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Exact tests for structural change in first-order dynamic models

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Abstract

Several finite-sample tests of parameter constancy against the presence of structural change are proposed for a linear regression model with one lagged dependent variable and independent normal disturbances. The procedures derived include analysis-of-covariance, CUSUM, CUSUM-of-squares, and predictive tests. The approach used to obtain the tests involves the application of three techniques: derivation of an exact confidence set for the autoregressive parameter (based on using an appropriately extended regression), a union-intersection technique, and (when required) randomization. The tests proposed are illustrated with some artificial data and applied to a dynamic trend model of gross private domestic investment in the U.S.

Key words: Finite-sample tests; Exact inference; First-order autoregressive model; Randomization; Structural change
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1. Introduction

Test and confidence set procedures for dynamic regression models are typically based on large-sample approximations whose reliability can be quite poor; see, for instance, Nankervis and Savin (1987) and Kiviet and Phillips (1993). A major problem in this context comes from the fact that usual test statistics have an indeterminate null distribution, since the latter depends on the values of unknown nuisance parameters. This is true, for example, even for tests of linear restrictions in a linear dynamic regression with one lagged dependent variable, a few exogenous regressors and Gaussian errors. Consequently, it is not surprising that to date finite-sample tests against the presence of structural change were not available for such models.

In this paper, we exploit recent results from Dufour (1989, 1990), Kiviet and Phillips (1990, 1992), and Dufour and Kiviet (1993) to derive exact finite-sample structural change tests in a basic linear dynamic model. Specifically, the model we shall consider is

$$y_t = \lambda y_{t-1} + x_t'\beta + u_t, \quad u_t \overset{\text{ind}}{\sim} \mathrm{N}(0, \sigma^2), \quad t = 1, \dots, T, \tag{1.1}$$

where y_t is the dependent variable (at time t), x_t is a $k \times 1$ vector of fixed (or strongly exogenous) regressors, $u_1, \dots, u_T$ are mutually independent random disturbances following a $\mathrm{N}(0, \sigma^2)$ distribution, and y_0 is either fixed or random but independent of $u_1, \dots, u_T$; the parameters λ, β, and σ^2 are unknown, and $\lambda \in \mathscr{D}_\lambda$, where $\mathscr{D}_\lambda \subseteq \mathbb{R}$ is a nonempty set of admissible values for λ. Depending on the context, the set $\mathscr{D}_\lambda$ may be $\mathbb{R}$ itself, the open interval $(-1, 1)$, the closed interval $[-1, 1]$, or any other appropriate subset of $\mathbb{R}$. It will be convenient to write model (1.1) in matrix form:

$$y = \lambda y_{-1} + X\beta + u, \quad u \sim \mathrm{N}(0, \sigma^2 I_T), \tag{1.2}$$

where $y = (y_1, y_2, \dots, y_T)'$, $y_{-1} = (y_0, y_1, \dots, y_{T-1})'$, $X = [x_1, x_2, \dots, x_T]'$, and $u = (u_1, u_2, \dots, u_T)'$.

In order to derive exact structural change tests for model (1.2), we will consider in turn two distinct cases, depending on whether the autoregressive parameter λ is assumed to be constant under the alternative or not. In the first case, we study two categories of tests: (1) generalizations of standard analysis-of-covariance (AOC) tests in static linear regressions (Kullback and Rosenblatt, 1957; Chow, 1960; Dufour, 1982a), which are built against alternatives where β may change at m known breakpoints ($m \geqslant 2$), and (2) generalizations of the CUSUM and CUSUM-of-squares tests (proposed by Brown, Durbin, and Evans, 1975), which are built against more general alternatives. In the second case, we study again two types of tests: (a) predictive tests which generalize those proposed by Chow (1960) and Dufour (1980, 1982c) for static linear regressions, and (b) AOC-type tests against alternatives where λ may change at a known breakpoint.

