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Exact inference and optimal invariant estimation for the stability parameter of symmetric α-stable distributions[☆]

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ABSTRACT

Hill estimation (Hill, 1975), the most widespread method for estimating tail thickness of heavy-tailed financial data, suffers from two drawbacks. One is that the optimal number of tail observations to use in the estimation is a function of the unknown tail index being estimated, which diminishes the empirical relevance of the Hill estimation. The other is that the hypothesis test of the underlying data lying in the domain of attraction of an α-stable law ($\alpha<2$) or of a normal law ($\alpha\geq 2$) for finite samples, is performed on the basis of the asymptotic distribution, which can be different from those for finite samples. In this paper, using the Monte Carlo technique, we propose an exact test method for the stability parameter of α-stable distributions which is based on the Hill estimator, yet is able to provide exact confidence intervals for finite samples. Our exact test method automatically includes an estimation procedure which does not need the assumption of a known number of observations on the distributional tail. Empirical applications demonstrate the advantages of our new method in comparison with the Hill estimation.

1. Introduction

Since the influential work by Mandelbrot (1963), α-stable distributions have often been considered a more realistic distribution for high-frequency variables, such as financial data, than the normal distribution, because asset returns, for example, are typically heavy-tailed and excessively peaked around zero—a phenomena that can be captured by α-stable distributions with $\alpha<2$.

Statistical inferences for estimations and hypothesis tests under the α-stable distributional assumption depend crucially on α (DuMouchel, 1971).[1] Therefore, one of the most important tasks in using the α-stable distribution is to precisely estimate α and to find exact confidence intervals for finite samples for estimated α. To discriminate $\alpha<2$ (domain of attraction of an α-stable law, Paretian case) from $\alpha\geq 2$ (domain of attraction of normal law), or rather $\alpha=2$ (Gaussian case), would be one example of how important it is to know what the exact confidence interval for the estimated α is.

Because of its simplicity and well-developed asymptotic properties, the Hill estimator (Hill, 1975) is the most popular method for estimating the tail thickness of empirical data, the stability parameter in the context of α-stable distributions. It is a simple nonparametric estimator based on order statistics. A severe drawback of the Hill estimator, however, is that it depends heavily on the number of tail observations used. In practice, the optimal number of observations on the distributional tail is generally unknown and depends on an unknown α. Another drawback is that its confidence interval for finite samples can be given only

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[1] Kurz-Kim and Loretan (2007), for example, revisit the CRSP data used in Fama and French (1992) and show that the empirical conclusion about the Capital Asset Pricing Model driven by Fama and French (1992) is not robust depending on the distributional assumption for the underlying data.

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based on the asymptotic distribution, which generally differ from the finite sample distributions. The Pickands (Pickands, 1975) and Dekkers, Einmahl and de Haan estimators (Dekkers et al., 1989) are variations on the Hill estimator. Some modifications are also considered by many authors. Huisman et al. (2001), for example, propose a weighted Hill estimator that takes into account the trade-off between bias and variance of the Hill estimator. However, for all the modified estimators of the Hill estimator, say, Hill-type estimators, the assumption of a known number of observations on the distributional tail is necessary, and confidence intervals for finite samples can be given only based on the asymptotic distribution.[2]

In this paper, we propose an estimation procedure based on the Hill estimator which automatically results from an exact test method via Monte Carlo (MC) technique (Dufour, 2006)[3] (henceforth referred to as the MC test or the MC estimation). This is because an exact confidence interval for finite samples can be constructed in the estimation procedures, or rather, an estimate in the test procedure. The novelty of the paper is, however, that we do not use the Hill estimator as a direct estimate of α, but rather as a pivotal statistic on which to base our estimate of α. In so doing, our MC method improves on the (direct) Hill estimation in two ways. First, the optimal number of observations on the distributional tail does not need (to be assumed) to be known for our estimator, i.e. it can vary optimally with the unknown α. Second, our estimator provides exact confidence intervals for finite samples.

The rest of the paper is structured as follows. Section 2 gives a brief summary of α-stable distributions and the Hill estimator. Consequently, the problem of choice of optimal k is discussed. Two practical problems for applying the Hill estimator, the two-tailed Hill estimator and the relocation of empirical data, are also discussed. In Section 3, after a brief summary of the MC technique, the MC estimation and test procedure are explained. In Section 4, we perform simulations to study the size of the usual asymptotic test and power of the MC test for finite samples. An empirical application is given in Section 5 to demonstrate the advantage of our new method over the Hill estimation. Section 6 summarizes the paper.

2. Framework

2.1. A brief summary of α-stable distributions

A random variable (*r.v.*) X is said to be stable if, for any positive numbers A and B, there is a positive number C and a real number D such that $AX_1 + BX_2 \stackrel{d}{=} CX + D$, where X_1 and X_2 are independent *r.v.*s with $X_i \stackrel{d}{=} X,\ i = 1,2$; and "$\stackrel{d}{=}$" denotes equality in distribution. Moreover, $C = (A^\alpha + B^\alpha)^{1/\alpha}$ for some $\alpha \in (0, 2]$, where the exponent α is called a stability parameter. A stable *r.v.*, X, with a stability parameter α is called α-stable. The α-stable distributions are described by four parameters denoted by $S(\alpha, \beta, \mu, \sigma)$. Although the α-stable laws are absolutely continuous, their densities can be expressed only by a complicated special function except in three special cases.[4] Therefore, the logarithm of the characteristic function of the α-stable distribution is the best way of characterizing all members of this family and is given as

$$ln \int_{-\infty}^{\infty} e^{ist} dP(X < x) = \begin{cases} -\sigma^\alpha |t|^\alpha \left[1 - i\beta \operatorname{sign}(t) \tan \frac{\pi\alpha}{2}\right] + i\mu t, & \text{for } \alpha \neq 1, \\ -\sigma|t| \left[1 + i\beta \frac{\pi}{2} \operatorname{sign}(t) \ln|t|\right] + i\mu t, & \text{for } \alpha = 1. \end{cases}$$

The shape of the α-stable distribution is determined by the stability parameter α. For $\alpha = 2$ the α-stable distribution reduces to the normal distribution, which is the only member of the α-stable family with finite variance. If $\alpha < 2$, moments of order α or higher do not exist and the tails of the distribution become thicker, i.e. the magnitude and frequency of outliers (from the viewpoint of the Gaussian) increase as α decreases. Skewness is governed by $\beta \in [-1,1]$. If $\beta = 0$, the distribution is symmetric. The location and scale of the α-stable distributions are denoted by μ and σ. The standardized version of the α-stable distribution is given by $S((x-\mu)/\sigma; \alpha, \beta, 0,1)$. The present paper considers only the symmetric case, i.e. we assume $\beta = 0$ throughout.

A strong argument in favor of the α-stable distribution as a distributional assumption for heavy-tailed empirical data is that only the α-stable distribution can serve as the limiting distribution of sums of independent identically distributed (i.i.d.) *r.v.*s proved by Zolotarev (1986). For more details on the α-stable distributions, see Zolotarev (1986) and Samorodnitsky and Taqqu (1994); and for a discussion of the role of the α-stable distributions in financial markets and macroeconomic modelling, see McCulloch (1996), Kim et al. (1997) and Rachev et al. (1999).

2.2. Hill estimator, choice of the number of tail observations and relocation

2.2.1. Hill estimator

The most popular method of estimating for α is the Hill estimator (Hill, 1975), which is a simple nonparametric estimator based on order statistics. Because of its simplicity and popularity, we use the Hill estimator for constructing our test statistic

[2] For a rough check, the quantile estimation of McCulloch (1986) may be used. The maximum likelihood (ML) estimation is also available: DuMouchel (1971) uses a binned approximate ML based on the cumulative distribution function. McCulloch (1998) considers symmetric stable ML using a numerical approximation to the density. But these are more complicated and computationally intensive than the Hill estimator.

