\ \frac{1}{\sigma_{mm}} X_m'y_m \end{bmatrix},$$

$$\left(X'\Sigma^{-1}X\right)^{-1} = \begin{bmatrix} \sigma_{11}(X_1'X_1)^{-1} & 0 & \cdots & 0 \\ 0 & \sigma_{22}(X_2'X_2)^{-1} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \sigma_{mm}(X_m'X_m)^{-1} \end{bmatrix},$$

   and

$$\hat{\beta}_Z = \left(X'\Sigma^{-1}X\right)^{-1} X'\Sigma^{-1}y = \begin{bmatrix} (X_1'X_1)^{-1} X_1'y_1 \\ (X_2'X_2)^{-1} X_2'y_2 \\ \vdots \\ (X_m'X_m)^{-1} X_m'y_m \end{bmatrix} = \begin{bmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \\ \vdots \\ \hat{\beta}_m \end{bmatrix}.$$

2. $X_1 = X_2 = \cdots = X_m \equiv \bar{X}$ (same regressors in the m equations).

In this case,

$$X = \begin{pmatrix} \bar{X} & 0 & \cdots & 0 \\ 0 & \bar{X} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \bar{X} \end{pmatrix} = I_m \otimes \bar{X} \ ,$$

hence

$$\begin{aligned} \hat{\beta}_Z &= \left[\left(I_m \otimes \bar{X} \right)' \left(\Sigma_c^{-1} \otimes I_T \right) \left(I_m \otimes \bar{X} \right) \right]^{-1} \left(I_m \otimes \bar{X} \right)' \left(\Sigma_c^{-1} \otimes I_T \right) y \\ &= \left[\Sigma_c^{-1} \otimes \left(\bar{X}' \bar{X} \right) \right]^{-1} \left(\Sigma_c^{-1} \otimes \bar{X}' \right) y \\ &= \left[\Sigma_c \otimes \left(\bar{X}' \bar{X} \right)^{-1} \right] \left(\Sigma_c^{-1} \otimes \bar{X}' \right) y = \left[I_m \otimes \left(\bar{X}' \bar{X} \right)^{-1} \bar{X}' \right] y \\ &= \begin{pmatrix} \left(\bar{X}' \bar{X} \right)^{-1} \bar{X}' & 0 & \cdots & 0 \\ 0 & \left(\bar{X}' \bar{X} \right)^{-1} \bar{X}' & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \left(\bar{X}' \bar{X} \right)^{-1} \bar{X}' \end{pmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix} \\ &= \begin{bmatrix} \left(\bar{X}' \bar{X} \right)^{-1} \bar{X}' y_1 \\ \left(\bar{X}' \bar{X} \right)^{-1} \bar{X}' y_2 \\ \vdots \\ \left(\bar{X}' \bar{X} \right)^{-1} \bar{X}' y_m \end{bmatrix} = \begin{bmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \\ \vdots \\ \hat{\beta}_m \end{bmatrix} . \end{aligned}$$

2.3. Estimation of the covariance matrix

In practice, Σ_c must be estimated. Two main methods have been proposed to do this.

1. Method based on OLS applied to individual equations.
   Given

$$\hat{u}_i = y_i - X_i \hat{\beta}_i \ ,$$
$$\hat{\beta}_i = \left(X_i' X_i \right)^{-1} X_i y_i \ , \quad i = 1, \ldots , m \ ,$$

   we can compute the estimators

$$\hat{\sigma}_{ij} = \hat{u}_i' \hat{u}_j / T \ , \quad i, j = 1, \ldots , m \ ,$$

$$\hat{\Sigma}_c = \left[\hat{\sigma}_{ij} \right]_{i,j=1,\ldots,m} \ , \ \hat{\Sigma} = \hat{\Sigma}_c \otimes I_T \ ,$$

which yields the following estimator of β :

$$b_Z = \left(X'\hat{\Sigma}^{-1}X\right)^{-1}X'\hat{\Sigma}^{-1}y \ .$$

For T large,

$$b_Z \overset{app}{\sim} N\left[\beta, \left(X'\Sigma^{-1}X\right)^{-1}\right] \ .$$

2. Iterative procedure.

   (a) $\hat{\Sigma}$ is estimated by OLS: $\hat{\beta}_0 \equiv b_Z$;

   (b) we reestimate Σ with the new residuals: this yields

$$\hat{\Sigma}_1 \rightarrow \hat{\beta}_1 = \left(X'\hat{\Sigma}_1^{-1}X\right)^{-1}X'\hat{\Sigma}_1^{-1}y \ ; \tag{2.3}$$

   (c) we reestimate Σ with the new residuals: this yields

$$\hat{\Sigma}_2 \rightarrow \hat{\beta}_2 = \left(X'\hat{\Sigma}_2X\right)^{-1}X'\hat{\Sigma}_2^{-1}y \, ; \tag{2.4}$$

   (d) and so on up to convergence.

This iterative procedure is equivalent to maximum likelihood and the resulting estimator is asymptotically normal.

3. Chronological list of references

1. Zellner (1962)
2. Zellner (1963)
3. Kmenta and Gilbert (1968)
4. Oberhofer and Kmenta (1974)
5. Revankar (1974)
6. Revankar (1976)
7. Mehta and Swamy (1976)
8. Kunitomo (1977)
9. Buse (1979)
10. Srivastava and Dwivedi (1979)
11. Breusch (1980)
12. Kariya (1981*b*)
13. Kariya (1981*a*)
14. Harvey and Phillips (1982)
15. Kariya, Fujikoshi and Krishnaiah (1984)
16. Rothenberg (1984)
17. Phillips (1985)
18. Hillier (1987)
19. Srivastava and Giles (1987)
20. Shiba and Tsurumi (1988)
21. Kiviet, Phillips and Schipp (1995)
22. Rilstone and Veall (1996)

23. van Garderen (1997)
24. Dufour and Torrès (1998)
25. Dufour and Khalaf (2001)
26. Dufour and Khalaf (2002*b*)
27. Dufour and Khalaf (2002*a*)
28. Holgersson and Shukur (2001)

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