Properties of moments of random variables *

Jean-Marie Dufour †
Université de Montréal

First version: May 1995
Revised: January 2003
This version: January 14, 2003
Compiled: January 14, 2003, 1:50pm

* This work was supported by the Canada Research Chair Program (Chair in Econometrics, Université de Montréal), the Alexander-von-Humboldt Foundation (Germany), the Canadian Network of Centres of Excellence [program on *Mathematics of Information Technology and Complex Systems* (MITACS)], the Canada Council for the Arts (Killam Fellowship), the Natural Sciences and Engineering Research Council of Canada, the Social Sciences and Humanities Research Council of Canada, and the Fonds FCAR (Government of Québec).

† Canada Research Chair Holder (Econometrics). Centre interuniversitaire de recherche en analyse des organisations (CIRANO), Centre interuniversitaire de recherche en économie quantitative (CIREQ), and Département de sciences économiques, Université de Montréal. Mailing address: Département de sciences économiques, Université de Montréal, C.P. 6128 succursale Centre-ville, Montréal, Québec, Canada H3C 3J7. TEL: 1 514 343 2400; FAX: 1 514 343 5831; e-mail: jean.marie.dufour@umontreal.ca. Web page: http://www.fas.umontreal.ca/SCECO/Dufour .

Contents

List of Definitions, Propositions and Theorems

Let X and Y be real random variables, and let r and s be real positive constants ($r > 0,\ s > 0$). The distribution functions of X and Y are denoted $F_X(x) = \mathsf{P}[X \le x]$ and $F_Y(x) = \mathsf{P}[Y \le x]$.

1. Existence of moments

1.1 EXISTENCE OF ABSOLUTE AND ORDINARY MOMENTS. $\mathsf{E}(|X|)$ always exists in the extended real numbers $\overline{\mathbb{R}} \equiv \mathbb{R} \cup \{\infty\} \cup \{-\infty\}$ and $\mathsf{E}(|X|) \in [0, \infty]$; *i.e.,* either $\mathsf{E}(|X|)$ is a non-negative real number or $\mathsf{E}(|X|) = \infty$.

1.2 $\mathsf{E}(X)$ exists and is finite $\Leftrightarrow \mathsf{E}(|X|) < \infty$.

1.3 $\mathsf{E}(|X|) < \infty \Rightarrow |\mathsf{E}(X)| \le \mathsf{E}(|X|) < \infty$.

1.4 If $0 < r \le s$, then

$$\mathsf{E}(|X|^s) < \infty \Rightarrow \mathsf{E}(|X|^r) < \infty\,. \tag{1.1}$$

1.5 MONOTONICITY OF L_r. $L_s \subseteq L_r$ for $0 < s \le r$.

1.6 $\mathsf{E}(|X|^r) < \infty \Rightarrow \mathsf{E}(X^k)$ exists and is finite for all integers k such that $0 < k \le r$.

2. Moment inequalities

2.1 c_r-INEQUALITY.

$$\mathsf{E}(|X + Y|^r) \le c_r[\mathsf{E}(|X|^r) + \mathsf{E}(|Y|^r)] \tag{2.1}$$

where

$$\begin{aligned} c_r &= 1, &&\text{if } 0 < r \le 1\,, \\ &= 2^{r-1}, &&\text{if } r > 1\,. \end{aligned} \tag{2.2}$$

2.2 MEAN FORM OF c_r-INEQUALITY.

$$\begin{aligned} \mathsf{E}(|\tfrac{1}{2}(X+Y)|^r) &\le \left(\tfrac{1}{2}\right)^r [\mathsf{E}(|X|^r) + \mathsf{E}(|Y|^r)]\,, &&\text{if } 0 < r \le 1, \\ &\le \tfrac{1}{2}[\mathsf{E}(|X|^r) + \mathsf{E}(|Y|^r)]\,, &&\text{if } r > 1. \end{aligned} \tag{2.3}$$