[3] An estimation procedure which is based on the MC method is termed a Hodges–Lehmann estimation in the literature. See Hodges and Lehmann (1963) for the basic idea.

[4] The three special cases, in which the densities are expressible via elementary functions, are i) the Gaussian distribution $S(2,0,\mu,\sigma) \equiv N(\mu,2\sigma^2)$, ii) the symmetric Cauchy distribution $S(1,0,\mu,\sigma)$, and iii) the Lévy distribution $S(0.5, \pm 1,\mu,\sigma)$; see Zolotarev (1986).

although any consistent estimator can be used to formulate our test statistic. Given a sample of n observations, $X_1, X_2, \ldots, X_n$, the (upper-tail) Hill estimator[5] is given as

$$\hat{\alpha}_H = \left[k^{-1} \sum_{j=1}^{k} \left(\ln X_{n+1-j:n} - \ln X_{n-k:n} \right) \right]^{-1}, \tag{1}$$

where k is the number of tail observations used and $X_{j:n}$ denotes the j-order statistic of the sample size n. If the tail of the distribution is asymptotically Pareto, which for $0<\alpha<2$ is in the domain of attraction of the α-stable Paretian law, $\hat{\alpha}_H$ is used as estimate for the stability parameter. The asymptotic properties of the Hill estimator have been studied by many authors and are well developed: Goldie and Smith (1987) prove asymptotic normality of the Hill estimator, i.e.

$$\sqrt{k}\left(\hat{\alpha}_H^{-1} - \alpha^{-1}\right) \sim N(0, \alpha^{-2}). \tag{2}$$

Mason (1982) and Hsing (1991) consider weak consistency of the Hill estimator for independent and dependent cases. The strong consistency is proved by Deheuvels et al. (1988). Although the asymptotic properties of the Hill estimator are well known, little can be said about its finite sample performance.

2.2.2. Choice of optimal k

Before we estimate the stability parameter using the Hill estimator, one practical problem needs to be solved: how to choose optimally the number of observations on the distributional tail, k, which are contained in the Hill estimator. Note that the choice of k involves a trade-off, because it must be small enough for the observation $X_{n-k:n}$ to characterize the asymptotic power tail of the distribution. If it is too small, however, the estimator will lack precision. For more analytic details on this issue see Segers (2002). In practice, k is determined more or less by intuition or rather arbitrarily in the Hill estimation. This is a severe drawback for the Hill estimation in terms of empirical relevance and, as far as we know, there is no statistical consensus to determine k. Some authors have looked into the problem. DuMouchel (1983) proposes setting the combined k from both tails equally to 10% of samples as k independent of n and α, which, as will be shown, can be optimal only when α is very small and n is very large. Beirlant et al. (1996), for instance, consider a couple of the quantile-plot methods that can be implemented and used in empirical work. Therefore, from the practical point of view, our MC estimation, for which k does not need to be known, is of empirical importance.

In order to demonstrate the problem of the dependence of k on the unknown α, we show the so-called Hill bias plot. Fig. 1 shows Hill estimates for which the data come from an α-stable distribution with $\alpha=1.75$ and a sample size of 1000.[6] The values of k plotted are in steps of 10 from 10 (the ten largest observations in absolute value) to 990. For each value of k, 100,000 replications are made.

The solid line shows the mean of the 100,000 estimates for each k and the dashed lines above and below of the decreasing solid line show 95% Monte Carlo confidence intervals. The straight solid line means the true value of α is 1.75.

Fig. 1 shows clearly that the Hill estimate is extremely sensitive to the choice of k. Depending on k, almost every value of α is possible for empirical data. But the true $\alpha=1.75$ can be mostly detected for a certain k, namely $k=430$.[7] Furthermore, Fig. 1 also shows that estimates of the stability parameter larger than 2 in empirical applications are not necessarily evidence against infinite-variance stable distributions with $\alpha<2$, as is pointed out in McCulloch (1997). A bad choice of k is often misleading with respect to the true tail thickness.

2.3. Two-tailed Hill estimator and median relocation

In this subsection, we discuss some aspects raised in practical applications of the Hill estimator in Eq. (1). The Hill estimator in Eq. (1) uses the largest k observations for estimating the stability parameter. For the symmetric case, especially for small samples, the two-tailed Hill estimator based on the order statistics of the absolute values can be also used in order to achieve high estimation efficiency, as:

$$\hat{\alpha}_{2H} = \left[k^{-1} \sum_{j=1}^{k} \left(\ln|X|_{n+1-j:n} - \ln|X|_{n-k:n} \right) \right]^{-1}, \tag{3}$$

where, for the same n, k here is two times the k in the one-tailed case in Eq. (1). For practical applications, the data must somehow be relocated, because the estimator in Eq. (3) is scale-invariant, but not location-invariant. Despite the existence of the first moment for $1<\alpha<2$, the mean often cannot serve optimally as a relocation parameter because of its fluctuation, especially when α is small. Therefore, the median is an alternative choice as a relocation parameter.

Regarding relocation, we perform a simulation study. The simulation shows the efficiency of the Hill estimator among three relocations; true mean, sample mean and sample median. The case for true mean is not of empirical relevance, but it serves as a

[5] Because we consider only the symmetric case, we will use the two tails later in the paper in order to obtain more tail observations and, hence, more efficiency in the estimation.

[6] The results for other α values and sample sizes are the same as that for 1000 with respect to the main conclusion.

[7] Because our MC procedure, as will be shown, uses such k determined by simulation under the null hypothesis (i.e. by known α), in this sense our MC estimate can be regarded as optimal.

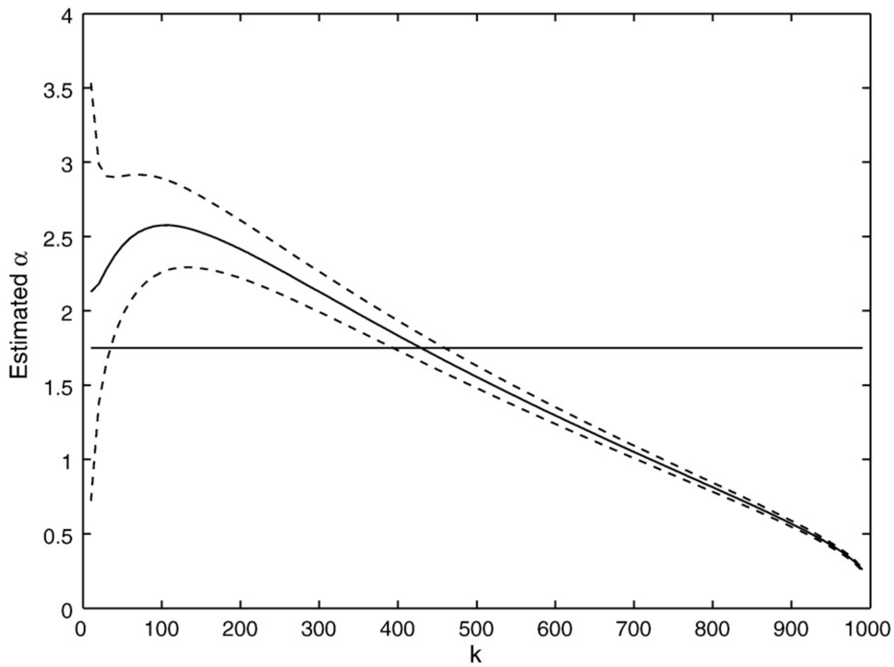

Fig. 1. Hill bias plot.

benchmark for the other two sample statistics. The simulation is designed as $\alpha = 1.01, 1.25, 1.5, 1.75, 1.95, 2$, $\beta = 0$, $\mu = 0$ and $\sigma = 1$ with sample size of $n = 100, 250, 500, 1000, 5000$.[8] For each combination, 10,000 replications were made. For estimation we use the usual Hill estimator, where the sample is relocated by true mean, by estimated sample mean and by the estimated sample median. The α-stable pseudo-*r.v.s* were generated with the algorithm of Chambers et al. (1976).[9] The results of the simulations are summarized in Table A in the Appendix. Table A shows that using the median as a relocation parameter is almost as efficient as using the true mean for all α and n adopted in the simulation. Furthermore, it is clearly shown that using the median as a relocation parameter is more efficient than using the mean in the sense of mean square error for all α and n adopted in the simulation. The difference of the two root mean squares for the median and mean cases is larger as α becomes smaller, which is expected because of the large fluctuation of sample means for small values of α, and the difference remains even for a large sample size ($n = 5000$). For this reason, in the literature a trimmed mean as a relocation is also recommended. However, the median relocation seems to be mostly appropriate for practical work.

