Stochastic Processes and their Applications 4 (1976) 107–120

ON SPECTRAL ESTIMATION FOR A HOMOGENEOUS RANDOM PROCESS ON THE CIRCLE *

Jean-Marie DUFOUR **
Université de Montréal, Montréal, P.Q., Canada

Roch ROY ***
Université du Québec à Montréal, Montréal, P.Q., Canada

Received 30 August 1974
Revised 19 September 1975

A homogeneous random process on the circle $\{X(P)\colon P \in S\}$ is a process whose mean is zero and whose covariance function depends only on the angular distance θ between the points, i.e. $\mathrm{E}\{X(P)\} \equiv 0$ and $\mathrm{E}\{X(P)\,X(Q)\} = R(\theta)$. We assume that the homogeneous process $X(P)$ is observed at a finite number of points, equally spaced on the circle. Given independent realizations of the process, we first propose unbiased estimates for the parameters of the aliased spectrum and for the covariance function. We assume further that the process is Gaussian. The exact distribution of the spectral estimates and the asymptotic distribution of the estimates of the covariance function are derived. Finally, it is shown that the estimates proposed are in fact the maximum likelihood estimates and that they have minimum variance in the class of unbiased estimates.

1. Introduction

Let $\{X(P)\colon P \in S\}$ be a real-valued process on the unit circle S of the two-dimensional space $\mathbf{R}^2$, which has finite second-order moment and which is continuous in quadratic mean (q.m.). Under these conditions, the process $X(P)$ can be expanded in a Fourier series which is convergent in q.m.:

$$X(P) = C_{01} + \sum_{n=1}^{\infty} \{C_{n1} \cos(nP) + C_{n2} \sin(nP)\}\,, \qquad (1.1)$$

* Research partially supported by the National Research Council of Canada under grant A 7269.
** Now graduate student in the Economics Department, University of Chicago.
*** Now at the Département d'Informatique, Université de Montréal.

where

$$C_{01} = \frac{1}{2\pi} \int_0^{2\pi} X(P)\,\mathrm{d}P\,,$$

$$C_{n1} = \frac{1}{\pi} \int_0^{2\pi} X(P)\cos(nP)\,\mathrm{d}P\,, \quad n \geqslant 1\,,$$

$$C_{n2} = \frac{1}{\pi} \int_0^{2\pi} X(P)\sin(nP)\,\mathrm{d}P\,, \quad n \geqslant 1\,. \tag{1.2}$$

The integrals in (1.2) are defined in the q.m. sense and the series (1.1) converges in q.m. (see [8]).

The process $X(P)$ is said to be homogeneous if its first- and second-order moments are invariant under the group of rotations of the circle. This is equivalent to say that the mean $\mathrm{E}\{X(P)\}$ is constant (and in this paper we will assume that $\mathrm{E}\{X(P)\} \equiv 0$) and that the covariance function $\mathrm{E}\{X(P)\,X(Q)\}$ depends only on the angular distance θ_{PQ} between the points P and Q. Obviously, $\mathrm{E}\{X(P)\} \equiv 0$ implies that $\mathrm{E}\{C_{ni}\} = 0$ and from [8, Theorem 5] (see [6] for a more elementary treatment) the restriction on the covariance function of $X(P)$ implies that the coefficients C_{ni} are uncorrelated, i.e.,

$$E\{C_{ni}\,C_{mj}\} = \delta_{ij}\,\delta_{nm}\,a_n \geqslant 0\,, \tag{1.3}$$

for all possible values of i, j, n and m, where δ is the Kronecker delta. From (1.1) and (1.3), it is easily deduced that

$$\mathrm{E}\{X(P)\,X(Q)\} = R(\theta_{PQ}) = \sum_{n=0}^{\infty} a_n \cos(n\theta_{PQ})\,, \qquad \theta_{PQ} \in [0, \pi]\,, \tag{1.4}$$

where the spectral parameters a_n are defined by (1.3) and satisfy

$$\sum_{n=0}^{\infty} a_n < \infty\,.$$

The purpose of this paper is to develop a spectral analysis when the process is observed at N points equally spaced on the circle:

