Complements on classical linear model *

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1. Formulas for partitioned regression

$$\begin{aligned} y &= X\beta + \varepsilon \\ &= (X_1,\, X_2)\begin{pmatrix}\beta_1\\ \beta_2\end{pmatrix} + \varepsilon \\ &= X_1\beta_1 + X_2\beta_2 + \varepsilon \end{aligned} \tag{1.1}$$

$$\hat{\beta} = (X'X)^{-1}X'y = \begin{pmatrix}\hat{\beta}_1\\ \hat{\beta}_2\end{pmatrix} \tag{1.2}$$

where

$$\begin{aligned} X &: T\times k\,,\ X_1 : T\times k_1\,,\ X_2 : T\times k_2\,, \\ \beta_1 &: k_1\times 1\,,\ \beta_2 : k_2\times 1\,,\ k = k_1 + k_2\,. \end{aligned}$$

$$\begin{aligned} \hat{\beta}_1 &= (X_1'X_1)^{-1}X_1'y - (X_1'X_1)^{-1}X_1'X_2D^{-1}X_2'M_1y \\ &= b_1 - (X_1'X_1)^{-1}X_1'X_2D^{-1}X_2'M_1y \end{aligned} \tag{1.3}$$

where

$$b_1 = (X_1'X_1)^{-1}X_1'y, \tag{1.4}$$

$$M_1 = I_T - X_1(X_1'X_1)^{-1}X_1', \tag{1.5}$$

$$D = X_2'M_1X_2\,; \tag{1.6}$$

$$\hat{\beta}_2 = D^{-1}X_2'M_1y = (X_2'M_1X_2)^{-1}X_2'M_1y; \tag{1.7}$$

$$\hat{\beta}_1 = (X_1'M_2X_1)^{-1}X_1'M_2y \tag{1.8}$$

where

$$M_2 = I_T - X_2(X_2'X_2)^{-1}X_2'\,. \tag{1.9}$$

For further discussion, the reader may con consult Schmidt (1976) and Seber (1977).

2. Updating formulas for linear regressions

$$y_t = x_t'\beta + \varepsilon_t \quad , \quad t = 1,...,T \tag{2.1}$$

where

$$x_t : k\times 1\,, \tag{2.2}$$

$$Y_r = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_r \end{pmatrix} \quad , \quad X_r = \begin{bmatrix} x_1' \\ x_2' \\ \vdots \\ x_r' \end{bmatrix} \quad , \quad r = k, k+1, ..., T \, . \tag{2.3}$$

$$b_r = (X_r'X_r)^{-1} X_r'Y_r \tag{2.4}$$

is the estimator of β based on the first r observations. Then the following updating formulas hold [see Brown, Durbin and Evans (1975)] :

$$b_r = b_{r-1} + \left(X_r'X_r\right)^{-1} x_r \left(y_r - x_r'b_{r-1}\right) \, , \, k+1 \leq r \leq T \, , \tag{2.5}$$

$$(X_r'X_r)^{-1} = (X_{r-1}'X_{r-1})^{-1} - \frac{\left(X_{r-1}'X_{r-1}\right)^{-1} x_r x_r' \left(X_{r-1}'X_{r-1}\right)^{-1}}{1 + x_r' \left(X_{r-1}'X_{r-1}\right)^{-1} x_r} \, . \tag{2.6}$$

Further,

$$\begin{aligned} V(b_r) - V(b_{r-1}) &= \sigma^2 (X_r'X_r)^{-1} - \sigma^2 (X_{r-1}'X_{r-1})^{-1} \\ &= -\sigma^2 \frac{\left(X_{r-1}'X_{r-1}\right)^{-1} x_r x_r' \left(X_{r-1}'X_{r-1}\right)^{-1}}{1 + x_r' \left(X_{r-1}'X_{r-1}\right)^{-1} x_r} \end{aligned} \tag{2.7}$$

is a negative semidefinite matrix.

3. Orthogonal decompositions of least squares estimators

Consider $\hat{\beta}$ and $\hat{\beta}_0$, respectively the unrestricted estimator of β and the restricted estimator of β under the constraint $R\beta = r$:

$$\hat{\beta} = \left(X'X\right)^{-1} X'y, \tag{3.1}$$

$$\hat{\beta}_0 = \hat{\beta} + Q_R \left[r - R\hat{\beta}\right] \tag{3.2}$$

where

$$Q_R = \left(X'X\right)^{-1} R' \left[R \left(X'X\right)^{-1} R'\right]^{-1}. \tag{3.3}$$