3. The Monte Carlo estimation and test

In this section, we first summarize the basic idea of the MC test. After that, we introduce the MC estimation and test procedure.

3.1. A brief summary of the Monte Carlo method

The technique of the MC method was originally proposed by Dwass (1957) for implementing permutation tests and was later extended by Barnard (1963), and has recently been revisited by Dufour (2006). It provides an attractive method of building exact tests from statistics whose finite sample distribution is intractable but can be simulated.

Let $S_1,\ldots,S_N$ be random samples with i.i.d. *r.v.s* with the same distribution S. It is assumed that $S_1,\ldots,S_N$ are also independent and interchangeable[10]. Suppose that the distribution of S under H_0 may not be easy to compute analytically but can be simulated. The method of MC tests provides a simple method which permits us to replace the theoretical distribution $F(x)$ by its sample analogue based on $S_1,\ldots,S_N$ as

$$\hat{F}_N[x; S(N)] = \frac{1}{N} \sum_{i=1}^{N} I_{[0,\infty)}(x - S_i) \tag{4}$$

[8] If $\alpha = 1$ the true mean doesn't exist. We therefore use $\alpha = 1.01$ as the smallest value.

[9] The same random generator will be used for all the following simulations.

[10] The interchangeability of $S_1,\ldots, S_N$ is sufficient for most of the results presented in this paper. The elements of a random vector $(S_1,S_2,\ldots,S_N)$ are interchangeable if $(S_{r_1},S_{r_2},\ldots,S_{r_N})' \sim (S_1,S_2,\ldots,S_N)'$ for any permutation $(r_1,r_2,\ldots,r_N)$ of the integers $(1,2,\ldots,N)$.

where $S(N)=(S_1,\ldots,S_N)$, and $I_A(x)$ is the usual indicator function associated with the set A, i.e. $I_A(x)=1$, when $x\in A$ and 0 otherwise. Furthermore, let

$$\hat{G}_N[x;S(N)] = \frac{1}{N}\sum_{i=1}^{N} I_{[0,\infty)}(S_i - x) \tag{5}$$

be the corresponding sample function of the tail area. The sample distribution function is related to the ranks $R_1,\ldots,R_N$ of the variables $S_1,\ldots,S_N$ (when put in ascending order) by the expression:

$$R_j = N\hat{F}_N[S_j;S(N)] = \sum_{i=1}^{N} s\left(S_j - S_i\right),\ j = 1,\ldots,N. \tag{6}$$

The main idea of the MC method is that one can obtain critical values and/or compute p-values by replacing the "theoretical" null distribution $F(x)$ through its simulation-based "estimate" $\hat{F}_N(x)$ in a way that will preserve the level of the test in finite samples, irrespective of the number N of replications used as follows. Let S_0 be an empirical sample of interest, $(S_1,\ldots,S_N)'$, a simulated $(N\times 1)$-, and consequently $(S_0,S_1,\ldots,S_N)'$, a $((N+1)\times 1)$-random vector of interchangeable real *r.v.s.*[11] Moreover, let $\hat{F}_N(x)\equiv\hat{F}_N[x;S(N)]$ and $\hat{G}_N(x)=\hat{G}_N[x;S(N)]$ be defined as in Eqs. (4)–(5) and $\hat{F}_N^{-1}(x)$ be the quantile function[12] and set

$$\hat{p}_N(x) = \frac{N\hat{G}_N(x)+1}{N+1}. \tag{7}$$

then

$$P\left[\hat{G}_N(S_0)\le\alpha_1\right] = P\left[\hat{F}_N(S_0)\ge 1-\alpha_1\right] = \frac{I[\alpha_1 N]+1}{N+1},\ \text{for } 0\le\alpha_1\le 1, \tag{8}$$

$$P\left[S_0\ge\hat{F}_N^{-1}(1-\alpha_1)\right] = \frac{I[\alpha_1 N]+1}{N+1},\ \text{for } 0<\alpha_1<1, \tag{9}$$

and

$$P[\hat{p}_N(S_0)\le\alpha] = \frac{I[\alpha(N+1)]}{N+1},\ \text{for } 0\le\alpha\le 1, \tag{10}$$

where $I[z]$is the largest integer less than or equal to z. For practical purposes, α_1 and N will be chosen as

$$\alpha = \frac{I[\alpha_1 N]+1}{N+1}, \tag{11}$$

which is the desired significance level. Provided N is reasonably large, α_1 will be very close to α; in particular, if $\alpha(N+1)$ is an integer, we can take

$$\alpha_1 = \alpha - \frac{(1-\alpha)}{N},$$

in which case we see easily that the critical region $\hat{G}_N(S_0)\le\alpha_1$ is equivalent to $\hat{G}_N(S_0)<\alpha$. For $0<\alpha<1$, the randomized critical region $S_0\ge\hat{F}_N^{-1}(1-\alpha_1)$ has the same level (α) as the non-randomized critical region $S_0\ge F^{-1}(1-\alpha)$, or equivalently the critical regions $\hat{p}_N(S_0)\le\alpha$ and $\hat{G}_N(S_0)\le\alpha_1$ have the same level as the critical region $G(S_0)\equiv 1-F(S_0)\le\alpha$. See Dufour (2006) for more details on proofs and discussions about the MC test.

[11] The zero probability of ties is assumed, but the results are still valid for a positive probability of ties.

[12] For any probability distribution function $F(x)$, the quantile function $F^{-1}(q)$ is defined as follows:

$$F^{-1}(q) = \begin{cases} \inf x : F(x)\ge q, & \text{if } 0<q<1, \\ \inf x : F(x)>0, & \text{if } q=0, \\ \sup x : F(x)<1, & \text{if } q=1. \end{cases}$$

See Reiss (1989, p. 13).

3.2. Finite sample estimation and test for the stability parameter

We now test our random sample, $\{X_1,X_2,\ldots,X_n\}$, from a symmetric α-stable ($S_\alpha S$) distribution[13] for

$$H_0(\alpha_0): \alpha = \alpha_0. \tag{12}$$

For our MC test the α_0 is not a unique value, but, depending on a preliminary choice, several values. To make this clear, we use the symbol $H_0(\alpha_0)$ instead of the usual H_0. To perform this test, we need a test statistic which is free of nuisance parameters under the null hypothesis. A possible statistic can be given as

$$ST(\alpha_0) = \hat{\alpha} - \alpha_0, \tag{13}$$

where $\hat{\alpha}$ may be any consistent estimator for α. The fact that $\hat{\alpha}$ may be any consistent estimator for α means that our MC method can provide any consistent estimator with an exact confidence interval for finite samples. Furthermore, we construct an estimator for each α_0 assumed based on Eq. (3) and median relocation which can be used as $\hat{\alpha}$ in Eq. (13):

$$\hat{\alpha}(\alpha_0) = \left[k(\alpha_0)^{-1}\sum_{j=1}^{k(\alpha_0)}\left(\ln|\tilde{X}|_{n+1-j:n} - \ln|\tilde{X}|_{n-k(\alpha_0):n}\right)\right]^{-1}, \tag{14}$$

with $\tilde{X}_i := X_i - X^{\text{med}}$ and $k(\alpha_0) = k(n,\alpha_0)$ being the number of tail observations under each respective null hypothesis. This construction enables us to use any optimal k in some sense which is determined by a theoretical and/or a simulative relationship between k and α. We adopt the k/n ratio tabulated by simulation in Rachev and Mittnik (2000, p. 114). For a given α, they choose the k as the optimal number of tail observations by which the mean of the estimates from the simulated samples with α is equal to α.[14] In this sense, we regard our MC estimation as optimal—at least for finite samples.[15]

To estimate the stability parameter using our MC method, the test statistic in Eq. (13) should be nuisance-free. Because the estimator in Eq. (14) is location and scale-invariant, the test statistic in Eq. (13) is pivotal as proved in the following lemma.