$$X(\delta r)\,, \qquad r = 0, 1, \ldots, N-1\,,$$

where $\delta = 2\pi/N$. The case of complete realizations has been studied in

[7]. Now, when the process is observed at a finite number of equally spaced points, it is not possible to estimate the function $R(\theta)$ for all values of θ. We have an aliasing phenomenon. More specifically, we can estimate $R(\theta)$ only for the values of θ which are a multiple of δ. In fact, for $r = 0, 1, ..., [\frac{1}{2}N]$ (where $[\frac{1}{2}N]$ denotes the largest integer smaller than or equal to $\frac{1}{2}N$), eq. (1.4) gives us

$$R(\delta r) = \sum_{k=0}^{\infty} a_k \cos(k\delta r) .$$

By writing $k = n + jN$, it follows that

$$R(\delta r) = \sum_{n=0}^{N-1} \sum_{j=0}^{\infty} a_{n+jN} \cos\{(n + jN)\delta r\}$$

$$= \sum_{n=0}^{N-1} \cos(n\delta r) \sum_{j=0}^{\infty} a_{n+jN}$$

Now, let

$$B_n = \sum_{j=0}^{\infty} a_{n+jN} , \quad n = 0, 1, ..., N-1 ,$$

$$A_n = \begin{cases} B_n , & n = 0, \frac{1}{2}N , \\ B_n + B_{N-n} , & 0 < n < \frac{1}{2}N . \end{cases} \tag{1.5}$$

Then we obtain

$$R(\delta r) = \sum_{n=0}^{[N/2]} A_n \cos(n\delta r) , \quad r = 0, 1, ..., [\tfrac{1}{2}N] . \tag{1.6}$$

So the only estimable parameters are $A_0, A_1, ..., A_{[N/2]}$. Note that eq. (1.5) applies to $n = \frac{1}{2}N$ only when $\frac{1}{2}N$ is an integer and this convention will be followed all along the paper.

The inverse of the relation (1.6) is given by

$$A_n = (\alpha_n/N) \sum_{r=0}^{[N/2]} \alpha_r R(\delta r) \cos(n\delta r) , \quad n = 0, 1, ..., [\tfrac{1}{2}N] , \tag{1.7}$$

where

$$\alpha_r = \begin{cases} 1 , & r = 0, \frac{1}{2}N , \\ 2 , & 0 < r < \frac{1}{2}N , \end{cases}$$

To verify (1.7), let

$$D_n = \begin{cases} A_n , & n = 0, \tfrac{1}{2}N , \\ \tfrac{1}{2}A_n , & 0 < n < \tfrac{1}{2}N , \\ \tfrac{1}{2}A_{N-n} , & \tfrac{1}{2}N < n \leqslant N - 1 . \end{cases} \tag{1.8}$$

This allows us to write

$$R(\delta r) = \sum_{n=0}^{N-1} D_n \cos(n\delta r) , \tag{1.9}$$

and extending the definition of $R(\delta r)$ to $r = 0, 1, ..., N - 1$ by the natural relation

$$R(\delta(N - r)) = R(\delta r) ,$$

it follows, after some algebraic operations, that

$$R(\delta r) = \sum_{n=0}^{N-1} D_n \mathrm{e}^{-in\delta r} , \qquad r = 0, 1, ..., N - 1 .$$

Now, from [2, eq. (3.4.17)], we have

$$D_n = \frac{1}{n} \sum_{r=0}^{N-1} \mathrm{e}^{in\delta r} R(\delta r) , \qquad n = 0, 1, ..., N - 1 .$$

Since D_n is real-valued,

$$\mathrm{Im}(D_n) = \sum_{r=0}^{N-1} R(\delta r) \sin(n\delta r) = 0 ,$$

and using (1.8), the relation (1.7) follows directly,

So, from (1.7), we see that the estimation of the parameters A_n is equivalent to the estimation of the parameters $R(\delta r)$.