The tests suggested are obtained by adapting to structural change problems the exact inference procedures proposed in Dufour and Kiviet (1993) for model (1.2), which are themselves generalizations of the approach suggested in Dufour (1990) for making exact inference in a linear regression with AR(1) errors. The basic building block in our approach is the possibility of finding exact confidence sets for the coefficient λ, from the full sample or from subsamples. In particular, this can be done by applying least squares to an extended regression, as set out in Kiviet and Phillips (1990, 1992) where two procedures are suggested. These confidence sets are similar and have the desired *size*, i.e., for any confidence level $1 - \alpha$, with $0 \leqslant \alpha < 1$, the probability that λ be contained in the set is precisely $1 - \alpha$. The confidence set for λ is then combined with 'conditional' tests on the other coefficients (which assume λ to be known) to obtain (unconditional) 'generalized bounds tests'. These bounds tests are exact in the sense that the probability of rejecting the null hypothesis does not exceed the chosen level. Concerning the definition of the *level* of a test or confidence set (as opposed to its *size*), the reader may consult Lehmann (1986, Sec. 3.1, p. 69). Finally, when the distribution of a test statistic (for given $\lambda = \lambda_0$) is not well tabulated or is not analytically tractable, we suggest using a 'randomized' (or Monte Carlo) version of the test which remains exact irrespective of the number N of replications and becomes equivalent to the original nonrandomized test as $N \to \infty$.

In Section 2, we give two lemmas which will be useful in later developments: the first one provides a simple way of deriving exact generalized bounds tests, while the second one shows how a genuinely exact test can be obtained when the distribution of a (similar) test statistic is simulated with an arbitrary number of replications. In Section 3, we show how an exact confidence set for λ can be built. Sections 4 and 5 describe the analysis-of-covariance and CUSUM tests, against alternatives where λ is assumed to be constant. Section 6 proposes predictive tests and analysis-of-covariance tests against alternatives where λ may change. In Section 7, the various procedures are illustrated with both artificial data and a dynamic trend model for U.S. real gross domestic private investment in nonresidential structures. Section 8 concludes.

2. Nuisance parameters, bounds procedures, and randomization

The most basic difficulty one meets in deriving finite-sample inference procedures for a dynamic model of the form (1.2) comes from the fact that usual inference procedures (such as Wald, likelihood ratio, or Lagrange multiplier tests) involve nuisance parameters: the null distribution of a test statistic for $\lambda = \lambda_0$ typically depends on the the unknown parameter β (and possibly σ^2), while the null distribution of a test statistic for some restriction on β depends on the unknown value of λ. Even though this dependence may tend to disappear in large samples, it does not in finite samples. In particular, the dependence on λ appears to be especially difficult to alleviate.

Recently, however, Kiviet and Phillips (1990, 1992) showed that exact tests and confidence sets for λ can be derived by extending appropriately the matrix X of fixed regressors in (1.2). The purpose of using an extended regressor matrix is precisely to eliminate nuisance parameters. We will describe in the next section how an exact confidence set for λ can be obtained.

Given an exact confidence set for λ, it is possible to obtain finite-sample tests and confidence sets for the vector β (or subvectors of it) by using the union-intersection approach proposed in Dufour (1990) for linear regressions with AR(1) errors; see Dufour and Kiviet (1993). In the present paper, we apply this approach to obtain exact structural change tests in the context of model (1.2). For that purpose, we will repeatedly exploit the following lemma, which generalizes some of the basic ideas used in Dufour (1990) and Dufour and Kiviet (1993).

Lemma 1. Let y be a random vector whose distribution depends on a parameter $\gamma \in A$, where A is a nonempty subset of $\mathbb{R}^p$, let $Q(y;\gamma)$ be a real-valued statistic and let $C(y) \subseteq A$ be a confidence set for γ. Define also

$$Q_L(y) = \inf\{Q(y;\gamma_0): \gamma_0 \in C(y)\}, \tag{2.1}$$

$$Q_U(y) = \sup\{(y;\gamma_0): \gamma_0 \in C(y)\}, \tag{2.2}$$

where we set $Q_L(y) = -\infty$ and $Q_U(y) = +\infty$ when $C(y)$ is empty. Then, for any $x \in \mathbb{R}$ and for any $\gamma_1 \in A$, we have the two following inequalities:

$$P[Q_L(y) \geqslant x] \leqslant P[Q(y;\gamma_1) \geqslant x] + P[\gamma_1 \notin C(y)], \tag{2.3}$$

$$P[Q_U(y) \leqslant x] \leqslant P[Q(y;\gamma_1) \leqslant x] + P[\gamma_1 \notin C(y)]. \tag{2.4}$$