Proposition 1. *[Invariance] Let $X_1,X_2,\ldots,X_n$ be i.i.d. random variables which follow an $S(\alpha,\beta,\mu,\sigma)$ distribution, and let*

$$\hat{\alpha} = a(X_1,X_2,\ldots,X_n) \tag{15}$$

be an estimator of α. If the estimator $\hat{\alpha}$ is scale-invariant, i.e.

$$\hat{\alpha} = a(cX_1,\ldots,cX_n) = a(X_1,X_2,\ldots,X_n), \text{for all } c > 0, \tag{16}$$

then the estimator $\hat{\alpha}$ has a distribution which depends only on α, β and μ/σ. If, furthermore, $\hat{\alpha}$ is location-scale-invariant, i.e.

$$\hat{\alpha} = a(cX_1 + d,\ldots,cX_n + d) = a(X_1,X_2,\ldots,X_n), \text{for all } c > 0 \text{ and } d \in \mathbf{R}, \tag{17}$$

then the estimator $\hat{\alpha}$ has a distribution which depends only on β.

[13] In case of asymmetric data, our procedure can be also applied in the same way, as in the Hill estimation, for only one tail. Based on the Kolmogorov–Smirnov statistic, Dufour et al. (2007) propose an estimation and test for the asymmetric parameter of α-stable distributions.

[14] The k/n values for the two-tailed case are reproduced for selected n in the following table, where the intermediate values needed in the paper are calculated by a linear interpolation.

	α									
n	1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9
100	.23	.29	.35	.37	.39	.41	.42	.43	.44	.44
250	.168	.240	.324	.348	.380	.408	.420	.424	.432	.440
500	.140	.214	.308	.348	.378	.404	.418	.424	.432	.440
1000	.121	.197	.295	.342	.378	.402	.417	.425	.431	.439
2000	.0715	.1845	.2880	.3405	.3765	.3995	.4160	.4245	.4315	.4380
5000	.0660	.1768	.2814	.3390	.3750	.3980	.4140	.4240	.4318	.4372
10,000	.0400	.1671	.2801	.3385	.3747	.3981	.4139	.4239	.4317	.4373

[15] When implementing our method, there are other rules that may be used for determining $k(n,\alpha)$. One can easily employ any asymptotic relationships between k and α (see de Haan and Ferreira, 2006, Ch. 3 for more details on the topic) which usually are determined, for instance, by minimizing the asymptotic mean squared error of the Hill estimate. The true tail behavior of α-stable laws, however, is visible only for extremely large data sets, possibly larger than 10^6 (see Borak et al., 2005) which implies that the optimal k/n for finite samples can be vastly different from those calculated from such asymptotic relationships. Furthermore, whatever rule is used for determining k/n, the results from our method will be unchanged, because the rule that is chosen provides the same condition in estimating for both the empirical data and the simulated data from every grid of α assumed under the null hypothesis. This footnote is the outcome of a discussion with an anonymous referee, whom we wish to thank very much.

Proof 1. To obtain the first result, we observe that

$$X_i/\sigma \sim S(\alpha,\beta,\mu/\sigma,1),\ i=1,\ldots,n. \quad (18)$$

Then, using the scale-invariance property (16) with $c=1/\sigma$, we can write

$$\hat{\alpha}=a(X_1/\sigma,\ldots,X_n/\sigma), \quad (19)$$

from which we see that the distribution of $\hat{\alpha}$ depends only on α, β, and μ/σ. Similarly, under the location-scale invariance condition (17), we observe the following:

$$(X_i-\mu)/\sigma \sim S(\alpha,\beta,0,1),\ i=1,\ldots,n. \quad (20)$$

Hence, taking $c=1/\sigma$ and $d=-\mu/\sigma$

$$\hat{\alpha}=a(X_1^*,\ldots,X_n^*), \quad (21)$$

where $X_i^*=(X_i-\mu)/\sigma,\ i=1,\ldots,n.$

Now, we introduce the MC estimation and test procedure for the stability parameter. Given a random sample $\{X_1,X_2,\ldots,X_n\}$ from a $S_\alpha S$ distribution, the following seven steps are needed:

1. Determine the set of possible α under the null hypothesis. From the viewpoint of empirical relevance it is reasonable to assume that $\alpha_0\in[1, 2]$.
2. Choose a rule $k(\alpha_0)=k(n, \alpha_0)$ for selecting k as a function of n and α_0.
3. Using the empirical data, calculate the estimator in Eq. (14) and the test statistic in Eq. (13) for every point in a fine grid of α_0 values in the set of possible values, in steps of 0.01, for example.

Table 1
Asymptotic test at 5% size.

		Choice of k					
		Random[a]		DuMouchel[b]		Known[c]	
α	n	Mean	Median	Mean	Median	Mean	Median
1	100	26.0	1.0	21.2	7.0	22.2	5.0
	250	29.6	0.5	23.0	3.0	21.8	4.9
	500	33.1	0.8	25.6	1.7	21.6	4.8
	1000	37.2	2.2	29.8	0.9	23.2	4.7
	5000	44.7	20.9	35.3	0.1	18.2	5.0
	10000	46.1	33.1	35.9	0.0	13.0	5.1
1.25	100	11.8	3.8	22.9	17.5	13.1	4.1
	250	13.6	5.8	25.1	17.1	14.8	4.5
	500	15.4	7.8	27.4	16.9	16.3	4.4
	1000	20.1	13.8	31.6	18.7	18.2	4.5
	5000	48.1	52.4	51.2	35.0	20.9	5.1
	10000	64.4	67.7	66.9	51.8	21.9	4.9
1.5	100	9.6	7.5	35.3	32.1	6.1	4.1
	250	14.4	12.2	47.5	43.2	5.8	4.3
	500	19.4	16.8	63.3	58.4	5.5	3.8
	1000	26.2	24.6	80.6	77.1	5.7	4.0
	5000	31.8	31.1	99.6	99.4	5.1	4.0
	10000	61.5	61.8	100.0	100.0	5.2	4.0
1.75	100	14.5	13.0	62.2	61.2	4.2	3.9
	250	25.0	24.3	85.9	84.5	4.7	4.3
	500	33.0	32.5	97.6	97.2	4.2	4.1
	1000	40.7	40.7	100.0	100.0	4.2	4.1
	5000	54.5	54.3	100.0	100.0	4.2	4.2
	10000	60.6	60.5	100.0	100.0	4.5	4.4
2	100	21.2	19.7	93.3	92.9	4.2	4.1
	250	35.5	35.0	99.9	99.9	4.0	4.0
	500	43.3	42.9	100.0	100.0	3.6	3.6
	1000	51.8	51.5	100.0	100.0	3.6	4.0
	5000	67.4	67.1	100.0	100.0	4.6	4.4
	10000	74.4	74.2	100.0	100.0	5.0	4.9

The usual two-tailed Hill estimator in (3) relocating the sample by estimated sample mean and by the estimated sample median (denoted mean and median in Table). For choice of k, we consider three cases: [a]choose randomly α between 1 and 2 and determine k using the Table in footnote 14, [b]a fixed ratio of 20% proposed by DuMouchel (1983) and [c]assuming known α and determine k using the Table in footnote 14.

