Hilbert spaces *

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List of Definitions, Propositions and Theorems

1. Inner-product spaces

1.1 Definition INNER PRODUCT SPACE. *Let $\mathcal{H}$ be a vector space on a set scalars S, where $S = \mathbb{R}$ or $S = \mathbb{C}$. An* inner product *on H is an application which associates to each pair of elements x and y in $\mathcal{H}$ a complex number $\langle x, y\rangle$ such that, for all $x, y, z \in \mathcal{H}$,*

(*a*) $\langle x, y\rangle = \overline{\langle y, x\rangle}$;

(*b*) $\langle x + y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$;

(*c*) $\langle \alpha x, y\rangle = \alpha \langle x, y\rangle$, *for all* $\alpha \in S$;

(*d*) $\langle x, x\rangle \geq 0$;

(*e*) $\langle x, x\rangle = 0$ *if and only if* $x = 0$.

If $\langle \cdot, \cdot\rangle$ is an inner product on $\mathcal{H}$, the pair $(\mathcal{H}, \langle \cdot, \cdot\rangle)$ is called an inner product space*, and the elements of $\mathcal{H}$ are also called elements of the inner-product space $(\mathcal{H}, \langle \cdot, \cdot\rangle)$. When there is no ambiguity on the definition of the inner product, the inner-product space $(\mathcal{H}, \langle \cdot, \cdot\rangle)$ may simply be denoted $\mathcal{H}$.*

1.2 Remark If $S = \mathbb{R}$, condition (a) reduces to

(a)′ $\langle x, y\rangle = \langle y, x\rangle$.

1.3 Definition NORM ASSOCIATED WITH AN INNER PRODUCT. *The* norm *of an element x of an inner-product space $(\mathcal{H}, \langle \cdot, \cdot\rangle)$ is defined by*

$$\|x\| = \sqrt{\langle x, x\rangle}\ . \tag{1.1}$$

1.4 Example EUCLIDEAN SPACE. $\mathbb{R}^n$ with the usual scalar product

$$\langle x, y\rangle = \sum_{i=1}^{n} x_i y_i \tag{1.2}$$

where $x = (x_1, \dots, x_x)'$ $y = (y_1, \dots, y_x)'$, where $x_i \in \mathbb{R}$ and $y_j \in \mathbb{R}$ for all i and j, is an inner-product space whose norm is the usual Euclidean norm

$$\|x\| = \left(\sum_{i=1}^{n} x_i^2\right)^{1/2} . \tag{1.3}$$

1.5 Example COMPLEX EUCLIDEAN SPACE. $\mathbb{C}^n$ with the scalar product

$$\langle x, y\rangle = \sum_{i=1}^{n} x_i \overline{y}_i \tag{1.4}$$

where $x = (x_1, \ldots, x_n)'$ $y = (y_1, \ldots, y_n)'$, where $x_i \in \mathbb{C}$ and $y_j \in \mathbb{C}$ for all i and j, is an inner-product space. The associated norm is:

$$\|x\| = \left(\sum_{i=1}^{n}|x_i|^2\right)^{1/2} . \tag{1.5}$$

1.6 Example REAL-VALUED SQUARE INTEGRABLE RANDOM VARIABLES. The set $L^2 = L^2(\Omega, \mathcal{A}, P)$ of all the random variables $X : \Omega \to \mathbb{R}$ such that $E(X^2) < \infty$ with, for any $X, Y \in L^2$,

$$\langle X, Y \rangle = E(XY) \tag{1.6}$$

is an inner-product space. The associated norm is:

$$\|X\| = [E(X^2)]^{1/2} . \tag{1.7}$$

1.7 Example COMPLEX-VALUED SQUARE INTEGRABLE RANDOM VARIABLES. The set $L^2_{\mathbb{C}}(\Omega, \mathcal{A}, P)$ of all the random variables $X : \Omega \to \mathbb{C}$ such that $E(|X|^2) < \infty$ with

$$\langle X, Y \rangle = E(X\overline{Y}) \tag{1.8}$$

is an inner-product space. The associated norm is:

$$\|X\| = [E(|X|^2)]^{1/2} . \tag{1.9}$$

1.8 Proposition CAUCHY-SCHWARZ INEQUALITY. *Let $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ be an inner-product space. Then, for all $x, y \in \mathcal{H}$,*

$$|\langle x, y \rangle| \leq \|x\| \, \|y\| \tag{1.10}$$

and

$$|\langle x, y \rangle| = \|x\| \, \|y\| \;\Leftrightarrow\; x = \left(\frac{\langle x, y \rangle}{\langle y, y \rangle}\right) y . \tag{1.11}$$