Proof. By definition

$$\gamma_1 \in C(y) \Rightarrow Q_L(y) \leqslant Q(y;\gamma_1) \leqslant Q_U(y),$$

where the 'event' $\gamma_1 \in \emptyset$ is interpreted as an 'impossible event' having probability zero, while $\gamma_1 \notin \emptyset$ is a 'sure event' having probability 1 ($\emptyset$ is the empty set). Then

$$\begin{aligned} P[Q_L(y) \geqslant x] &= P[Q_L(y) \geqslant x \text{ and } \gamma_1 \in C(y)] + P[Q_L(y) \geqslant x \text{ and } \gamma_1 \notin C(y)] \\ &\leqslant P[Q_L(y) \geqslant x \text{ and } \gamma_1 \in C(y)] + P[\gamma_1 \notin C(y)] \\ &\leqslant P[Q(y;\gamma_1) \geqslant x] + P[\gamma_1 \notin C(y)], \\ P[Q_U(y) \leqslant x] &= P[Q_U(y) \leqslant x \text{ and } \gamma_1 \in C(y)] + P[Q_U(y) \leqslant x \text{ and } \gamma_1 \notin C(y)] \\ &\leqslant P[Q_U(y) \leqslant x \text{ and } \gamma_1 \in C(y)] + P[\gamma_1 \notin C(y)] \\ &\leqslant P[Q(y;\gamma_1) \leqslant x] + P[\gamma_1 \notin C(y)]. \quad \text{Q.E.D.} \end{aligned}$$

This lemma will be applied as follows. Consider the case where

(i) $\gamma_1 = \gamma$, where γ is the true value of the parameter vector,

(ii) $Q(y; \gamma)$ has a unique known distribution,

(iii) $P[\gamma \in C(y)] \geqslant 1 - \alpha_1$, with $0 \leqslant \alpha_1 < 1$,

and let $c(\alpha)$ be a point such that

$$P[Q(y; \gamma) \geqslant c(\alpha)] = \alpha, \tag{2.5}$$

where $0 \leqslant \alpha \leqslant 1$. Then

$$P[Q_L(y) \geqslant c(\alpha_2)] \leqslant \alpha_2 + \alpha_1, \tag{2.6}$$

$$P[Q_U(y) \leqslant c(\alpha'_2)] \leqslant 1 - \alpha'_2 + \alpha_1 = 1 - (\alpha'_2 - \alpha_1). \tag{2.7}$$

Let us now interpret $Q(y; \gamma_0) \geqslant c(\alpha)$ as a critical region for testing a hypothesis H_0, which usually concerns parameters other than γ and has size α when $\gamma = \gamma_0$. Taking

$$0 \leqslant \alpha_1 < \alpha < 1, \quad \alpha_2 = \alpha - \alpha_1, \quad \alpha'_2 = \alpha + \alpha_1 < 1, \tag{2.8}$$

we then have

$$P[Q_L(y) \geqslant c(\alpha_2)] \leqslant \alpha, \tag{2.9}$$

$$P[Q_U(y) \leqslant c(\alpha'_2)] \leqslant 1 - \alpha. \tag{2.10}$$

This suggests the following (unconditional) generalized bounds test with level α for H_0:

$$\begin{aligned} &\text{reject } H_0 \text{ when } Q_L(y) \geqslant c(\alpha_2), \\ &\text{accept } H_0 \text{ when } Q_U(y) < c(\alpha'_2), \\ &\text{consider the test inconclusive otherwise.} \end{aligned} \tag{2.11}$$

For further discussion of such procedures, see Dufour (1989, 1990).

