Linear models with nonscalar covariance matrix and generalized least squares

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1. Generalized least squares

1.1. Best linear unbiased estimator

\[ y = X\beta + u \] (1.1)

where \( y \) is a \( T \times 1 \) vector of observations on a dependent variable, \( X \) is a \( T \times k \) nonstochastic matrix of rank \( k \), and \( u \) is a \( T \times 1 \) vector of disturbances (errors) such that

\[
\begin{align*}
E(u) &= 0 \\
V(u) &= \sigma^2 V
\end{align*}
\] (1.2)

and \( V \) is a known \( T \times T \) positive definite matrix. Then the least-squares estimator

\[ \hat{\beta} = (X'X)^{-1}X'y \] (1.3)

is unbiased but does not have minimal variance. The covariance matrix of \( \hat{\beta} \) is

\[ V(\hat{\beta}) = \sigma^2 (X'X)^{-1}X'VVX(X'X)^{-1} \] (1.4)

so that the usual formula

\[ V(\hat{\beta}) = \sigma^2 (X'X)^{-1} \] (1.5)

is not valid.

The fact \( V \) is positive definite entails that \(|V| \neq 0\), so there is no perfect correlation between the disturbances. Further, there exists a nonsingular \( T \times T \) matrix \( P \) such that

\[
\begin{align*}
PVP' &= I_T, \\
(P')^{-1}V^{-1}P^{-1} &= (PVP')^{-1} = I_T.
\end{align*}
\] (1.6) (1.7)

Multiply both sides of (1.1) by \( P \):

\[ Py = PX\beta + Pu. \] (1.8)
We get in this way the transformed model

\[ y_* = X_* \beta + u_* \]  \hspace{1cm} (1.9)

where

\[ y_* = Py, \quad X_* = PX, \quad u_* = Pu \]  \hspace{1cm} (1.10)

\[ \mathbb{E}(u_*) = 0, \]  \hspace{1cm} (1.11)

\[ \mathbb{V}(u_*) = \mathbb{E}[Puu'P'] = \sigma^2 PV P' = \sigma^2 I_T. \]  \hspace{1cm} (1.12)

Then

\[ \hat{\beta}_G = \left( X_*'X_* \right)^{-1} X_*'y_* \]  \hspace{1cm} (1.13)

is the best linear unbiased estimator of \( \beta \):

\[ \mathbb{E}(\hat{\beta}_G) = \beta \]
\[ \mathbb{V}(\hat{\beta}_G) = \sigma^2 (X_*'X_*)^{-1}. \]  \hspace{1cm} (1.14)

We can also write:

\[ \hat{\beta}_G = (X'P'PX)^{-1} X'P'Py = (X'V^{-1}X)^{-1} X'V^{-1}y \]  \hspace{1cm} (1.15)

for

\[ PV P' = I_T \Rightarrow V = P^{-1}(P')^{-1} = (P'P)^{-1} \]
\[ \Rightarrow V^{-1} = P'P. \]  \hspace{1cm} (1.16)

\( \hat{\beta}_G \) is called the generalized least squares estimator of \( \beta \):

\[ \mathbb{E}(\hat{\beta}_G) = \beta, \]
\[ \mathbb{V}(\hat{\beta}_G) = \sigma^2 (X_*'X_*) = \sigma^2 (X'V^{-1}X)^{-1}. \]  \hspace{1cm} (1.17)

We know that \( \hat{\beta} \) minimizes

\[ (y - X\beta)'(y - X\beta). \]  \hspace{1cm} (1.18)
Similarly, $\hat{\beta}_G$ minimizes

$$(y^* - X^* \beta)' (y^* - X^* \beta) = (Py - PX \beta)' (Py - PX \beta)$$

$$= (y - X \beta)' P' P (y - X \beta)$$

$$= (y - X \beta)' V^{-1} (y - X \beta)$$

This is why $\hat{\beta}_G$ is also called a *weighted least squares* estimator of $\beta$. 
1.2. Gaussian case

Suppose
\[
    u \sim N \left[ 0, \sigma^2 V \right] \tag{1.19}
\]
Then
\[
    \hat{\beta}_G \sim N \left[ \beta, \sigma^2 \left( X'V^{-1}X \right)^{-1} \right] \tag{1.20}
\]
is the best mean squares unbiased estimator of $\beta$.

We can build tests and confidence intervals in the usual manner by using the transformed model
\[
    (Py) = (PX)\beta + (Pu) \tag{1.21}
\]
instead of
\[
    y = X\beta + u. \tag{1.22}
\]
2. Estimation with heteroskedasticity

2.1. Known variance structure

Suppose

\[
E[uu'] = \sigma^2 \begin{bmatrix}
  d_1^2 & 0 & \cdots & 0 \\
  0 & d_2^2 & \cdots & 0 \\
  \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & \cdots & d_T^2
\end{bmatrix} = \sigma^2 V.
\]  

(2.1)

The variance of each element of \( u \) is then

\[
V(u_t) = \sigma_t^2 = d_t^2 \sigma^2
\]

(2.2)

and we have:

\[
y_t = x_t' \beta + u_t, \quad t = 1, \ldots, T
\]

\[
\frac{y_t}{d_t} = \frac{1}{d_t} x_t' \beta + \frac{u_t}{d_t}, \quad t = 1, \ldots, T
\]

(2.3)

\[
y_{st} = x_{st}' \beta + u_{st}, \quad t = 1, \ldots, T
\]

(2.4)

\[
V(u_{st}^2) = V\left(\frac{u_t}{d_t}\right) = \sigma^2 \frac{d_t^2}{d_t^2} = \sigma^2
\]

(2.5)

\[
P = \begin{bmatrix}
  1/d_1 & 0 & \cdots & 0 \\
  0 & 1/d_2 & \cdots & 0 \\
  \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & \cdots & 1/d_T
\end{bmatrix}
\]

(2.6)

2.2. Unknown variance structure

It is rare that \( d_1, \ldots, d_T \) are known.

It is impossible to estimate \( T + k \) parameters with \( T \) observations (incodental parameter problem).

One must make hypotheses on the form of the variance structure.

1. \( d_t^2 = c (x_{ik})^2 \)
where $x_k$ is one of the explanatory variables or another variable. Then

$$\frac{y_t}{x_{tk}} = \frac{1}{x_{tk}} x_t' \beta + \frac{u_t}{x_{tk}} , \quad t = 1, \ldots, T$$

$$\text{V} \left( \frac{u_t}{x_{tk}} \right) = \sigma^2 c = c \sigma^2$$ \hspace{1cm} (2.7)

2. $\sigma_t^2 = c (E y_t)^2 = c (x_t' \beta)^2$

Then

$$\frac{y_t}{E(y_t)} = \frac{1}{E(y_t)} x_t' \beta + \frac{u_t}{E(y_t)} , \quad t = 1, \ldots, T$$ \hspace{1cm} (2.8)

A difficulty here is that $E(y_t) = x_t' \beta$ is unknown. This suggests a two-step procedure.

1. Estimate $\beta$ par OLS. This is reasonable because $\hat{\beta}$ is unbiased.

2. The model is then transformed according to:

$$\frac{y_t}{x_t' \hat{\beta}} = \left( \frac{1}{x_t' \hat{\beta}} x_t' \right) \beta + \frac{u_t}{x_t' \hat{\beta}} .$$ \hspace{1cm} (2.9)

In this way, the model becomes “approximately homoskedastic”. For $T$ large this leads to efficient estimators and valid tests and confidence intervals.