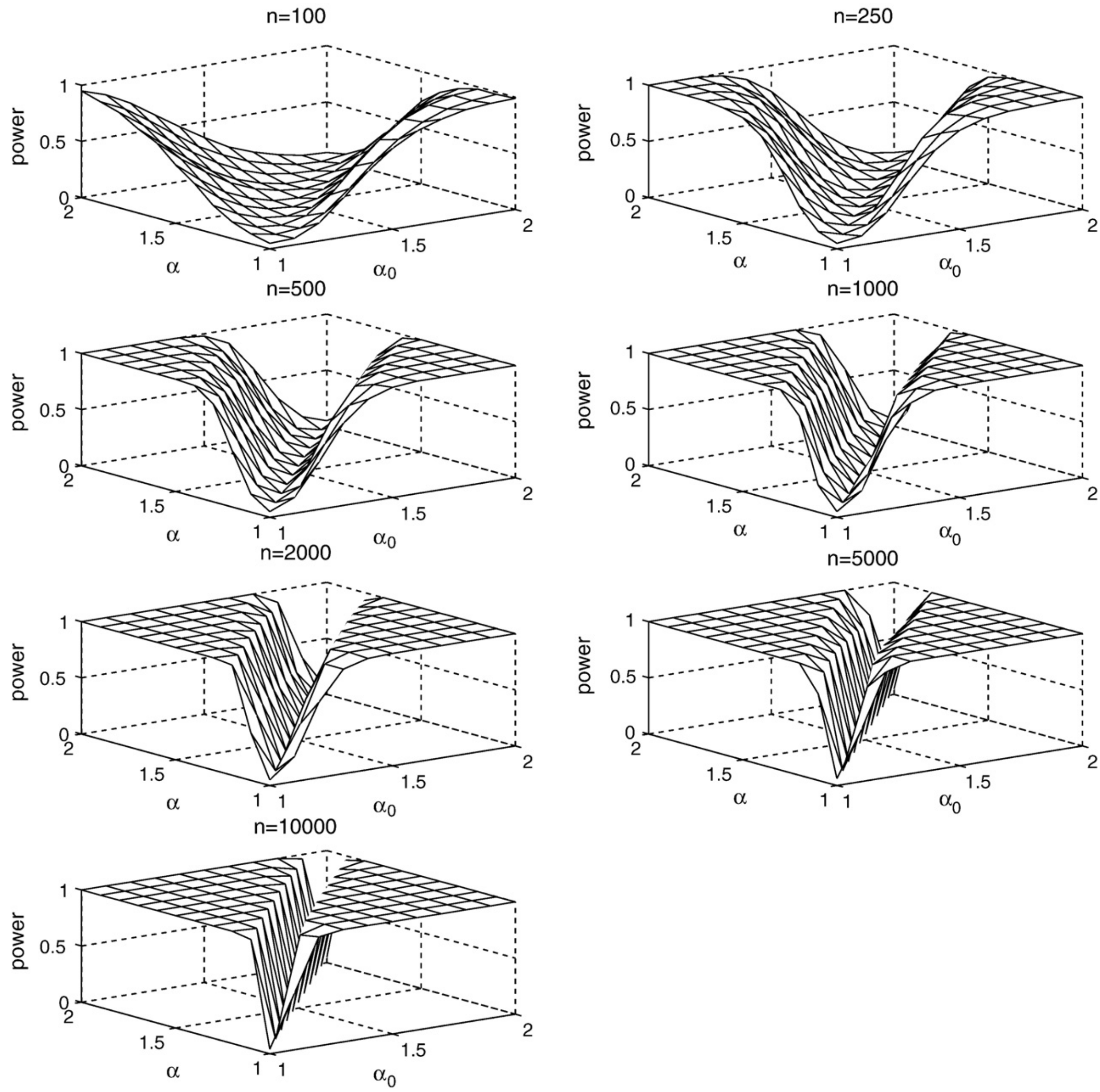

Fig. 2. Power function for selected values of α and *n*.

4. For every value of α_0 in the grid, generate *N* Monte Carlo samples of size *n* and calculate the test statistic in Eq. (13), and sort. Typically, $N=99$ or 999 (for the convenience of calculating *p*-values) and $n=$ the size of empirical data (for the sake of purpose).
5. Compute *p*-values under all possible null hypotheses as described in the previous subsection.
6. Take the $\hat{\alpha}(\alpha_0)$ as the estimate of α at which the $(1-p)$ value has its minimum (often zero). This is our MC estimate for stability parameter of the $S_\alpha S$ distribution, denoted $\hat{\alpha}_{MC}$.
7. Take the $\hat{\alpha}_l$ and $\hat{\alpha}_r$ as the left and right limit of the η% confidence interval at which the *p*-value is $(1-\eta)/100$, where $\hat{\alpha}_l < \hat{\alpha}_r$. This is now the exact confidence interval for $\hat{\alpha}_{MC}$ in step 6.

4. A simulation study: size and power

4.1. Size distortion of the asymptotic confidence interval

A strong advantage of the MC method is that it provides exact confidence intervals for finite samples. In practice, the asymptotic normality as given in Eq. (2) is usually used for finite samples. In order to see the size distortion of the asymptotic test,

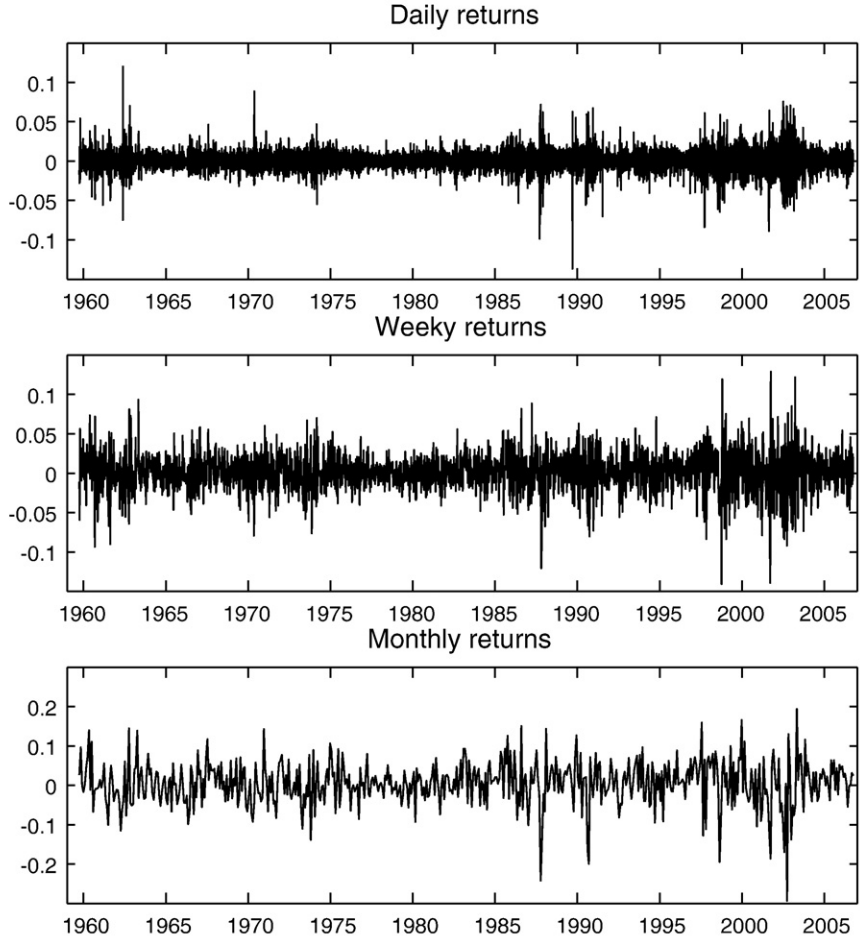

Fig. 3. DAX returns in different frequencies.

we perform a simulation study. The simulation is designed as $\alpha = 1, 1.25, 1.5, 1.75, 2$[16], $\beta = 0$, $\mu = 0$ and $\sigma = 1$with a sample size of n = 100, 250, 500, 1000, 5000, 10,000. For each combination, 10,000 replications were made. We use the usual two-tailed Hill estimator in Eq. (3) relocating the sample by the estimated sample mean and by the estimated sample median. For our choice of k, we consider three cases: i) choose randomly α between 1 and 2 and determine k using the table in footnote 14, ii) a fixed ratio of 20% proposed by DuMouchel (1983) and iii) assume a known α and determine k using the table in footnote 14. The result of the simulation is summarized in Table 1, where the numbers in the table are percentage points of rejection. Although we only report for the 5% size, the other usual confidence levels show very similar results.