1.9 Definition ANGLE AND ORTHOGONALITY. *Let x and y be two elements of an inner product space $(\mathcal{H}, \langle\cdot,\cdot\rangle)$. The* angle *between the vectors x and y is defined by*

$$\theta = \cos^{-1}[\langle x, y \rangle / (\|x\| \, \|y\|)] . \tag{1.12}$$

x and y are said to be orthogonal *(denoted $x \perp y$) if and only if*

$$\langle x, y \rangle = 0 . \tag{1.13}$$

1.10 Proposition PROPERTIES OF THE NORM. *Let $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ be an inner-product space with the norm $\|x\| = \sqrt{\langle x, x \rangle}$. Then the following properties hold for all $x, y \in \mathcal{H}$:*

(a) $\|x + y\|^2 = \|x\|^2 + \|y\|^2 + \langle x, y \rangle + \langle y, x \rangle$;

(b) $\|x+y\| \leq \|x\| + \|y\|$; (*Triangle inequality*)

(c) $\|\alpha x\| \leq |\alpha| \, \|x\|$, *for all* $\alpha \in \mathbb{C}$;

(d) $\|x\| \geq 0$;

(e) $\|x\| = 0$ *if and only if* $x = 0$;

(f) $\|x+y\|^2 + \|x-y\|^2 = 2\,\|x\|^2 + 2\,\|y\|^2$. (*Parallelogram law*)

1.11 Proposition CONVERGENCE IN NORM. *Let* $\{x_n : n = 1, 2, ...\}$ *be a sequence of elements of an iner product space* $(\mathcal{H}, \langle \cdot, \cdot \rangle)$. *We say that* x_n *converges to* $x \in \mathcal{H}$ *if and only if*

$$\|x_n - x\| \underset{n\to\infty}{\longrightarrow} 0 \,. \tag{1.14}$$

1.12 Proposition CONTINUITY OF INNER PRODUCT. *Let* $\{x_n : n = 1, 2, ...\}$ *and* $\{y_n : n = 1, 2, ...\}$ *be two sequences of elements of an inner-product space* $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ *such that* $\|x_n - x\| \underset{n\to\infty}{\longrightarrow} 0$ *and* $\|y_n - y\| \underset{n\to\infty}{\longrightarrow} 0$, *where* $x,\, y \in \mathcal{H}$ *.Then*

$$\|x_n\| \underset{n\to\infty}{\longrightarrow} \|x\| \tag{1.15}$$

and

$$\langle x_n, y_n \rangle \underset{n\to\infty}{\longrightarrow} \langle x, y \rangle \ . \tag{1.16}$$

1.13 Example MEAN SQUARE CONVERGENCE. Let $\{X_n : n = 1, 2, ...\}$ a sequence of real random variables in $L^2 \equiv L^2(\Omega, \mathcal{A}, P)$. If we define $\langle X, Y \rangle = E(XY)$, then the convergence in norm represents mean-square convrgence:

$$\|X_n - X\| \underset{n\to\infty}{\longrightarrow} 0 \ \Leftrightarrow \ E[(X_n - X)^2] \underset{n\to\infty}{\longrightarrow} 0 \,. \tag{1.17}$$

2. Hilbert spaces

2.1 Definition CAUCHY SEQUENCE. *Let* $\{x_n : n = 1, 2, ...\}$ *be a sequence of elements of an inner-product space* $(\mathcal{H}, \langle \cdot, \cdot \rangle)$. *Then* x_n *is a* Cauchy sequence *of* $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ *if and only if*

$$\|x_n - x_m\| \underset{m,\,n\to\infty}{\longrightarrow} 0 \,. \tag{2.1}$$

2.2 Definition HILBERT SPACE. *Let* $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ *an inner-product space. Then* $(\mathcal{H}, \langle \cdot, \cdot \rangle)$*is a* Hilbert space *if and only if every Cauchy sequence of* $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ *converges in norm to an element of* $\mathcal{H}$,

$$\|x_n - x_m\| \underset{m,n\to\infty}{\longrightarrow} 0 \ \Rightarrow \ \exists x \in \mathcal{H} \text{ such that } \|x_n - x\| \underset{n\to\infty}{\longrightarrow} 0 \,. \tag{2.2}$$