A second problem one meets in deriving exact tests in the context of model (1.2) comes from the fact that the test statistics used may have fairly complex forms, even though their distribution under the null hypothesis does not depend on nuisance parameters. In such cases, the analytical evaluation of the distribution of a test statistic can be quite difficult, but the same distribution may be easy to simulate by Monte Carlo methods. In such cases, it is possible to consider a 'randomized' version of the test that can have any desired level. Even though the basic property used to derive such tests is well-known (see Dwass, 1957; Barnard, 1963; Birnbaum, 1974; Edgington, 1980; Foutz, 1980; Jöckel, 1986), we state it here in a lemma for future reference (for a proof, see Dufour and Kiviet, 1993).

Lemma 2. Let Z_j, $j = 1, \ldots, N$, be independent and identically distributed (i.i.d.) real random variables with a continuous distribution, and let R_j be the rank of Z_j when $Z_1, \ldots, Z_N$ are ranked in nondecreasing order ($j = 1, \ldots, N$), i.e.,

$$R_j = \sum_{i=1}^{N} U(Z_j - Z_i), \tag{2.12}$$

where $U(x) = 1$ if $x \geqslant 0$ and $U(x) = 0$ if $x < 0$. Then, for $j = 1, \ldots, N$,

$$\begin{aligned} P[R_j/N \geqslant x] &= 1, && \text{if} \quad x \leqslant 0, \\ &= (I[(1-x)N] + 1)/N, && \text{if} \quad 0 < x \leqslant 1, \\ &= 0, && \text{if} \quad x > 1, \end{aligned} \tag{2.13}$$

where $I[x]$ is the largest integer less than or equal to x.

This lemma will be used in the following way. Let Z_N be the value of a test statistic computed from an observed sample, let $Z_1, \ldots, Z_{N-1}$ be i.i.d. random variables which are distributed like Z_N under H_0, and let $0 < \alpha < 1$. Then, by selecting $c_N(\alpha)$ to be a positive real number such that

$$I[(1 - c_N(\alpha))N] + 1 \leqslant N\alpha, \tag{2.14}$$

the critical region $R_N/N \geqslant c_N(\alpha)$ has size not larger than α. Thus, R_N/N may be viewed as a modified test statistic for H_0. From (2.13), it is easy to see that the critical point

$$c_N(\alpha) = 1 - \frac{I[N\alpha]}{N} + \frac{1}{N} \tag{2.15}$$

yields a test of size $I[N\alpha]/N$, so that $\alpha - (1/N) \leqslant P[R_N/N \geqslant c_N(\alpha)] \leqslant \alpha$, and thus provides the desired result; in particular, when $N\alpha$ is an integer, we get $P[R_N/N \geqslant c_N(\alpha)] = \alpha$ by taking $c_N(\alpha) = 1 - \alpha + (1/N)$. With $c_N(\alpha)$ defined as in (2.15), the critical region $R_N/N \geqslant c_N(\alpha)$ can be rewritten in the intuitively attractive form $p_N \leqslant I[N\alpha]/N$, where

$$p_N = 1 - \frac{R_N}{N} + \frac{1}{N} \tag{2.16}$$

can be interpreted as a 'randomized' or 'Monte Carlo' p-value. Because the function $I[\cdot]$ is discrete, several values of $c_N(\alpha)$ may yield the same critical region: all critical points c such that $\{I[(1-c)N] + 1\}/N = \alpha$ correspond to the same test with size α. The only levels for which the equality in (2.14) can hold exactly are j/N, $j = 0, 1, \ldots, N-1$, but it is easy to find a critical point $c_N(\alpha')$ such that we have both $P[R_N/N \geqslant c_N(\alpha')] \leqslant \alpha$ and $|\alpha' - \alpha| \leqslant 1/N$: clearly, by taking N sufficiently large, the difference $|\alpha' - \alpha|$ can be made arbitrarily small, and, if αN is an integer, we can have $P[R_N/N \geqslant c_N(\alpha)] = \alpha$. Note also that the test $R_N/N \geqslant c_N(\alpha)$ is not equivalent to the nonrandomized test $Z_N \geqslant c(\alpha)$ where $P[Z_N \geqslant c(\alpha)] = \alpha$. But, as $N \to \infty$, the two tests become equivalent under weak regularity conditions. For further discussion, see Birnbaum (1974), Dwass (1957), Foutz (1980), and Jöckel (1986); for applications of Monte Carlo tests in time series contexts, see Dufour and Hallin (1987, p. 426) and Theil and Shonkwiler (1986).