Estimates of the parameters A_n were first proposed by Hannan [4]. In Section 2 of this paper, these estimates are studied in more detail. They are shown to be unbiased and in the case of a Gaussian process their exact distribution is derived. Also estimates of $R(\delta r)$, $r = 0, 1, ..., [\tfrac{1}{2}N]$, are proposed.

In Section 3 it is shown that the estimates considered, again in the Gaussian case, are in fact the maximum likelihood estimates and have minimum variance among unbiased estimates.

2. Construction of the estimates

The basic statistic to be used for the estimation of the spectral parameters is the finite Fourier transform

$$d_X^{(N)}(\lambda) = \sum_{r=0}^{N-1} X(\delta r)\, \mathrm{e}^{-ir\lambda} , \quad \lambda \in \mathbf{R} ,$$

where $X(0), X(\delta), ..., X(\delta(N-1))$ are observations belonging to the same realization of the Gaussian process $X(P)$. The frequencies of interest in this case are those of the form δn, $n = 0, 1, ..., [\frac{1}{2}N]$. In the following, $\mathcal{N}(\mu, \sigma^2)$ will denote a real normal variable with mean μ and variance σ^2. Similarly, $\mathcal{N}^c(\mu, \sigma^2)$ will denote a complex normal variable with mean μ and variance σ^2.

Theorem 2.1. *Let $X(P)$ be a Gaussian homogeneous process with mean zero. Then the random variables $d_X^{(N)}(\delta n)$, $n = 0, 1, ..., [\frac{1}{2}N]$, are mutually independent with $d_X^{(N)}(\delta n)$ being distributed as a $\mathcal{N}^c(0, \frac{1}{2}N^2 A_n)$ for $0 < n < \frac{1}{2}N$ and as a $\mathcal{N}(0, N^2 A_n)$ for $n = 0, \frac{1}{2}N$.*

Proof. This theorem can be proven by showing that the cumulants of the finite Fourier transform are those associated with the alleged distributions. For an analogous proof, see [2, Theorem 4.4.1]. The main difference here consists in the fact that we obtain the exact distribution rather than the asymptotic one; this is particular to a process on the circle. This point is explained by looking at the behavior of the covariance structure of the finite Fourier transform. Since $\mathbf{E}\{d_X^{(N)}(\delta n)\} = 0$, $n = 0, 1, ..., [\frac{1}{2}N]$, we have

$$\begin{aligned}
&\mathrm{Cov}(d_X^{(N)}(\delta n_1), d_X^{(N)}(\delta n_2)) = \\
&\quad = \mathbf{E}\{d_X^{(N)}(\delta n_1)\, \overline{d_X^{(N)}(\delta n_2)}\} \\
&\quad = \sum_{r=0}^{N-1} \sum_{s=0}^{N-1} \exp[-i\delta(n_1 r - n_2 s)]\, \mathbf{E}\{X(\delta r)\, X(\delta s)\} \\
&\quad = \sum_{s=0}^{N-1} \exp[-i\delta(n_1 - n_2)s] \left\{ \sum_{r=0}^{N-1} \exp[-i\delta n_1 (r-s)]\, R(\delta\, |r-s|) \right\} .
\end{aligned}$$

Writing $u = r - s$, using the fact that $R(\delta(N-u)) = R(\delta u)$ and eq. (1.9), it follows that

$$\sum_{r=0}^{N-1} \exp[-i\delta n_1 (r-s)]\ R(\delta|r-s|) = \sum_{u=0}^{N-1} \exp[-i\delta n_1 u]\ R(\delta u)$$

$$= N\, D_{n_1} .$$

From this relation we deduce that

$$\operatorname{Cov}(d_X^{(N)}(\delta n_1), d_X^{(N)}(\delta n_2)) = N\, D_{n_1} \sum_{s=0}^{N-1} \exp[-i\delta(n_1 - n_2)s]$$

$$= \begin{cases} N^2 D_{n_1} & \text{if } n_1 = n_2 , \\ 0 & \text{if } n_1 \neq n_2 , \end{cases}$$

since

$$\sum_{s=0}^{N-1} e^{-i\delta s} = \begin{cases} N & \text{if } s \equiv 0 \pmod N , \\ 0 & \text{if } s \not\equiv 0 \pmod N . \end{cases} \tag{2.1}$$

In the same manner it is seen that

$$E\{d_X^{(N)}(\delta n_1)\, d_X^{(N)}(\delta n_2)\} = 0 ,$$

if $n_1, n_2 \in \{1, 2, ..., [\frac{1}{2}N]\}$ with $n_1, n_2 \neq \frac{1}{2}N$, and this completes the proof of the theorem.