2.3 Proposition NORM CONVERGENCE AND CAUCHY CRITERION. *Let* $\{x_n : n = 1, 2, ...\}$ *be a sequence of elements of a Hilbert space* $(\mathcal{H}, \langle\cdot,\cdot\rangle)$. *Then* x_n *converges in norm if and only if* x_n *is a Cauchy sequence.*

2.4 Theorem COMPLETENESS OF $L^2(\Omega, \mathcal{A}, P)$. *The inner-product space* $L^2(\Omega, \mathcal{A}, P)$ *with* $\langle X, Y\rangle = E(XY)$ *is complete and thus constitutes a Hilbert space.*

PROOF. See Brockwell and Davis (1991, Section 2.10). ∎

3. Projection theorems

3.1 Definition CLOSED SUBSPACE. *A linear subspace* $\mathcal{M}$ *of a Hilbert space* $\mathcal{H}$ *is a* closed subspace *of* $\mathcal{H}$ *if and only if* $\mathcal{M}$ *contains all its limit points, i.e.*

$$x_n \in \mathcal{M}\,,\ \forall n\,,\ \text{and}\ \|x_n - x\| \underset{n\to\infty}{\longrightarrow} 0\ \Rightarrow\ x \in \mathcal{M}\,. \tag{3.1}$$

3.2 Definition ORTHOGONAL COMPLEMENT. *The orthogonal complement* $\mathcal{M}^{\perp}$ *of a subset* $\mathcal{M}$ *of* $\mathcal{H}$ *is the set of all* $x \in \mathcal{H}$ *which are orthogonal to all the elements of* $\mathcal{M}$, *i.e.*

$$x \in \mathcal{M}^{\perp} \Leftrightarrow \langle x, y\rangle = 0\ ,\ \forall y \in \mathcal{M}\,. \tag{3.2}$$

3.3 Proposition CLOSURE OF ORTHOGONAL COMPLEMENT. *If* $\mathcal{M}$ *is any subset of a Hilbert space* $\mathcal{H}$, *then* $\mathcal{M}^{\perp}$ *is a closed subspace of* $\mathcal{H}$.

3.4 Theorem PROJECTION THEOREM. *Let* $\mathcal{M}$ *be a closed subspace of a Hilbert space* $\mathcal{H}$. *Then, for any* $x \in \mathcal{H}$,

(*a*) *there is a unique element* $\widehat{x} \in \mathcal{M}$ *such that*

$$\|x - \widehat{x}\| = \inf_{y\in\mathcal{M}} \|x - y\| \tag{3.3}$$

and

(*b*)

$$\begin{aligned} &\widehat{x} \in \mathcal{M}\ \text{and}\ \|x - \widehat{x}\| = \inf_{y\in\mathcal{M}} \|x - y\| \\ &\Leftrightarrow \widehat{x} \in \mathcal{M}\ \text{and}\ x - \widehat{x} \in \mathcal{M}^{\perp}\,. \end{aligned} \tag{3.4}$$

3.5 Corollary EXISTENCE OF PROJECTION MAPPING OF $\mathcal{H}$ ONTO $\mathcal{M}$. *Let* $\mathcal{M}$ *be a closed subspace of a Hilbert space* $\mathcal{H}$, *and let* I *be the identity mapping on* $\mathcal{H}$. *Then there is a unique mapping* $P_{\mathcal{M}}$ *of* $\mathcal{H}$ *onto* $\mathcal{M}$ *such that* $I - P_{\mathcal{M}}$ *maps* $\mathcal{H}$ *onto* $\mathcal{M}^{\perp}$. $P_{\mathcal{M}}$ *is called the projection mapping of* $\mathcal{H}$ *onto* $\mathcal{M}$.