3. Exact confidence sets for λ

To apply Lemma 1, we will need an exact confidence set for λ. For this purpose, we shall use the approach developed in Kiviet and Phillips (1990, 1992), which is based on deriving first exact similar tests for the hypothesis $\lambda = \lambda_0$. Dufour (1990) also proposed a related approach to obtain an exact confidence set for the autoregressive parameter in a linear regression with AR(1) errors from an exact test.

Kiviet and Phillips (1990, 1992) give two procedures for testing the null hypothesis $\lambda = \lambda_0$ exactly in model (1.2); a third one is given in Dufour and Kiviet (1993). The null distributions of these test statistics are free of the nuisance parameters β and σ; moreover, they are invariant with respect to the value and the (stochastic) nature of y_0. However, the null distribution of these tests does depend on both λ_0 and X, and so it is not feasible to produce general tables of exact critical values. The actual application of these tests requires considerable computational efforts, but by adapting them in the form of simulation tests according to Lemma 2, these procedures are operational and relatively easy to execute. Exact confidence sets for λ can be constructed by 'inversion' of these test procedures for $\lambda = \lambda_0$. These three particular exact tests are based on straightforward least-squares results in a regression model which corresponds to (1.2) augmented by a number of redundant strongly exogenous regressors:

$$y = \lambda y_{-1} + X(\lambda_0)\beta_* + u, \tag{3.1}$$

where $X(\lambda_0)$ is a full column rank matrix whose columns span the same space as the space spanned by the columns of $[X \,\vdots\, \iota_T(\lambda_0) \,\vdots\, J_T(\lambda_0)X]$, and

$$\iota_T(\lambda) = \begin{bmatrix} 1 \\ \lambda \\ \lambda^2 \\ \vdots \\ \lambda^{T-1} \end{bmatrix}, \qquad J_T(\lambda) = \begin{bmatrix} 0 & \cdots & \cdots & \cdots & 0 \\ 1 & 0 & & & \vdots \\ \lambda & 1 & 0 & & \vdots \\ \lambda^2 & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ \lambda^{T-2} & \cdots & \lambda & 1 & 0 \end{bmatrix} \tag{3.2}$$

Extending a model by including particular redundant regressors in order to achieve test invariance has also been suggested in Dagenais and Dufour (1985) to test serial correlation with missing observations, and Dufour and King (1991, p. 125) with respect to testing hypotheses about the autocorrelation coefficient in a linear regression with AR(1) errors.

For the least-squares estimator of λ in (3.1), we have

$$\hat{\lambda}(\lambda_0) = (y'_{-1}M[X(\lambda_0)]y_{-1})^{-1}y'_{-1}M[X(\lambda_0)]y. \tag{3.3}$$

where $M[X(\lambda_0)] = I - X(\lambda_0)[X(\lambda_0)'X(\lambda_0)]^{-1}X(\lambda_0)'$. In order to test $\lambda = \lambda_0$ exactly, Kiviet and Phillips (1990, 1992) have suggested the following statistics:

$$c_\lambda^*(\lambda_0) = \hat{\lambda}(\lambda_0) - \lambda_0, \tag{3.4}$$

and the t-ratio

$$t_\lambda^*(\lambda_0) = c_\lambda^*(\lambda_0)/\hat{\sigma}[\hat{\lambda}(\lambda_0)], \tag{3.5}$$