2.3 CLOSURE OF L_r. Let a and b be real numbers. Then

$$X \in L_r \text{ and } Y \in L_r \Rightarrow aX + bY \in L_r\,. \tag{2.4}$$

2.4 HÖLDER INEQUALITY. If $r > 1$ and $\dfrac{1}{r} + \dfrac{1}{s} = 1$, then

$$\mathsf{E}(|XY|) \le [\mathsf{E}(|X|^r)]^{1/r}[\mathsf{E}(|Y|^s)]^{1/s} . \tag{2.5}$$

2.5 CAUCHY-SCHWARZ INEQUALITY.

$$\mathsf{E}(|XY|) \le [\mathsf{E}(X^2)]^{1/2}[\mathsf{E}(Y^2)]^{1/2} . \tag{2.6}$$

2.6 MINKOWSKI INEQUALITY. If $r \ge 1$, then

$$\mathsf{E}(|X+Y|^r)^{1/r} \le [\mathsf{E}(|X|^r)]^{1/r} + [\mathsf{E}(|Y|^r)]^{1/r} . \tag{2.7}$$

2.7 MOMENT MONOTONICITY. $[\mathsf{E}(|X|^r)]^{1/r}$ is a non-decreasing function of r, *i.e.*

$$0 < r \le s \Rightarrow [\mathsf{E}(|X|^r)]^{1/r} \le [\mathsf{E}(|X|^s)]^{1/s} . \tag{2.8}$$

2.8 Theorem LIAPUNOV THEOREM. $\log[\mathsf{E}(|X|^r)]$ *is a convex function of* r, i.e. *for any* $\lambda \in [0,1]$,

$$\log[\mathsf{E}(|X|^{\lambda r+(1-\lambda)s})] \le \lambda \log[\mathsf{E}(|X|^r)] + (1-\lambda)\log[\mathsf{E}(|X|^s)] . \tag{2.9}$$

2.9 LOWER BOUNDS ON THE MOMENTS OF A SUM. If $\mathsf{E}(|X|^r) < \infty$, $\mathsf{E}(|Y|^r) < \infty$ and $\mathsf{E}(Y \mid X) = 0$, then

$$\mathsf{E}(|X+Y|^r) \ge \mathsf{E}(|X|^r) , \quad \text{for } r \ge 1. \tag{2.10}$$

2.10 JENSEN INEQUALITY. If $g(x)$ is a convex function on $\mathbb{R}$ and $\mathsf{E}(|X|) < \infty$, then, for any constant $c \in \mathbb{R}$,

$$g(c) \le \mathsf{E}[g(X - EX + c)] \tag{2.11}$$

and, in particular,

$$g(EX) \le \mathsf{E}[g(X)] . \tag{2.12}$$

3. Markov-type inequalities

3.1 MARKOV INEQUALITIES. Let $g : \mathbb{R} \to \mathbb{R}$ be a function such that $g(X)$ is a real random variable, $\mathsf{E}(|g(X)|) < \infty$ and

$$\mathsf{P}[0 \le g(X) \le M] = 1 \tag{3.1}$$

where $M \in [0,\, \infty]$. If $g(x)$ is a non-decreasing function on $\mathbb{R}$, then, for all $a \in \mathbb{R}$,

$$\frac{\mathsf{E}\left[g\left(X\right)\right] - g\left(a\right)}{M} \leq \mathsf{P}[X \geq a] \leq \frac{\mathsf{E}\left[g\left(X\right)\right]}{g\left(a\right)}\,. \tag{3.2}$$

If $g(x)$ is a non-decreasing function on $[0, \infty)$ and $g(x) = g(-x)$ for any x, then, for all $a \geq 0$,

$$\frac{\mathsf{E}\left[g\left(X\right)\right] - g\left(a\right)}{M} \leq \mathsf{P}[|X| \geq a] \leq \frac{\mathsf{E}\left[g\left(X\right)\right]}{g\left(a\right)} \tag{3.3}$$

where $0/0 \equiv 1$.