3.6 Theorem PROPERTIES OF PROJECTION MAPPINGS. *Let $\mathcal{H}$ be a Hilbert space and $P_{\mathcal{M}}$ the projection mapping onto the closed subspace $\mathcal{M}$. Then the following properties hold for all $x, y \in \mathcal{H}$:*

(*a*) *each element $x \in \mathcal{H}$ has a unique representation as the sum of an element of $\mathcal{M}$ and an element of $\mathcal{M}^{\perp}$,*

$$x = P_{\mathcal{M}}x + (I - P_{\mathcal{M}})x \; ; \tag{3.5}$$

(*b*) $\|x\|^2 = \|P_{\mathcal{M}}x\|^2 + \|(I - P_{\mathcal{M}})x\|^2$;

(*c*) $P_{\mathcal{M}}(\alpha x + \beta y) = \alpha P_{\mathcal{M}}x + \beta P_{\mathcal{M}}y$, *for all* α , $\beta \in S$;

(*d*) $\|x_n - x\| \underset{n\to\infty}{\longrightarrow} 0 \Rightarrow P_{\mathcal{M}}x_n \underset{n\to\infty}{\longrightarrow} P_{\mathcal{M}}x$;

(*e*) $x \in \mathcal{M} \Leftrightarrow P_{\mathcal{M}}x = x$;

(*f*) $x \in \mathcal{M}^{\perp} \Leftrightarrow P_{\mathcal{M}}x = 0$;

(*g*) *if $\mathcal{M}_1$ and $\mathcal{M}_2$ are two subspaces of $\mathcal{H}$ such that $\mathcal{M}_1 \subseteq \mathcal{M}_2$, then*

$$P_{\mathcal{M}_1}(P_{\mathcal{M}_2}x) = P_{\mathcal{M}_1}x \; . \tag{3.6}$$

4. Orthonormal sets

4.1 Definition CLOSED SPAN. *Let $\mathcal{M}$ be a subset of a Hilbert space $\mathcal{H}$. The closed span of $\mathcal{M}$, denoted $\overline{sp}(\mathcal{M})$, is the smallest closed subspace of $\mathcal{H}$ that contains $\mathcal{M}$. If $\mathcal{M} = \{e_t : t \in T\}$, we can also denote $\overline{sp}(\mathcal{M})$ by $\overline{sp}\{e_t : t \in T\}$.*

4.2 Proposition CLOSED SPAN OF A FINITE SET. *The closed span of a finite set $\mathcal{M} = \{x_1, \dots, x_n\}$ is the set of all linear combinations*

$$y = \alpha_1 x_1 + \dots + \alpha_n x_n \; , \; \alpha_1 \, , \, \dots \, , \, \alpha_n \in S \tag{4.1}$$

where $S = \mathbb{R}$ or $S = \mathbb{C}$.

4.3 Definition ORTHOGONAL AND ORTHONORMAL SETS. *Let $\{e_t : t \in T\}$ be a subset of an inner-product space $(\mathcal{H}, \langle\cdot,\cdot\rangle)$. Then $\{e_t : t \in T\}$ is said to be* orthogonal *if and only if*

$$\langle e_s, e_t \rangle = 0 \, , \text{ for } s \neq t \; . \tag{4.2}$$

Further, $\{e_t : t \in T\}$ is said to be orthonormal *if and only it is orthogonal and $\langle e_t, e_t \rangle = 1$ for all $t \in T$.*

4.4 Theorem PROJECTIONS ON FINITE ORTHONORMAL SETS. *If* $\{e_1, \ldots, e_n\}$ *is an orthonormal subset of a Hilbert space* $\mathcal{H}$ *and* $\mathcal{M} = \overline{sp}\{e_1, \ldots, e_n\}$, *then the following properties hold for all* $x \in \mathcal{H}$:

(*a*) $P_{\mathcal{M}}x = \sum_{i=1}^{n} \langle x, e_i \rangle \, e_i$;

(*b*) $\|P_{\mathcal{M}}x\|^2 = \sum_{i=1}^{n} | \langle x, e_i \rangle |^2$;

(*c*) $\left\| x - \sum_{i=1}^{n} \langle x, e_i \rangle \, e_i \right\| \le \left\| x - \sum_{i=1}^{n} c_i e_i \right\|$, *for all* $c_1, \ldots, c_n \in S$;

(*d*)
$$\left\| x - \sum_{i=1}^{n} \langle x, e_i \rangle \, e_i \right\| = \left\| x - \sum_{i=1}^{n} c_i e_i \right\| \;\Leftrightarrow\; c_i = \langle x, e_i \rangle \;,\; i = 1, \ldots, n\,; \tag{4.3}$$

(*e*) $\sum_{i=1}^{n} | \langle x, e_i \rangle |^2 \le \|x\|^2$. (*Bessel's inequality*)

4.5 Remark The coefficients $c_i = \langle x, e_i \rangle$, $i = 1, \ldots, n$, are called the *Fourier coefficients* relative to the set $\{e_1, \ldots, e_n\}$.