From Theorem 2.1 we see that an unbiased estimate of A_n is given by

$$\hat{A}_n = \begin{cases} (1/N^2)\, |d_X^{(N)}(\delta n)|^2 , & n = 0, \frac{1}{2}N , \\ (2/N^2)\, |d_X^{(N)}(\delta n)|^2 , & 0 < n < \frac{1}{2}N . \end{cases} \tag{2.2}$$

The distribution of $\hat{A}_n$ is given by:

$$\hat{A}_n \sim \begin{cases} A_n \chi_1^2 , & n = 0, \frac{1}{2}N , \\ \frac{1}{2} A_n \chi_2^2 , & 0 < n < \frac{1}{2}N , \end{cases}$$

(here the symbol $\sim$ is used for "distributed as" and χ_n^2 denotes a chi-square random variable with n degrees of freedom).

Given T independent realizations $\{X_t(\delta r)$: $r = 0, 1, ..., N-1, t = 1, ..., T\}$, of the process $X(P)$ and if $\hat{A}_{n,t}$ is the estimate (2.2) corresponding to the t^{th} realization, then a simple unbiased estimate of A_n is given by

$$A_n^{(T)} = (1/T) \sum_{t=1}^{T} \hat{A}_{n,t} , \quad n = 0, 1, ..., [\tfrac{1}{2}N] . \tag{2.3}$$

By the independence of the realizations of the process $X(P)$ it follows that the estimates $A_n^{(T)}$ are mutually independent and that

$$A_n^{(T)} \sim \begin{cases} (A_n/T)\chi_T^2 , & n = 0, \tfrac{1}{2}N , \\ (A_n/2T)\chi_{2T}^2 , & 0 < n < \tfrac{1}{2}N . \end{cases}$$

From this we see that

$$\operatorname{Var}(A_n^{(T)}) = \begin{cases} 2A_n^2/T , & n = 0, \tfrac{1}{2}N , \\ A_n^2/T , & 0 < n < \tfrac{1}{2}N , \end{cases}$$

which means that $A_n^{(T)}$ is consistent (as $T \to \infty$) for A_n.

In order to construct an estimate of the covariance function $R(\delta r)$ suppose for a while that only one realization of the process $X(P)$ is available. Then we consider

$$\hat{R}(\delta r) = \sum_{n=0}^{[N/2]} \hat{A}_n \cos(n\delta r) , \quad r = 0, 1, ..., [\tfrac{1}{2}N] , \tag{2.4}$$

with $\hat{A}_n$ being given by (2.2).

For a Gaussian process $X(P)$, $\hat{R}(\delta r)$ is unbiased for $R(\delta r)$ and

$$\operatorname{Cov}(\hat{R}(\delta r), \hat{R}(\delta s)) = \sum_{n=0}^{[N/2]} \beta_n A_n^2 \cos(n\delta r) \cos(n\delta s) ,$$

$r, s = 0, 1, ..., [\tfrac{1}{2}N]$, where

$$\beta_n = \begin{cases} 2 , & n = 0, \tfrac{1}{2}N , \\ 1 , & 0 < n < \tfrac{1}{2}N . \end{cases}$$

The estimate $\hat{R}(\delta r)$ can be written in a more familiar form, from which we can show that it is unbiased for any homogeneous process. For this we need the following lemma.