3.2 CHEBYSHEV INEQUALITIES. If $\mathsf{P}[|X| \leq M] = 1$, where $M \in [0,\, \infty]$, then, for all $a \geq 0$,

$$\frac{\mathsf{E}\left(|X|^r\right) - a^r}{M^r} \leq \mathsf{P}[|X| \geq a] \leq \frac{\mathsf{E}\left(|X|^r\right)}{a^r}\,. \tag{3.4}$$

3.3 Theorem REFINED MARKOV INEQUALITIES. *Let $g : \mathbb{R} \to \mathbb{R}$ be a function such that $g(X)$ is a real random variable, $\mathsf{E}(|g(X)|) < \infty$ and*

$$0 \;\leq\; g(x) \leq M_U \text{ for } x \geq A_U\,, \tag{3.5}$$

$$0 \;\leq\; g(x) \leq M_L \text{ for } x \leq A_L\,, \tag{3.6}$$

where $0 \leq M_U \leq \infty$, $0 \leq M_L \leq \infty$, $0 \leq A_U \leq \infty$ and $0 \leq A_L \leq \infty$. Let also

$$C_U(g,\, a) = \int\limits_{[a,\,\infty)} g(x)\, dF_X(x)\,, \quad C_L(g,\, a) = \int\limits_{(-\infty,\,a]} g(x)\, dF_X(x)\,. \tag{3.7}$$

(a) *If $g(x)$ is nondecreasing on $[A_U, \infty)$, then, for $a \geq A_U$,*

$$\frac{C_U(g,\, a)}{M_U} \leq \mathsf{P}[X \geq a] \leq \frac{C_U(g,\, a)}{g\left(a\right)}\,. \tag{3.8}$$

(b) *If $g(x)$ is nonincreasing on $(-\infty,\, A_L]$, then, for $a \leq A_L$,*

$$\frac{C_L(g,\, a)}{M_L} \leq \mathsf{P}[X \leq a] \leq \frac{C_L(g,\, a)}{g\left(a\right)}\,. \tag{3.9}$$

(c) *If $g(x)$ is nondecreasing on $[A_U, \infty)$ and nonincreasing on $(-\infty,\, A_L]$, then, for $a \geq$*

$\max\{|A_U|,\, |A_L|\}$,

$$\begin{aligned} \mathsf{P}[|X| \geq a] &\leq \frac{C_U(g,\, a)}{g\,(a)} + \frac{C_L(g,\, a)}{g\,(-a)} \\ &\leq \frac{C_U(g,\, a) + C_L(g,\, a)}{\min\{g\,(a)\,,\, g\,(-a)\}}\,, \end{aligned} \tag{3.10}$$

$$\begin{aligned} \mathsf{P}[|X| \geq a] &\geq \frac{C_U(g,\, a)}{M_U} + \frac{C_L(g,\, a)}{M_L} \\ &\geq \frac{C_U(g,\, a) + C_L(g,\, a)}{\max\{M_U,\, M_L\}}\,. \end{aligned} \tag{3.11}$$

4. Moments and behavior of tail areas

4.1 Proposition MOMENT EXISTENCE AND TAIL AREA DECAY. *Let* $r > 0$. *If* $\mathsf{E}(|X|^r) < \infty$, *then*

$$\begin{aligned} \lim_{x\to\infty} \{x^r\, \mathsf{P}\,[X \geq x]\} &= \lim_{x\to-\infty} \{|x|^r\ \mathsf{P}\,[X \leq x]\} \\ &= \lim_{x\to\infty} \{x^r\, \mathsf{P}\,[|X| \geq x]\} = 0\,. \end{aligned} \tag{4.1}$$

In particular, if $\mathsf{E}(|X|) < \infty$, *then*

$$\begin{aligned} \lim_{x\to\infty} \{x\, \mathsf{P}\,[X \geq x]\} &= \lim_{x\to-\infty} \{|x|\ \mathsf{P}\,[X \leq x]\} \\ &= \lim_{x\to\infty} \{x\, \mathsf{P}\,[|X| \geq x]\} = 0\,. \end{aligned} \tag{4.2}$$