4.6 Corollary PROJECTION ON A UNIDIMENSIONAL SUBSPACE. *Suppose* $\mathcal{M} = \overline{sp}\{e_1\}$ *where* $e_1 \in \mathcal{H}$ *and* $e_1 \neq 0$. *Then*
$$P_{\mathcal{M}}x = \frac{\langle x, e_1 \rangle}{\langle e_1, e_1 \rangle} e_1 = \frac{\langle x, e_1 \rangle}{\|e_1\|^2} e_1 \,. \tag{4.4}$$

4.7 Corollary PROJECTIONS ON FINITE ORTHOGONAL SETS. *Let* $\{e_1, \ldots, e_n\}$ *be a finite set of non-zero orthogonal vectors in a Hilbert space* $\mathcal{H}$, *let* $\mathcal{M} = \overline{sp}\{e_1, \ldots, e_n\}$, *and let* $\beta_i(x) = \langle x, e_i \rangle \,/\, \langle e_i, e_i \rangle$, $i = 1, \ldots, n$. *Then the following properties hold for all* $x \in \mathcal{H}$:

(*a*) $P_{\mathcal{M}}x = \sum_{i=1}^{n} \beta_i(x) e_i$;

(*b*) $\|P_{\mathcal{M}}x\|^2 = \sum_{i=1}^{n} | \langle x, e_i \rangle |^2 / \|e_i\|^2 = \sum_{i=1}^{n} | \langle x, e_i \rangle |^2 / \langle e_i, e_i \rangle = \sum_{i=1}^{n} |\beta_i(x)|^2 \, \|e_i\|^2$;

(*c*) $\left\| x - \sum_{i=1}^{n} \beta_i(x) e_i \right\| \le \left\| x - \sum_{i=1}^{n} c_i e_i \right\|$, *for all* $c_1, \ldots, c_n \in S$;

(*d*)
$$\left\| x - \sum_{i=1}^{n} \beta_i(x) e_i \right\| = \left\| x - \sum_{i=1}^{n} c_i e_i \right\| \;\Leftrightarrow\; c_i = \beta_i \,,\; i = 1, \ldots, n\,; \tag{4.5}$$

(e) $\sum\limits_{i=1}^{n} |\langle x, e_i\rangle|^2 / \|e_i\|^2 = \sum\limits_{i=1}^{n} |\beta_i(x)|^2 \,\|e_i\|^2 \le \|x\|^2$. (*Bessel's inequality*)

4.8 Definition SEPARABILITY. *The Hilbert space $\mathcal{H}$ is* separable *if and only if $\mathcal{H} = \overline{sp}\{e_t : t \in T\}$ where T is a finite or countable infinite set.*

4.9 Theorem ORTHONORMAL REPRESENTATIONS IN SEPARABLE HILBERT SPACES. *Let $\mathcal{H} = \overline{sp}\{e_1, e_2, \ldots\}$ be a separable Hilbert space where $\{e_1, e_2, \ldots\}$ is an orthonormal set. Then the following properties hold for all $x \in \mathcal{H}$:*

(*a*) *the set of all finite linear combinations of $\{e_1, e_2, \ldots\}$ is dense in $\mathcal{H}$, i.e., for any $\varepsilon > 0$, there is an $n > 0$ and constants $c_1, \ldots, c_n \in S$ such that*

$$\left\| x - \sum_{i=1}^{n} c_i e_i \right\| < \varepsilon \; ; \tag{4.6}$$

(*b*) $x = \sum\limits_{i=1}^{\infty} \langle x, e_i\rangle \, e_i$, *i.e.*

$$\left\| x - \sum_{i=1}^{n} \langle x, e_i\rangle \, e_i \right\| \underset{n\to\infty}{\longrightarrow} 0 \; ; \tag{4.7}$$

(*c*) $\|x\|^2 = \sum\limits_{i=1}^{n} |\langle x, e_i\rangle|^2$;

(*d*) $\langle x, y\rangle = \sum\limits_{i=1}^{\infty} \langle x, e_i\rangle \langle e_i, y\rangle$, $\forall\, y \in \mathcal{H}$; (*Parseval's identity*)

(*e*) $x = 0 \Leftrightarrow \langle x, e_i\rangle = 0$, , $i = 1, 2, \ldots$.