Lemma 2.2. *Let* $\{Y(r): r = 0, 1, ..., N - 1\}$ *be a sequence of N real num-*

bers. Then

$$\sum_{s=0}^{N-1-r} Y(s+r)\,Y(s) + \sum_{s=0}^{r-1} Y(s+N-r)\,Y(s)$$

$$= \frac{1}{n} \sum_{s=0}^{N-1} |d_Y^{(N)}(\delta s)|^2 \cos(\delta s r)\,, \tag{2.5}$$

for $r = 0, 1, \ldots, [\frac{1}{2}N]$, *where* $\delta = 2\pi/N$. *(The second term in the left-hand side of* (2.5) *is taken to be zero when* $r = 0$.)

This lemma can be proven first by expanding the right-hand side of (2.5) using the fact that

$$\sum_{s=0}^{N-1} |d_Y^{(N)}(\delta s)|^2 \cos(\delta s r) = \sum_{s=0}^{N-1} |d_Y^{(N)}(\delta s)|^2\, e^{i\delta s r}\,, \tag{2.6}$$

and taking advantage of the relation (2.1). Eq. (2.6) is obtained by using the property $|d_Y^{(N)}(\delta(N-s))| = |d_Y^{(N)}(\delta s)|$.

Now, from Lemma 2.2, it follows that

$$\hat{R}(\delta r) = \frac{1}{n} \sum_{s=0}^{N-1-r} X(\delta(s+r))\, X(\delta s)$$

$$+ \sum_{s=0}^{r-1} X(\delta(s+N-r))X(\delta s)\,, \quad r = 0, 1, \ldots, [\tfrac{1}{2}N]\,. \tag{2.7}$$

For any homogeneous process $X(P)$ (not necessarily Gaussian), we deduce from this last relation that $\hat{R}(\delta r)$ is unbiased for $R(\delta r)$, $r = 0, 1, \ldots, [\frac{1}{2}N]$.

Also, since the estimates $\hat{R}(\delta r)$, $r = 0, 1, \ldots, [\frac{1}{2}N]$, are related to the estimates $\hat{A}_n$, $n = 0, 1, \ldots, [\frac{1}{2}N]$, by the same one-to-one correspondence that relates the parameters $R(\delta r)$ and A_n (see (1.6), (1.7) and (2.4)), we have also

$$\hat{A}_n = \frac{\alpha_n}{N} \sum_{r=0}^{[N/2]} \alpha_r\, \hat{R}(\delta r) \cos(n\delta r)\,, \quad n = 0, 1, \ldots, [\tfrac{1}{2}N]\,, \tag{2.8}$$

which allows us to conclude that $\hat{A}_n$ is unbiased for A_n, $n = 0, 1, \ldots, [\frac{1}{2}N]$.

If T independent realizations are available, a consistent estimate of $R(\delta r)$ will be given by

$$R^{(T)}(\delta r) = \frac{1}{T} \sum_{t=1}^{T} \hat{R}_t(\delta r) , \tag{2.9}$$

where $\hat{R}_t(\delta r)$ is the estimate (2.4) corresponding to the t^{th} realization. From (2.9) we see that $(R^{(T)}(0), R^{(T)}(\delta), \ldots, R^{(T)}([\frac{1}{2}N]\,\delta))'$ forms an asymptotically normal vector.

3. Optimality of the estimates

The estimation of the covariance function $R(\delta r)$, $r = 0, 1, \ldots, [\frac{1}{2}N]$ is equivalent to the estimation of the covariance matrix Σ of the vector $X = (X(0), X(\delta), \ldots, X(\delta(N-1)))'$ knowing that Σ has the form

$$\Sigma = (\sigma(|r-s|))_{r,s=0,1,\ldots,N-1} ,$$

where

$$\sigma(|r-s|) = \sigma(N - |r-s|) \quad \text{if } |r-s| > \tfrac{1}{2}N . \tag{3.1}$$

Identity (3.1) implies that Σ is a circulant matrix.