Some comments on Table 1 are in order. (1) the median relocation generally shows a better result for all α and n selected. Even in the known case, the median relocation works better when α is small, because of the large fluctuation of the sample mean for small α, as discussed in Subsection 2.3. (2) In a comparison of the three different choice methods, the ratio of over-rejection is the worst for the fixed k, followed by the random choice, and is, as expected, best for the known case. The fixed percentage of 20% is, as can be seen in the table in Footnote 14, only appropriate for a small α, say 1.1, or even smaller α when n is very small. Because of that, the fixed percentage method is not appropriate in almost all cases considered, while the random choice can be from time to time a proper one, just *randomly*. (3) Contrary to the general intuition, the general aggravation of the size distortion when n becomes large, at least for the random and the fixed method, can be explained because the large outliers for a small α which occur seldom have a better chance of being included in the sample when n is sufficiently large. (4) The size distortion is more dramatic as α approaches 2, because the thinner the tail becomes, the more difficult to estimate it precisely, and it is well known that the Hill-type estimates have a bias as α approaches 2. This phenomenon has a severe consequence in empirical studies, because most empirical data have an α between 1.5 and 2 (see Kim et al., 1997) meaning that a careful choice of k, or rather, using a method without assuming known k, is of importance for empirical work. The results of the asymptotic test show that it suffers generally from size distortion, even for the known case if α is small. Note that the sizes from our MC method are, by construction, exact.

[16] If necessary, we choose the same k/n ratio $\alpha = 1.9$ for $\alpha = 2$.

4.2. Power function of the Monte Carlo test

The theoretical size and power of the MC test is considered in Dufour (2006). Although the discrepancy of the correct size and the superior power of the MC test over the conventional test go to zero as the sample size approaches infinity, the behavior of the power function for the finite sample is usually of interest.

To check the power of our MC test, we perform a simulation study by drawing from symmetric α-stable pseudo-*r.v.*s relocated by the median. As pseudo-empirical data we use the same α-stable random sample generated earlier, and test H_0: $\alpha = \alpha_0$, where α_0 is assumed to take on values from 1.0 to 2.0 in steps of 0.1. Sample sizes of $n = 100$, 250, 500, 1000, 2000, 5000 and 10,000 are selected, and the number of replications is 10,000. To demonstrate the power function, we select a usual significance level of 95%. Fig. 2 shows the power functions for the selected α, n and percentage points as described above.

As expected, the power converges to the corresponding ideal value for each given significance level as the sample size grows. A sample size of 2000 gives a rather satisfactory power. A large loss of power can be observed for extremely small sample sizes.

5. Empirical applications

To illustrate the use of the Monte Carlo method in practice, we employ the German stock index from its beginning (1 October 1959) to 30 September 2006 (47 years). For a deeper look, we consider them in three different frequencies, namely daily (11,796 observations), weekly (2453 observations) and monthly (564 observations), where the observations for the weekly and monthly data are those of the end of the period, i.e. the Friday values for the weekly data and the value at end of the each month for the monthly data. Fig. 3 shows the empirical data.

The volatility cluster looks, to a large extent, similar in three different frequencies. However, a careful look reveals that many of the single outliers in the daily returns are no longer seen in the weekly returns and *vice versa*. The same also applies between the weekly returns and the monthly ones. This is because the weekly and/or monthly data do not come from a moving average of the daily data. (Even if the low frequency data come from a moving average of a higher frequency data, the two dynamics are not necessarily the same or very similar.)

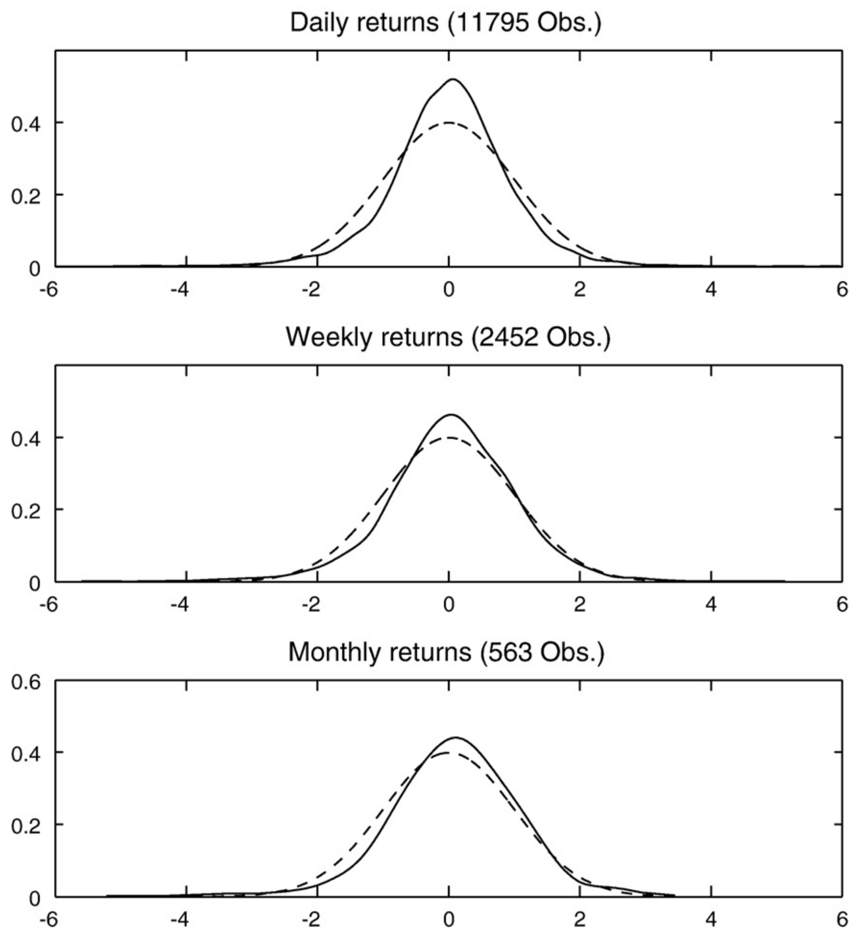

Fig. 4. Empirical densities of different frequencies.