4.10 Corollary ORTHOGONAL REPRESENTATIONS IN SEPARABLE HILBERT SPACES. *Let $\mathcal{H} = \overline{sp}\{e_1, e_2, \ldots\}$ be a separable Hilbert space where $\{e_1, e_2, \ldots\}$ is a set of non-zero orthogonal vectors, and let $\beta_i(x) = \langle x, e_i\rangle / \langle e_i, e_i\rangle$, $i = 1, \ldots, n$. Then the following properties hold for all $x \in \mathcal{H}$:*

(*a*) *the set of all finite linear combinations of $\{e_1, e_2, \ldots\}$ is dense in $\mathcal{H}$, i.e., for any $\varepsilon > 0$, there is an $n > 0$ and constants $c_1, \ldots, c_n \in S$ such that*

$$\left\| x - \sum_{i=1}^{n} c_i e_i \right\| < \varepsilon \; ; \tag{4.8}$$

(b) $x = \sum_{i=1}^{\infty} \beta_i(x) e_i$, i.e.

$$\Big\| x - \sum_{i=1}^{n} \beta_i e_i \Big\| \underset{n\to\infty}{\longrightarrow} 0 \; ; \tag{4.9}$$

(c) $\|x\|^2 = \sum_{i=1}^{n} |\langle x, e_i\rangle|^2 / \|e_i\|^2 = \sum_{i=1}^{n} |\langle x, e_i\rangle|^2 / \langle e_i, e_i\rangle = \sum_{i=1}^{n} |\beta_i(x)|^2 \, \|e_i\|^2$;

(d)

$$\begin{aligned} \langle x, y\rangle &= \sum_{i=1}^{\infty} \langle x, e_i\rangle \langle e_i, y\rangle / \|e_i\|^2 = \sum_{i=1}^{\infty} \langle x, e_i\rangle \langle e_i, y\rangle / \langle e_i, e_i\rangle \\ &= \sum_{i=1}^{\infty} \beta_i(x) \overline{\beta_i(y)} \, \|e_i\|^2 \ , \ \forall\, y \in \mathcal{H} \; ; \qquad \text{(Parseval's identity)} \end{aligned}$$

(e) $x = 0 \Leftrightarrow \langle x, e_i\rangle = 0\,, \ i = 1, 2, \ldots$.

4.11 Corollary PROJECTIONS IN SEPARABLE HILBERT SPACES. *Under the assumptions of Theorem* **4.9**, *we have for each* $x \in \mathcal{H}$:

(a) *for any* $\varepsilon > 0$, *there is an* $n > 0$ *such that*

$$\|x - P_{E_n} x\| < \varepsilon \tag{4.10}$$

where $E_n = \overline{sp}\{e_1, \ldots, e_n\}$;

(b) $P_{E_n} x \underset{n\to\infty}{\longrightarrow} x$.

5. Conditional expectations

5.1 Definition CONDITIONAL EXPECTATION WITH RESPECT TO A CLOSED SUBSPACE. *Let* $\mathcal{M}$ *be a closed subspace of* $L^2 \equiv L^2(\Omega, \mathcal{A}, P)$ *containing the constant functions and let* $Y \in L^2$. *Then the conditional expectation of* Y *given* $\mathcal{M}$ *is the projection of* Y *on* $\mathcal{M}$:

$$E_{\mathcal{M}} Y = P_{\mathcal{M}} Y \ . \tag{5.1}$$

5.2 Definition CONDITIONAL EXPECTATION WITH RESPECT TO A RANDOM VECTOR. *Let* $X = (X_1, \ldots, X_k)'$ *be a vector of random variables in* $L^2 \equiv L^2(\Omega, \mathcal{A}, P)$ *and let* $L^2(X)$ *be the closed subspace of* $L^2(\Omega, \mathcal{A}, P)$ *consisting of all the random variables in* L^2 *of the form* $\phi(X)$ *for some function* ϕ. *Then the conditional expectation of* Y *given* X *is the projection of* Y *on* $L^2(X)$:

$$E(Y|X) = E(Y|X_1, \ldots, X_k) = P_{L^2(X)} Y \ . \tag{5.2}$$

6. Sources and additional references

A good summary of Hilbert space theory aimed at applications in time series analysis may be found in Brockwell and Davis (1991, Chapter 2). Other good reviews appear in: Debnath and Mikusiński (1990) for general applications, Small and McLeish (1994) for applications in statistical theory, and Young (1988) for a more mathematically oriented presentation.

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