4.2 Theorem DISTRIBUTION DECOMPOSITION OF r-MOMENTS. *For any* $r > 0$,

$$\int_0^\infty x^r dF_X(x) = r\int_0^\infty x^{r-1}[1 - F_X(x)]dx\,, \tag{4.3}$$

$$\begin{aligned} \mathsf{E}(|X|^r) &= r\int_0^\infty x^{r-1}\mathsf{P}(|X| \geq x)dx \\ &= r\int_0^\infty x^{r-1}[1 - F_X(x) + F_X(-x)]dx\,, \end{aligned} \tag{4.4}$$

and

$$\mathsf{E}(|X|^r) < \infty \Leftrightarrow x^{r-1}\mathsf{P}(|X| \geq x) \text{ is integrable on } (0,\, +\infty)$$

$$\Leftrightarrow |x|^{r-1}\,[1 - F_X(x) + F_X(-x)] \text{ is integrable on } (0,\, +\infty)$$
$$\Leftrightarrow \int_0^\infty x^{r-1}[1 - F_X(x)]dx\, < \infty \text{ and } \int_{-\infty}^0 |x|^{r-1}\, F_X(x)dx < \infty\,. \tag{4.5}$$

4.3 Proposition DISTRIBUTION DECOMPOSITION OF THE FIRST ABSOLUTE MOMENT.

$$\int_0^\infty x dF_X(x) = \int_0^\infty [1 - F_X(x)]dx\,, \tag{4.6}$$

$$\mathsf{E}|X| = \int_0^\infty \mathsf{P}(|X| \geq x)dx\,, \tag{4.7}$$

and

$$\mathsf{E}(|X|) < \infty \Leftrightarrow \mathsf{P}(|X| \geq x) \text{ is integrable on } (0,\, +\infty)$$
$$\Leftrightarrow [1 - F_X(x) + F_X(-x)] \text{ is integrable on } (0,\, +\infty)$$
$$\Leftrightarrow \int_0^\infty [1 - F_X(x)]dx\, < \infty \text{ and } \int_{-\infty}^0 F_X(x)dx < \infty\,. \tag{4.8}$$

4.4 Proposition MOMENT-TAIL AREA INEQUALITIES. *Let $g(x)$ be a nonnegative strictly increasing function on $[0,\, \infty)$ and let $g^{-1}(x)$ be the inverse function of g. Then,*

$$\sum_{n=1}^\infty \mathsf{P}\big[|X| \geq g^{-1}(n)\big] \leq \mathsf{E}[g(X)] \leq \sum_{n=0}^\infty \mathsf{P}\big[|X| > g^{-1}(n)\big]\,. \tag{4.9}$$

In particular, for any $r > 0$,

$$\begin{aligned} \sum_{n=1}^\infty \mathsf{P}(|X| \geq n^{1/r}) \quad &\leq \quad \mathsf{E}(|X|^r) \leq \sum_{n=0}^\infty \mathsf{P}(|X| > n^{1/r}) \\ &\leq \quad 1 + \sum_{n=1}^\infty \mathsf{P}(X > n^{1/r})\,. \end{aligned} \tag{4.10}$$

4.5 Corollary MEAN-TAIL AREA INEQUALITIES. *If X is a positive random variable,*

$$\sum_{n=1}^\infty \mathsf{P}(X \geq n) \leq \mathsf{E}(X) \leq 1 + \sum_{n=1}^\infty \mathsf{P}(X > n)\,. \tag{4.11}$$