where

$$\hat{\sigma}[\hat{\lambda}(\lambda_0)]^2 = \frac{1}{T - \operatorname{rank}[X(\lambda_0)]}\frac{[y - \hat{\lambda}(\lambda_0)y_{-1}]'M[X(\lambda_0)][y - \hat{\lambda}(\lambda_0)y_{-1}]}{y'_{-1}M[X(\lambda_0)]y_{-1}} \tag{3.6}$$

is the estimated variance of $\hat{\lambda}(\lambda_0)$. In addition to those, we shall also consider here two other statistics suggested in Dufour and Kiviet (1993) which have the nice feature of being derived as monotonic transformations of particular likelihood ratio statistics, namely,

$$\mathscr{L}_\lambda^*(\lambda_0) = \frac{S_0^*(\lambda_0)}{S_1^*(\lambda_0)} \tag{3.7}$$

and

$$\mathscr{L}_\lambda^{**}(\lambda_0) = \frac{S_0(\lambda_0)}{S_1^*(\lambda_0)}, \tag{3.8}$$

where

$$S_0^*(\lambda_0) = \min_{\beta_*}\, [y(\lambda_0) - X(\lambda_0)\beta_*]'[y(\lambda_0) - X(\lambda_0)\beta_*], \tag{3.9}$$

$$S_0(\lambda_0) = \min_{\beta}\, [y(\lambda_0) - X\beta]'[y(\lambda_0) - X\beta], \tag{3.10}$$

$$S_1^*(\lambda_0) = \min_{\lambda,\beta_*}\, [y - \lambda y_{-1} - X(\lambda_0)\beta_*]'[y - \lambda y_{-1} - X(\lambda_0)\beta_*], \tag{3.11}$$

with $y(\lambda_0) = y - \lambda_0 y_{-1}$. It is easily verified that

$$[t_\lambda^*(\lambda_0)]^2 = (T - \text{rank}[X(\lambda_0)])\,[\mathscr{L}_\lambda^*(\lambda_0) - 1].$$

Clearly, tests based on $\mathscr{L}_\lambda^{**}(\lambda_0)$ are likely to be more powerful than those based on $\mathscr{L}_\lambda^*(\lambda_0)$ because $\mathscr{L}_\lambda^{**}(\lambda_0)$ takes into account a wider set of restrictions entailed by $\lambda = \lambda_0$ in the extended model (3.1).

Under $\lambda = \lambda_0$, the three residual sums of squares defined in (3.9)–(3.11) reduce to

$$S_0^*(\lambda_0) = u'M[X(\lambda_0)]u, \tag{3.12}$$

$$S_0(\lambda_0) = u'M[X]u, \tag{3.13}$$

$$\begin{aligned} S_1^*(\lambda_0) &= u'M[y_{-1} \vdots X(\lambda_0)]u \\ &= u'M[X(\lambda_0)]u - \frac{\{u'M[X(\lambda_0)]\,J_T(\lambda_0)u\}^2}{u'J_T(\lambda_0)'M[X(\lambda_0)]\,J_T(\lambda_0)u}. \end{aligned} \tag{3.14}$$

For the derivation of the latter result, see Dufour and Kiviet (1993). It is important to note here that, under $\lambda = \lambda_0$, the coefficient vector β and the start-up value y_0 do not appear in the residual sum of squares. Further, the statistics $\mathscr{L}_\lambda^*(\lambda_0)$ and $\mathscr{L}_\lambda^{**}(\lambda_0)$ are then functions of ratios of quadratic forms in u, so that their null distributions do not depend on σ either.