Keeping to $X(P)$ Gaussian, if T realizations of the process $X(P)$ are available, the matrix Σ is to be estimated from the T vectors

$$X_t = (X_t(0), X_t(\delta), \ldots, X_t((N-1)\delta))' , \quad t = 1, \ldots, T ,$$

which are independent and identically distributed $\mathcal{N}(\mathbf{0}, \Sigma)$, where $\mathbf{0}$ denotes the vector $(0, \ldots, 0)'$.

For Σ non-singular, we first obtain that the estimates previously defined are the maximum likelihood estimates of the corresponding parameters.

Lemma 3.1. *A sufficient condition for the matrix Σ to be non-singular is that*

$$A_n > 0 , \quad n = 0, 1, \ldots, [\tfrac{1}{2}N] . \tag{3.2}$$

Proof. Since Σ is a covariance matrix, to show that Σ is non-singular is equivalent to show that Σ is positive definite. By (1.9), one can write for $r, s = 0, 1, ..., N-1$,

$$\sigma(|r-s|) = \sum_{n=0}^{N-1} D_n \cos(n\delta |r-s|)$$

where the D_n's are defined by (1.8).
Now for any vector $\boldsymbol{a} = (a_0, a_1, ..., a_{N-1})'$ it is not difficult to see that

$$\boldsymbol{a}'\Sigma\boldsymbol{a} = \sum_{n=0}^{N-1} D_n \, |b_n|^2 ,$$

where

$$b_n = \sum_{r=0}^{N-1} a_r \mathrm{e}^{-in\delta r} , \quad n = 0, 1, ..., N-1 .$$

If $\boldsymbol{b} = (b_0, b_1, ..., b_{N-1})'$, then one can write $\boldsymbol{b} = F\boldsymbol{a}$, where the matrix F which is defined by

$$F = (\mathrm{e}^{-i\delta rs})_{r,\, s=0,1,...,N-1} ,$$

is non-singular. Then $\boldsymbol{a} \neq \mathbf{0}$ implies that $\boldsymbol{b} \neq \mathbf{0}$, and it follows that $\boldsymbol{a}'\Sigma\boldsymbol{a} > 0$. Thus the proof is complete.

Theorem 3.2. *If $X(P)$ is a Gaussian homogeneous process whose mean is zero and if Σ is non-singular, then the vectors $(R^{(T)}(0), ..., R^{(T)}([\frac{1}{2}N]\delta))'$ and $(A_0^{(T)}, ..., A_{[N/2]}^{(T)})'$ are the maximum likelihood estimates of $(R(0), ..., R([\frac{1}{2}N]\delta))'$ and $(A_0, ..., A_{[N/2]})'$, respectively.*

Proof. Since Σ is non-singular, the likelihood function of the vectors X_t, $t = 1, ..., T$, is given by

$$L(x_1, ..., x_T; \Sigma) = \frac{|\Sigma|^{-T/2}}{(2\pi)^{NT/2}} \exp\{-\tfrac{1}{2} \operatorname{tr} \Sigma^{-1} A\} , \tag{3.3}$$

where

$$A = \sum_{t=1}^{T} x_t x_t' ,$$

(see [1, p. 45]).

From the fact that the inverse of a circulant matrix is circulant (see [3]), it follows that Σ^{-1} is a circulant matrix and since Σ^{-1} is also symmetric, we deduce that Σ^{-1} has the form

$$\Sigma^{-1} = (d(|r - s|))_{r, s = 0, 1, \dots, N-1} ,$$

where

$$d(u) = d(N - u) \quad \text{if } u > \tfrac{1}{2}N . \tag{3.4}$$

Now we have

$$\operatorname{tr} \Sigma^{-1} A = \sum_{t=1}^{T} \sum_{r=0}^{N-1} \sum_{s=0}^{N-1} d(|r - s|)\, X_t(\delta r)\, X_t(\delta s) .$$