5. Moments of sums of random variables

In this section, we consider a sequence $X_1, \ldots, X_n$ of random variables, and study the moments of the corresponding sum and average:

$$S_n = \sum_{i=1}^{n} X_i \,, \quad \bar{X}_n = S_n/n \,. \tag{5.1}$$

5.1 Proposition BOUNDS ON THE ABSOLUTE MOMENTS OF A SUM OF RANDOM VARIABLES.

$$\begin{aligned} \mathsf{E}(|S_n|^r) &\leq \sum_{i=1}^{n} \mathsf{E}(|X_i|^r)\,, && \text{if } 0 < r \leq 1\,, \\ &\leq n^{r-1} \sum_{i=1}^{n} \mathsf{E}(|X_i|^r)\,, && \text{if } r > 1\,, \end{aligned} \tag{5.2}$$

and

$$\begin{aligned} \mathsf{E}(|\bar{X}_n|^r) &\leq \left(\tfrac{1}{n}\right)^r \sum_{i=1}^{n} \mathsf{E}(|X_i|^r)\,, && \text{if } 0 < r \leq 1\,, \\ &\leq \tfrac{1}{n} \sum_{i=1}^{n} \mathsf{E}(|X_i|^r)\,, && \text{if } r > 1\,. \end{aligned} \tag{5.3}$$

5.2 Proposition MINKOWSKI INEQUALITY FOR n VARIABLES. *If $r \geq 1$, then*

$$[\mathsf{E}(|S_n|^r)]^{1/r} \leq \sum_{i=1}^{n} [\mathsf{E}(|X_i|^r)]^{1/r} \tag{5.4}$$

and

$$\begin{aligned} [\mathsf{E}(|\bar{X}_n|^r)]^{1/r} &\leq \frac{1}{n} \sum_{i=1}^{n} [\mathsf{E}(|X_i|^r)]^{1/r} \\ &\leq \left\{ \frac{1}{n} \sum_{i=1}^{n} \mathsf{E}(|X_i|^r) \right\}^{1/r}. \end{aligned} \tag{5.5}$$

5.3 Proposition BOUNDS ON THE ABSOLUTE MOMENTS OF A SUM OF RANDOM VARIABLES UNDER CONDITIONAL SYMMETRY. *If the distribution of X_{k+1} given S_i is symmetric about zero for $k = 1, \ldots, n-1$, and $\mathsf{E}(|X_i|^r) < \infty$, $i = 1, \ldots, n$, then*

$$\mathsf{E}(|S_n|^r) \leq \sum_{i=1}^{n} \mathsf{E}(|X_i|^r) \quad \textit{for } 1 \leq r \leq 2\,, \tag{5.6}$$

and

$$\mathsf{E}(|\bar{X}_n|^r) \leq \left(\frac{1}{n}\right)^r \sum_{i=1}^{n} \mathsf{E}(|X_i|^r) \quad \textit{for } 1 \leq r \leq 2\,, \tag{5.7}$$

with equality holding when $r = 2$.

5.4 Proposition BOUNDS ON THE ABSOLUTE MOMENTS OF A SUM OF RANDOM VARIABLES UNDER MARTINGALE CONDITION. *If*

$$\mathsf{E}(X_{k+1} \mid S_k) = 0 \quad \textit{a.s.,} \quad k = 1, \ldots, n-1, \tag{5.8}$$

and $\mathsf{E}(|X_i|^r) < \infty$, $i = 1, \ldots, n$, *then*

$$\mathsf{E}(|S_n|^r) \leq 2 \sum_{i=1}^{n} \mathsf{E}(|X_i|^r)\,, \quad \textit{for } 1 \leq r \leq 2\,, \tag{5.9}$$

and

$$\mathsf{E}(|\bar{X}_n|^r) \leq\ 2 \left(\frac{1}{n}\right)^r \sum_{i=1}^{n} \mathsf{E}(|X_i|^r)\,, \quad \textit{for } 1 \leq r \leq 2\,. \tag{5.10}$$

Furthermore, for $r = 2$,

$$\mathsf{E}({S_n}^2) = \sum_{i=1}^{n} \mathsf{E}(X_i^2)\,. \tag{5.11}$$

5.5 Proposition BOUNDS ON THE ABSOLUTE MOMENTS OF A SUM OF RANDOM VARIABLES UNDER TWO-SIDED MARTINGALE CONDITION. *Let*

$$S_{m(k)} = \sum_{i=1,\ i\neq k}^{m+1} X_i\,, \quad 1 \leq k \leq m+1 \leq n. \tag{5.12}$$