An exact confidence set with level $1 - \alpha$ for λ can be built by 'inverting' either one of the above tests. To see how this is done, let us focus on the construction of a confidence set obtained (for example) from the $\mathscr{L}_\lambda^{**}(\lambda_0)$ test procedure and by making use of the simulation procedure set out in Lemma 2. We first generate $N - 1$ mutually independent $T \times 1$ vectors η_j, $j = 1, \ldots, N-1$, with $\eta_j \sim \mathrm{N}(0, I_T)$. For particular values $\lambda_0 \in \mathscr{D}_\lambda$, to be determined below, we can calculate

$$T_j(\lambda_0) = \frac{\eta_j' M[X]\eta_j}{\eta_j' M[X(\lambda_0)]\eta_j - \dfrac{\{\eta_j' M[X(\lambda_0)]\,J_T(\lambda_0)\eta_j\}^2}{\eta_j' J_T(\lambda_0)' M[X(\lambda_0)]\,J_T(\lambda_0)\eta_j}}, \tag{3.15}$$

for $j = 1, \ldots, N-1$, and set $T_N(\lambda_0) = \mathscr{L}_\lambda^{**}(\lambda_0)$ which is obtained by formula (3.8) from the actual data. If $c_N(\alpha)$ is defined as in (2.15) and $R_N(\lambda_0)$ is the rank of $T_N(\lambda_0)$ among $T_j(\lambda_0)$, $j = 1, \ldots, N$, then the set

$$\begin{aligned} C_\lambda(\alpha) &= \{\lambda_0 \mid \lambda_0 \in \mathscr{D}_\lambda \text{ and } R_N(\lambda_0)/N \geqslant c_N(\alpha)\} \\ &= \{\lambda_0 \mid \lambda_0 \in \mathscr{D}_\lambda \text{ and } R_N(\lambda_0) \geqslant N - I[N\alpha] + 1\} \end{aligned} \tag{3.16}$$

is a confidence set for λ with size $1 - (I[N\alpha]/N)$. In particular, if $N\alpha$ is an integer, the size of the confidence set is precisely $1 - \alpha$. The actual establishment of such a set requires iterative numerical procedures such as grid search, bi-section, etc. First, $T_j(\lambda_0)$, $j = 1, \ldots, N$, are calculated for a series of λ_0 values;

for each λ_0 value, the same $[\eta_1, \ldots, \eta_{N-1}]$ vectors are used and $R_N(\lambda_0)$ is determined. This provides an initial location of the confidence set. The set is not necessarily compact, but if it is, the confidence bounds are rather straightforwardly obtained after a series of further refinements by which one checks whether or not particular λ_0 values belong to the confidence set.

4. Analysis of covariance tests when λ is constant

In this section and the next one, we study the problem of testing the stability of model (1.2) against alternatives where λ and σ^2 are assumed constant. We will consider in turn two cases: (1) m possible breakpoints for changes in β are known (or assumed), and (2) the form of the structural changes is unknown. In the first case, we will propose generalizations of analysis-of-covariance tests similar to those proposed in Kullback and Rosenblatt (1957), Chow (1960), and Dufour (1982a). In particular, we will extend to model (1.2) the general analysis-of-covariance tests given in Dufour (1982a). In the second case, which we consider in the next section, we will propose analogues of the CUSUM tests suggested by Brown, Durbin, and Evans (1975).

Let us consider the following partitions of y, y_{-1}, X, and u defined in (1.2) into m subvectors or submatrices ($m \geqslant 2$):

$$y = \begin{bmatrix} y_{(1)} \\ y_{(2)} \\ \vdots \\ y_{(m)} \end{bmatrix}, \quad y_{-1} = \begin{bmatrix} y_{-1}^{(1)} \\ y_{-1}^{(2)} \\ \vdots \\ y_{-1}^{(m)} \end{bmatrix}, \quad X = \begin{bmatrix} X_{(1)} \\ X_{(2)} \\ \vdots \\ X_{(m)} \end{bmatrix}, \quad u = \begin{bmatrix} u_{(1)} \\ u_{(2)} \\ \vdots \\ u_{(m)} \end{bmatrix}, \tag{4.1}$$