Setting $u = |r - s|$, we obtain

$$\sum_{r=0}^{N-1} \sum_{s=0}^{N-1} d(|r - s|)\, X_t(\delta r)\, X_t(\delta s)$$
$$= d(0) \sum_{r=0}^{N-1} X_t^2(\delta r) + 2 \sum_{u=1}^{N-1} d(u) \sum_{r=0}^{N-1-u} X_t(\delta(r+u))\, X_t(\delta r) ,$$

and using (3.4) this last expression reduces to

$$d(0) \sum_{r=0}^{N-1} X_t^2(\delta r)$$
$$+ \sum_{u=1}^{N-1} d(u) \{ \sum_{r=0}^{N-1-u} X_t(\delta(r+u))\, X_t(\delta r) + \sum_{r=0}^{u-1} X_t(\delta(r+N-u))\, X_t(\delta r) \} .$$

By (2.7) and (2.9) we see that

$$\operatorname{tr} \Sigma^{-1} A = N \sum_{t=1}^{T} \sum_{u=0}^{N-1} d(u)\, \hat{R}_t(\delta u)$$
$$= NT \sum_{u=0}^{N-1} d(u)\, R^{(T)}(\delta u)$$

$$= T \sum_{r=0}^{N-1} \sum_{s=0}^{N-1} d(|r-s|)\, R^{(T)}(\delta |r-s|)$$

$$= T \operatorname{tr} \Sigma^{-1} \hat{\Sigma} ,$$

where the matrix $\hat{\Sigma}$ is defined by

$$\hat{\Sigma} = (R^{(T)}(|r-s|))_{r,s=0,1,\ldots,N-1} \cdot$$

Thus the likelihood function can be rewritten

$$L(x_1, \ldots, x_T; \Sigma) = \frac{|\Sigma|^{-T/2}}{(2\pi)^{NT/2}} \exp\{-\tfrac{1}{2} T \operatorname{tr} \Sigma^{-1} \hat{\Sigma}\} . \tag{3.5}$$

If $A_0, \ldots, A_{[N/2]}$ are positive, arguing as in the proof of Lemma 3.1, one can show that $\hat{\Sigma}$ is positive definite with probability 1. Then, from [1, Lemma 3.2.2], the likelihood function (3.5) attains its maximum with respect to Σ when $\Sigma = \hat{\Sigma}$. Since the matrix $\hat{\Sigma}$ has the same form as the matrix Σ, i.e.

$$\hat{\Sigma} = (R^{(T)}(\delta |r-s|)) ,$$

where $R^{(T)}(\delta u) = R^{(T)}(\delta(N-u))$ if $u > \frac{1}{2}N$, it follows that $(R^{(T)}(0), \ldots, R^{(T)}([\frac{1}{2}N]\delta))'$ is the maximum likelihood estimate of $(R(0), \ldots, R([\frac{1}{2}N]\delta))'$.

Finally, since the estimates $A_n^{(T)}$, $n = 0, 1, \ldots, [\frac{1}{2}N]$, are related to the estimates $R^{(T)}(\delta r)$, $r = 0, 1, \ldots, [\frac{1}{2}N]$, by the same one-to-one correspondence that relates the parameters A_n and $R(\delta r)$ (see (1.6), (1.7) and (2.8)), we can also conclude that $(A_0^{(T)}, \ldots, A_{[N/2]}^{(T)})'$ is the maximum likelihood estimate of $(A_0, \ldots, A_{[N/2]})'$. Thus the proof is complete.

In the following we shall show that the estimates considered have minimum variance among unbiased estimates.

Theorem 3.3. *Under the assumptions of Theorem 3.2, the estimates $R^{(T)}(\delta r)$ and $A_n^{(T)}$ are uniformly of minimum variance in the class of unbiased estimates of $R(\delta r)$ and A_n respectively, $r, n = 0, 1, \ldots, [\frac{1}{2}N]$.*

Proof. Let the true value of $R(\delta r)$ be $R_0(\delta r)$. From Section 2 we know that $\operatorname{Var}(R^{(T)}(\delta r) \mid R_0(\delta r)) < \infty$. Now let $f(X_1, \ldots, X_T)$ be an unbiased estimate of zero such that $\mathrm{Var}(f(X_1, ..., X_T) \mid R_0(\delta r)) < \infty$. In order to show that $R^{(T)}(\delta r)$ has minimum variance at the value $R(\delta r) = R_0(\delta r)$, it is sufficient to prove that

$$\mathrm{Cov}(f(X_1, ..., X_T), R^{(T)}(\delta r) \mid R_0(\delta r)) = 0 , \tag{3.6}$$

(see [5, p. 317]).