If

$$\mathsf{E}(X_k \mid S_{m(k)}) = 0 \quad \textit{a.s.,} \quad \textit{for } 1 \leq k \leq m+1 \leq n, \tag{5.13}$$

and $\mathsf{E}(|X_i|^r) < \infty$, $i = 1, \ldots, n$, *then*

$$\mathsf{E}(|S_n|^r) \leq \left(2 - \frac{1}{n}\right) \sum_{i=1}^{n} \mathsf{E}(|X_i|^r)\,, \quad \textit{for } 1 \leq r \leq 2\,, \tag{5.14}$$

and

$$\mathsf{E}(|\bar{X}_n|^r) \leq \left(\frac{1}{n}\right)^r \left(2 - \frac{1}{n}\right) \sum_{i=1}^{n} \mathsf{E}(|X_i|^r)\,, \quad \textit{for } 1 \leq r \leq 2\,. \tag{5.15}$$

5.6 Proposition BOUNDS ON THE ABSOLUTE MOMENTS OF A SUM OF INDEPENDENT RANDOM VARIABLES. *Let the random variables $X_1, \ldots, X_n$ be independent with $\mathsf{E}(X_i) = 0$ and $\mathsf{E}(|X_i|^r) < \infty$, $i = 1, \ldots, n$, and let*

$$D(r) = [13.52/(2.6\pi)^r]\, \Gamma(r) \sin(r\pi/2)\,. \tag{5.16}$$

If $D(r) < 1$ and $1 \le r \le 2$, then

$$\mathsf{E}(|S_n|^r) \le [1 - D(r)]^{-1} \sum_{i=1}^{n} \mathsf{E}(|X_i|^r)\,, \tag{5.17}$$

and

$$\mathsf{E}(|\bar{X}_n|^r) \le \left(\frac{1}{n}\right)^r [1 - D(r)]^{-1} \sum_{i=1}^{n} \mathsf{E}(|X_i|^r)\,, \quad \text{for } 1 \le r \le 2\,. \tag{5.18}$$

6. Proofs

1.1 to **3.2**. See Loève (1977, Volume I, Sections 9.1 and 9.3, pp. 151-162). For Jensen inequality, see also Chow and Teicher (1988, Section 4.3, pp. 103-106). Hannan (1985), Lehmann and Shaffer (1988), Piegorsch and Casella (1988) and Khuri and Casella (2002) discussed conditions for the existence of the moments of $1/X$.

2.9. See von Bahr and Esseen (1965, Lemma 3).

PROOF OF THEOREM **3.3** (a) For $x \geq a \geq A_U$, we have $g(x) \geq g(a)$ and $g(x) \leq M_U$, hence

$$C_U(g,\, a) = \int_{[a,\, \infty)} g(x)\; dF_X(x) \geq g(a) \int_{[a, \infty)} dF_X(x) = g(a)\, P[X \geq a]$$

and

$$\int_{[a,\, \infty)} g(x)\; dF_X(x) \leq M_U P[X \geq a]\ ,$$

from which we get the inequality

$$\frac{C_U(g,\, a)}{M_U} \leq P[X \geq a] \leq \frac{C_U(g,\, a)}{g(a)}\ .$$

(b) For $x \leq a \leq A_L$, we have $g(x) \geq g(a)$ and $g(x) \leq M_L$, hence

$$C_L(g,\, a) = \int_{[-\infty, a)} g(x)\; dF_X(x) \geq g(a) \int_{[-\infty, a)} dF_X(x) = g(a)\, P[X \leq a]$$