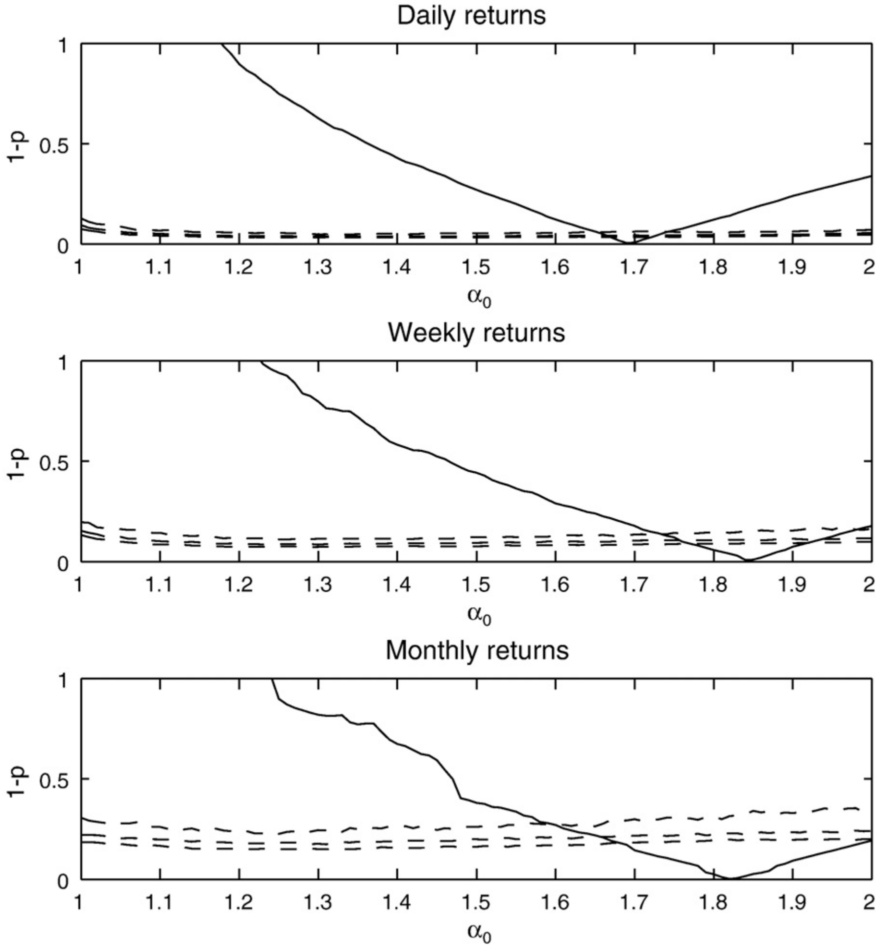

Fig. 5. MC estimates and their exact confidence intervals.

Fig. 4 shows the empirical densities of the three time series (solid line) compared with the normal density (dashed line). Each of the empirical densities appears excessively peaked around the mean and, at the same time, the tails are thicker than those of the normal density, which are the typical features of α-stable densities. This phenomenon is the most striking in the daily data, namely high-frequency data, as usually observed and reported in the literature. The kurtosis[17] for the three types of data is 10.67 for the daily returns, 5.36 for the weekly returns and 5.60 for the monthly returns.

Next, we estimate the stability parameter and the confidence interval of the three times series by means of our MC estimation and test procedure via the seven steps as described in Section 3.2. Fig. 5 illustrates the estimates and the corresponding confidence intervals, where the solid line gives $1-p$ values at given $H_0(\alpha_0)$: $\alpha=\alpha_0$ of the empirical data and the three dashed lines (from bottom to top) give simulated (one-tailed) quantiles of 90%, 95% and 99% for the estimate $\hat{\alpha}=\alpha_0$.[18]

The results of the estimates are numerically summarized in Table 2.

Some comments on the empirical results are in order. First, the changes in the probability at given $H_0(\alpha_0)$: $\alpha=\alpha_0$ shown in Fig. 5 become smoother as the sample size increases. For small sample sizes, the smoothness decreases. This is because (although the same unit and exponential *r.v.* are used in the transformation into α-stable *r.v.* for all α_0 selected) the number of k changes for different values of α. Second, the increase and decrease in the probability around $H_0(\alpha_0)$: $\alpha=\alpha_0$ are not symmetric. This means that the exact confidence intervals from the MC test for finite samples can be asymmetric, which is another advantage of our exact confidence interval for finite samples. Note that the asymptotic distribution of the Hill estimate, namely the normal distribution, is symmetric for all sample sizes and all quantiles. Also, there is no statistical background that the distribution of estimates for finite samples must be symmetric. Third, the estimated stability parameter for the daily data is 1.69, but it increases to 1.84 when the observation frequency decreases to a weekly interval, where both of the two estimates are via our exact confidence intervals highly significant for $\alpha<2$. Theoretically, the stability parameter should not change with observation frequency if the data are i.i.d.

[17] Under the assumption of α-stable distributions with $\alpha<2$, there exists no fourth moment, but the sample kurtosis exists numerically for any data set.

[18] Consequently, they are the 80%, 90% and 98% confidence interval. But, because we have in mind the null hypothesis as $\alpha=2$, the right side (one-tailed test) of the interval is usually the side that is of interest.

Table 2
Estimated stability parameters and their exact confidence intervals.[a]

Quantile / Data	0.5%	2.5%	5%	$\hat{\alpha}_{MC}$	95%	97.5%	99.5%
Daily	1.66	1.67	1.67	**1.69**	1.72	1.72	1.74
	(1.65)	(1.66)	(1.66)		(1.72)	(1.72)	(1.73)
Weekly	1.74	1.76	1.78	**1.84**	1.91	1.93	1.98
	(1.74)	(1.77)	(1.78)		(1.90)	(1.91)	(1.94)
Monthly	1.61	1.66	1.69	**1.82**	2	2	2
	(1.62)	(1.67)	(1.69)		(1.95)	(1.97)	(2)

[a] The values in parentheses are corresponding confidence intervals based on the asymptotic test given in Eq. (2).

However, most of the empirical financial data do not conform to this condition and are highly correlated in the second moment, as is well known. Because of this dependence in the second moment, the stability parameter from different frequencies can vary in finite samples.[19] The changes in frequency from weekly to monthly, however, show no further increase in $\hat{\alpha}$. Fourth, the difference of the confidence intervals, especially in the right side, between the asymptotic distribution and the exact method become larger as the sample size decreases from the daily data (11,795 observations) to the monthly data (563 observations). This leads us to one of the main results from the empirical application, which should be emphasized with respect to our exact confidence intervals. The hypothesis of $\alpha = 2$ for the monthly data cannot be rejected by the exact confidence intervals; however, the asymptotic test still enables the hypothesis to be rejected at a significance level of 97.5%. This means that results for or against $\alpha < 2$ for finite samples, especially small sample sizes, can be misleading if they are conducted by the asymptotic normal distributions. As already discussed, for large samples (here, daily data with a size of 11,795) the usual confidence intervals from the exact test method and the normal distribution are almost the same.

In the second part of our empirical applications, we apply the test for constancy of tail thickness suggested by Quintos et al. (2001) to the same weekly return data as before. The aim of this part is again to demonstrate that empirical results can crucially depend on the choice of k. In other words, because our MC procedure automatically uses an optimal k, it is able to provide correct results regarding the underlying tail thickness parameter.

Quintos et al. (2001) propose three tests for the constancy of tail thickness, namely a recursive, rolling and sequential test. We only employ the rolling test,[20] in which we can expect the largest fluctuation in the estimates for the tail thickness. This is because the Hill estimator is conditional on the largest k observations, which means that, for the recursive and the sequential test, the outlier behavior that appears in the initial sample remains in the selection of the k largest observations in the latter part of the sample, whereas, for the rolling test, the k largest observations can more easily change from one subsample to the next subsample. Under the assumption of an optimal k, they formulate a statistic for the rolling test as

$$V_n(i) = \frac{n_i k(n_i)}{n}\left(\frac{\hat{\alpha}_i}{\hat{\alpha}_n} - 1\right), \tag{22}$$

where n_i, $i = 1,2,\ldots,N$ is the size of N subsamples; $k(n_i)$ is the number of the observations on the distributional tail in a subsample i; n is the size of the whole sample; $\hat{\alpha}_i$ is the estimate of α for the i-th sample; and $\hat{\alpha}_n$ is the estimate of α for the whole sample. The test statistic for the rolling test focuses on the maximum of the statistic (22) for $k(n_i)/n \in \mathbb{R}_\pi := [\pi, 1-\pi]$, where π represents some small value (usually 0.1) commonly used in the construction of structure constancy tests–see Andrews (1993)–and converges to a non-standard distribution and the critical values from the non-standard distribution are tabulated in Quintos et al. (2001).