The joint density function of $X_1, ..., X_T$ is the function L defined by eq. (3.3). Given $R(\delta r) = R_0(\delta r)$, since $f(X_1, ..., X_T)$ is an unbiased estimate of zero, we have

$$\int f L \, d\mu = 0 , \tag{3.7}$$

where μ denotes the Lebesgue measure on the TN-dimensional euclidian space. Now, arguing as in [5, p. 318] we differentiate (3.7) with respect to $d(r)$. Permuting the differentiation and integration signs, we obtain

$$\int f \frac{\partial L}{\partial d(r)} \, d\mu = 0 . \tag{3.8}$$

From (3.5) we find

$$\frac{\partial L}{\partial d(r)} = -\tfrac{1}{2} T L \left\{ |\Sigma|^{-1} \frac{\partial |\Sigma|}{\partial d(r)} + \frac{\partial (\mathrm{tr}\, \Sigma^{-1} \hat{\Sigma})}{\partial d(r)} \right\}. \tag{3.9}$$

However, we have

$$\begin{aligned} \mathrm{tr}\, \Sigma^{-1} \hat{\Sigma} &= N \sum_{u=0}^{N-1} d(u) \, R^{(T)}(\delta u) \\ &= N \sum_{u=0}^{[N/2]} \alpha_u \, d(u) \, R^{(T)}(\delta u) , \end{aligned}$$

by (3.4) and from the fact that $R^{(T)}(\delta(N-u)) = R^{(T)}(\delta u)$. Since the values $d(u)$, $u = 0, 1, ..., [\frac{1}{2}N]$ are not functionally related, we obtain

$$\frac{\partial \, \mathrm{tr}\, \Sigma^{-1} \hat{\Sigma}}{\partial d(r)} = N \alpha_r \, R^{(T)}(\delta r) . \tag{3.10}$$

Replacing now $\partial L/\partial d(r)$ by its expression in (3.7), we deduce that

$$\int f R^{(T)}(\delta r)\, L\, d\mu = 0\,,$$

and (3.6) follows.

Moreover, since the proof is valid whatever the chosen value $R_0(\delta r)$, we can conclude that $R^{(T)}(\delta r)$ is uniformly of minimum variance in the class of unbiased estimates of $R(\delta r)$, $r = 0, 1, ..., [\frac{1}{2}N]$.

Finally, using (1.6) and (2.8), it follows from result (e) in [5, p. 318] that $A_n^{(T)}$ is also of minimum variance for A_n, $n = 0, 1, ..., [\frac{1}{2}N]$.

References

[1] T.W. Anderson, An Introduction to Multivariate Statistical Analysis (Wiley, New York, 1958).

[2] D.R. Brillinger, Time Series: Data Analysis and Theory (Holt, Rinehart and Winston, New York, 1975).

[3] L.J. Good, On the inversion of circulant matrices, Biometrika 37 (1950) 185–186.

[4] E.J. Hannan, The statistical analysis of hydrological time series, in: Proc. Symp. on Water Resources, Use and Management (Melbourne Univ. Press, Melbourne, 1963) 233–243.

[5] C.R. Rao, Linear Statistical Inference and its Application, 2nd ed. (Wiley, New York, 1973).

[6] R. Roy, Spectral analysis for a random process on the circle, J. Appl. Prob. 9 (1972) 745–757.

[7] R. Roy and J.-M. Dufour, Exact properties of spectral estimates for a Gaussian process on the circle, Utilitas Math. 5 (1974) 281–291.

[8] A.M. Yaglom, Second-order homogeneous random fields; in: Proc. 4th Berkeley Symp. on Mathematical Statistics and Probability, Vol. 2 (Univ. of California Press, Berkeley, Calif. 1961) 593–622.