and

$$\int_{[-\infty, a)} g(x)\; dF_X(x) \leq M_L P[X \leq a]$$

from which we get the inequality

$$\frac{C_L(g,\, a)}{M_L} \leq P[X \leq a] \leq \frac{C_L(g,\, a)}{g(a)}\ .$$

(c) For $a \geq \max(|A_U|, |A_L|)$, we have $a \geq A_U$ and $-a \leq A_L$, hence

$$\begin{aligned} P[|X| \geq a] &= P[X \geq a] + P[X \leq -a] \\ &\leq \frac{C_U(g,\, a)}{g(a)} + \frac{C_L(g,\, a)}{g(a)} \end{aligned}$$

$$\leq \frac{C_U(g,\, a) + C_L(g,\, a)}{\min\{g(a), g(-a)\}}$$

and

$$\begin{aligned} P[|X| \geq a] &\geq \frac{C_U(g,\, a)}{M_U} + \frac{C_L(g,\, a)}{M_L} \\ &\geq \frac{C_U(g,\, a) + C_L(g,\, a)}{\max(M_U, M_L)}. \end{aligned}$$

∎

4.1 to **4.3**. See Feller (1966, Section V.6, Lemma 1), Chung (1974, Exercises 17-18), Serfling (1980, Section 1.14, pp. 46-47) and Chow and Teicher (1988, Section 4.3, pp. 103-106). For other inequalities involving absolute moments, the reader may consult Beesack (1984).
4.4. See Chow and Teicher (1988, Section 4.1, Corollary 3, p. 90).
4.5. The inequality (4.11) is given by Chung (1974, Theorem 3.2.1) and Serfling (1980, Section 1.3, p. 12).
5.1. See von Bahr and Esseen (1965), Chung (1974, p. 48) and Chow and Teicher (1988, p. 108).
5.2. See Chung (1974, p. 48).
PROOF OF PROPOSITION **5.2** The first inequality follows by recursion on applying the Minkowski inequality for two variables. The first part of the second inequality is obtained by multiplying both sides of the first one by $(1/n)$. The second part follows on observing that the function $x^{1/r}$ is concave in x for $x > 0$ when $r > 1$. ∎

5.3. See von Bahr and Esseen (1965, Theorem 1).
5.4. See von Bahr and Esseen (1965, Theorem 2).
5.5. See von Bahr and Esseen (1965, Theorem 3).
5.6. See von Bahr and Esseen (1965, Theorem 4).

References

BEESACK, P. R. (1984): "Inequalities for Absolute Moments of a Distribution: From Laplace to von Mises," *Journal of Mathematical Analysis and Applications*, 98, 435–457.

CHOW, Y. S., AND H. TEICHER (1988): *Probability Theory. Independence, Interchangeability, Martingales. Second Edition*. Springer-Verlag, New York.

CHUNG, K. L. (1974): *A Course in Probability Theory*. Academic Press, New York, second edn.

FELLER, W. (1966): *An Introduction to Probability Theory and its Applications, Volume II*. John Wiley & Sons, New York.

HANNAN, E. J. (1985): "Letter to the Editor: "Sufficient and Necessary Conditions for Finite Negative Moments"," *The American Statistician*, 39, 326.

KHURI, A., AND G. CASELLA (2002): "The Existence of the First Negative Moment Revisited," *The American Statistician*, 56, 44–47.

LEHMANN, E. L., AND J. P. SHAFFER (1988): "Inverted Distributions," *The American Statistician*, 42, 191–194.

LOÈVE, M. (1977): *Probability Theory, Volumes I and II*. Springer-Verlag, New York, 4th edn.

PIEGORSCH, W. W., AND G. CASELLA (1988): "The Existence of the First Negative Moment," *The American Statistician*, 39, 60–62.

SERFLING, R. J. (1980): *Approximation Theorems of Mathematical Statistics*. John Wiley & Sons, New York.

VON BAHR, B., AND C.-G. ESSEEN (1965): "Inequalities for the rth Absolute Moment of a Sum of Random Variables, $1 \leq r \leq 2$," *The Annals of Mathematical Statistics*, 36, 299–303.