Specifically, we choose two rolling subsamples: a 10-year and a 20-year subsample, i.e. our first subsample for the 10-year (20-year) subsample contains the weekly returns from October 1959 to September 1969 (September 1979), and our last subsample contains the weekly returns from October 1996 (October 1986) to September 2006. Therefore, the number of subsamples is 37 and 27 for the 10-year and 20-year subsample, respectively. The size of subsamples is either 523 (27 times) or 522 (10 times) for the 10-year subsample and (1044) (14 times) or 1043 (13 times) for the 20-year subsample. For calculation of the test statistic, therefore, we take 523 and 1044 as the size of a rolling sample for the 10-year subsample and the 20-year subsample. Figs. 6 and 7 show empirical test statistics based on the (direct) Hill estimation and our MC estimation, where Fig. 6 (7) corresponds to the 10-years (20-years) subsample.

For both graphs, the mountain-shaped solid curve shows the empirical test statistic based on the 4001 (direct) Hill estimates for $k/n = [0.1:0.001:0.5]$ as the ratio of tail observations for both subsamples and the whole sample and the dashed line based on

[19] Resnick and Stâ*ricâ* (1998) show that consistency of the Hill estimator is given not only under independence (as proved also in Hsing, 1991) but also under quite general forms of dependence, including an ARCH-type structure. See also Quintos et al. (2001) for this topic.
[20] The recursive and sequential tests show mainly the same result.

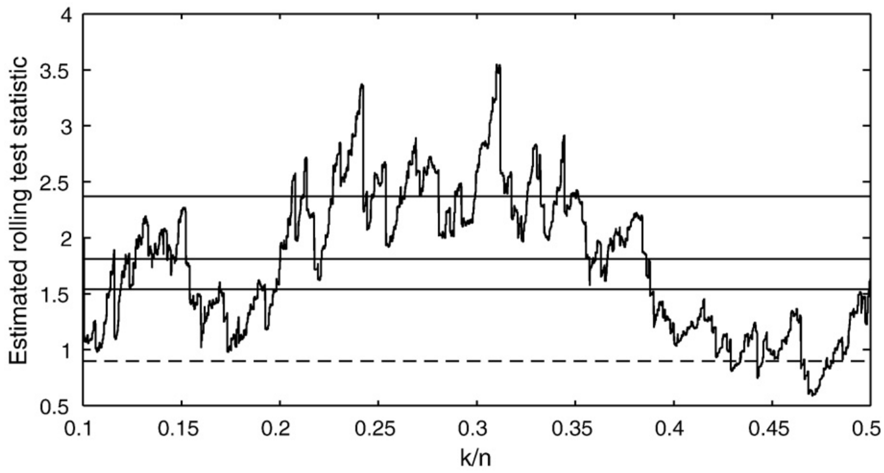

Fig. 6. Rolling test for 10-year subsamples.

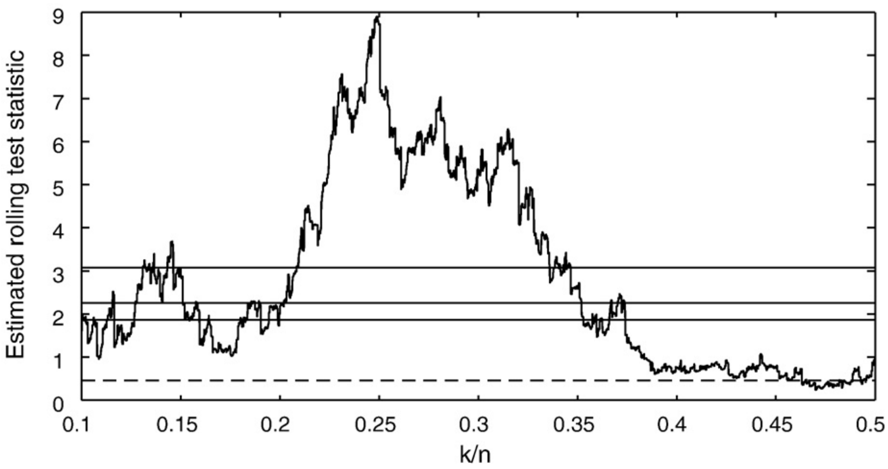

Fig. 7. Rolling test for 20-year subsamples.

the MC estimates. The three straight solid lines show 99%, 95% and 90% critical values of 2.37, 1.81 and 1.54 for the 10-year subsample in Fig. 6 and 3.075, 2.25 and 1.865 for the 20-year subsample in Fig. 7.[21]

Figs. 6 and 7 show that, according our MC estimate, the stability of tail thickness of the DAX weekly returns seems to be constant during the whole period for both subsample lengths. However, the results of the Hill estimate for both cases depend crucially on k/n. If k/n is chosen to be smaller than 0.3862 and 0.3740 (except some very small k/n values for both cases) for the 10-year and 20-year subsamples, respectively, we would not accept the null hypothesis of constancy of the tail thickness at a significance level of 95%.

The empirical applications demonstrate that empirical conclusions about the hypothesis test for $\alpha < 2$ for finite samples and/or for constancy of tail thickness depend crucially on the choice of the number of observations on the distributional tail and the confidence intervals used. Because our MC method automatically uses the optimal number of observations on the distributional tail and provides exact confidence intervals, it is of practical use for finite samples with unknown α.

6. Summary

In this paper we have considered an exact test and estimation method for stability parameter of α-stable distributions using the MC technique. Specifically, we have employed the Hill estimator for constructing the MC test statistic. Our MC estimation and

[21] The critical value is calculated by a linear interpolation from the table in Quintos et al. (2001) p. 662.

test procedure improve on the Hill estimation in two ways: our MC estimation does not need to assume that the optimal number of observations on the distributional tail (or, rather the underlying α) is known, and our MC test provides exact confidence intervals for finite samples.

The empirical applications show the sensitive dependency of empirical conclusions on the choice of observations in the tail and the confidence intervals used, and emphasize the practical meaning of our MC estimation and test procedure.

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Appendix A

Table A
Root mean square error of Hill estimate with different relocations.

α	Relocation Sample size	True mean	Sample mean	Sample median
1.01	100	0.1660	0.3338	0.1654
	250	0.1235	0.2890	0.1234
	500	0.0953	0.2697	0.0952
	1000	0.0727	0.2711	0.0727
	5000	0.0442	0.1763	0.0442
1.25	100	0.1616	0.2488	0.1625
	250	0.1063	0.2090	0.1063
	500	0.0762	0.1845	0.0764
	1000	0.0546	0.1771	0.0545
	5000	0.0253	0.1263	0.0253
1.5	100	0.1808	0.1958	0.1813
	250	0.1126	0.1306	0.1135
	500	0.0806	0.0949	0.0809
	1000	0.0579	0.0761	0.0578
	5000	0.0263	0.0368	0.0262
1.75	100	0.2048	0.2053	0.2068
	250	0.1291	0.1306	0.1299
	500	0.0909	0.0912	0.0903
	1000	0.0640	0.0655	0.0641
	5000	0.0290	0.0294	0.0290
1.95	100	0.2258	0.2261	0.2259
	250	0.1421	0.1414	0.1419
	500	0.0985	0.0987	0.0992
	1000	0.0696	0.0698	0.0699
	5000	0.0318	0.0317	0.0317
2.0	100	0.2303	0.2292	0.2313
	250	0.1463	0.1452	0.1455
	500	0.1004	0.1010	0.1016
	1000	0.0716	0.0714	0.0711
	5000	0.0330	0.0328	0.0327

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