Then, we see easily that

$$R\hat{\beta} - r = R\left[\beta + \left(X'X\right)^{-1} X'\varepsilon\right] - r \tag{3.4}$$

$$= (R\beta - r) + R_X \varepsilon \tag{3.5}$$

where

$$R_X = R \left(X'X\right)^{-1} X' \, , \tag{3.6}$$

$$\hat{\beta} - \hat{\beta}_0 = Q_R \left[R\hat{\beta} - r\right]$$

$$
\begin{aligned}
&= Q_R\big[(R\beta - r) + R_X\varepsilon\big] \\
&= Q_R(R\beta - r) + Q_R R_X\varepsilon \\
&= Q_R(R\beta - r) + Q\varepsilon \qquad (3.7)
\end{aligned}
$$

and

$$
\begin{aligned}
\hat{\beta}_0 &= \hat{\beta} + (\hat{\beta}_0 - \hat{\beta}) \\
&= \beta + \left(X'X\right)^{-1}X'\varepsilon - Q_R(R\beta - r) - Q\varepsilon \\
&= \beta + Q_R(r - R\beta) + [\left(X'X\right)^{-1}X' - Q]\varepsilon \qquad (3.8)
\end{aligned}
$$

where

$$
Q = Q_R R_X = Q_R R\left(X'X\right)^{-1}X'. \qquad (3.9)
$$

Since

$$
R_X X\left(X'X\right)^{-1} = R\left(X'X\right)^{-1}X'X\left(X'X\right)^{-1} = R\left(X'X\right)^{-1}, \qquad (3.10)
$$

$$
R_X R_X' = R\left(X'X\right)^{-1}X'X\left(X'X\right)^{-1}R' = R\left(X'X\right)^{-1}R' \qquad (3.11)
$$

and

$$
\begin{aligned}
R_X Q' &= R_X R_X' Q_R' \\
&= R\left(X'X\right)^{-1}R'\big[R\left(X'X\right)^{-1}R'\big]^{-1}R\left(X'X\right)^{-1} \qquad (3.12) \\
&= R\left(X'X\right)^{-1}, \qquad (3.13)
\end{aligned}
$$

it follows that

$$
\begin{aligned}
\mathsf{C}(R\hat{\beta} - r,\, \hat{\beta}_0) &= \mathsf{C}\big(R_X\varepsilon,\, [\left(X'X\right)^{-1}X' - Q]\varepsilon\big) \\
&= \mathsf{E}\big[R_X\varepsilon\varepsilon'[\left(X'X\right)^{-1}X' - Q]'\big] \\
&= \sigma^2 R_X[\left(X'X\right)^{-1}X' - Q]' \\
&= \sigma^2 R_X[X\left(X'X\right)^{-1} - Q'] \\
&= \sigma^2[R\left(X'X\right)^{-1} - R\left(X'X\right)^{-1}] = 0. \qquad (3.14)
\end{aligned}
$$

and

$$
\begin{aligned}
\mathsf{C}\big(\hat{\beta} - \hat{\beta}_0,\, \hat{\beta}_0\big) &= \mathsf{C}\big(Q_R\big[R\hat{\beta} - r\big],\, \hat{\beta}_0\big) \\
&= Q_R\mathsf{C}\big(R\hat{\beta} - r,\, \hat{\beta}_0\big) = 0.
\end{aligned}
$$

Thus $\hat{\beta}_0$ and $R\hat{\beta} - r$ are uncorrelated under the assumptions of the classical linear model, and similarly for $\hat{\beta}_0$ and $\hat{\beta} - \hat{\beta}_0$. This holds even if the normality assumption or the restriction $R\beta = r$

do not hold. Consequently, the identity

$$\hat{\beta} = \hat{\beta}_0 + (\hat{\beta} - \hat{\beta}_0) \tag{3.15}$$

provides a decomposition of $\hat{\beta}$ as the sum of two uncorrelated random vectors, so that

$$\mathsf{V}(\hat{\beta}) = \mathsf{V}(\hat{\beta}_0) + \mathsf{V}(\hat{\beta} - \hat{\beta}_0)\,. \tag{3.16}$$

More explicitly, we have

$$\hat{\beta} = \hat{\beta}_0 + Q_R(r - R\beta) - Q\varepsilon \tag{3.17}$$

where

$$\mathsf{C}\big[\hat{\beta}_0, Qy\big] = \mathsf{C}\big[\hat{\beta}_0, Q\varepsilon\big] = 0\,. \tag{3.18}$$

An interesting special case of the latter results is the one where

$$y = X_1\beta_1 + X_2\beta_2 + \varepsilon \tag{3.19}$$

and the restrictions take the form

$$\beta_2 = 0\,, \tag{3.20}$$

with

$$R = [0, I_{k_2}]\,, r = 0\,. \tag{3.21}$$

Then

$$\hat{\beta} = \begin{pmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \end{pmatrix}, \quad \hat{\beta}_0 = \begin{pmatrix} \hat{\beta}_{10} \\ \hat{\beta}_{20} \end{pmatrix} = \begin{pmatrix} (X_1'X_1)^{-1}X_1'y \\ 0 \end{pmatrix} \tag{3.22}$$

and

$$\hat{\beta}_1 = \hat{\beta}_{10} - Q_{20}R\hat{\beta} = \hat{\beta}_{10} - Q_{20}\hat{\beta}_2 \tag{3.23}$$

where

$$\hat{\beta}_2 = (X_2'M_1X_2)^{-1}X_2'M_1y \tag{3.24}$$

and $\hat{\beta}_2$ is independent of $\hat{\beta}_{10}$.[1]

[1] See Magnus and Durbin (1999) and Danilov and Magnus (2001) for further discussion.

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