where $y_{(i)}$, $y_{-1}^{(i)}$, and $u_{(i)}$ are $T_i \times 1$ vectors, $X_{(i)}$ is a $T_i \times k$ matrix, $T_i \geqslant 1$, and $r_i \equiv \operatorname{rank}(X_{(i)}) \geqslant 0$, $i = 1, \ldots, m$, and $T = \sum_{i=1}^{m} T_i \geqslant m \geqslant 2$. We *do not assume* here that the matrices $X_{(1)}, \ldots, X_{(m)}, X$ have full column ranks. As the alternative to model (1.2), we consider the extended model

$$y_{(i)} = \lambda y_{-1}^{(i)} + X_{(i)}\beta_i + u_{(i)}, \quad u_{(i)} \overset{\text{ind}}{\sim} \mathrm{N}(0, \sigma^2 I_{T_i}), \quad i = 1, \ldots, m, \tag{4.2}$$

where β_i is a $k \times 1$ vector of unknown coefficients and y_0 is independent of u. Thus, the ith regression in (4.2) represents a model of the form (1.2) for the observations $T_{i-1} + 1, \ldots, T_i$ ($i = 1, \ldots, m$, where $T_0 = 0$). $\beta_1, \ldots, \beta_m$ may differ but we will assume that λ and σ remain constant across the m subsamples. We want to test

$$\mathrm{H}_0\colon \quad \beta_1 = \beta_2 = \cdots = \beta_m. \tag{4.3}$$

To develop a test of H_0, it will be convenient to rewrite (4.2) in the more compact form:

$$y = \lambda y_{-1} + \bar{X}\bar{\beta} + u, \quad u \sim N(0, \sigma^2 I_T), \tag{4.4}$$

where $\bar{X}$ is a $T \times (mk)$ matrix and $\bar{\beta}$ is a $(mk) \times 1$ vector defined by

$$\bar{X} = \operatorname{diag}(X_{(i)}) = \begin{bmatrix} X_{(1)} & 0 & \cdots & 0 \\ 0 & X_{(2)} & \cdots & 0 \\ \vdots & \vdots & & \\ 0 & 0 & \cdots & X_{(m)} \end{bmatrix}, \quad \bar{\beta} = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_m \end{bmatrix}. \tag{4.5}$$

Let us now suppose that $\lambda = \lambda_0$, with λ_0 known. Then the model

$$y(\lambda_0) = \bar{X}\bar{\beta} + u, \tag{4.6}$$

where $y(\lambda_0) = y - \lambda_0 y_{-1}$ satisfies all the assumptions of the classical linear model (conditional on $\bar{X}$ and y_0), without possibly the assumption that $\bar{X}$ has full column rank. The problem of testing H_0 then has the form considered in Dufour (1982a), and the generalized Chow statistic for testing H_0 is

$$D(\lambda_0) = \frac{v}{v_0} \frac{S_0(\lambda_0) - S_1(\lambda_0)}{S_1(\lambda_0)}, \tag{4.7}$$

where $S_0(\lambda_0)$ and $S_1(\lambda_0)$ are the restricted and unrestricted minimum sum of squares, i.e.,

$$S_0(\lambda_0) = \min_{\beta} [y(\lambda_0) - X\beta]'[y(\lambda_0) - X\beta], \tag{4.8}$$

$$S_1(\lambda_0) = \min_{\bar{\beta}} [y(\lambda_0) - \bar{X}\bar{\beta}]'[y(\lambda_0) - \bar{X}\bar{\beta}], \tag{4.9}$$

while v and v_0 are the appropriate degrees of freedom:

$$v = \sum_{i=1}^{m} (T_i - r_i), \qquad v_0 = \sum_{i=1}^{m} r_i - r_0, \tag{4.10}$$

with $r_0 = \operatorname{rank}(X)$ and $r_i = \operatorname{rank}(X_{(i)})$, $i = 1, \ldots, m$. Under H_0 (with $\lambda = \lambda_0$), $D(\lambda_0)$ is distributed like $F(v_0, v)$, a Fisher random variable with (v_0, v) degrees of freedom, and the critical region $D(\lambda_0) \geqslant F(\alpha;\ v_0, v)$ has level α, where $P[F(v_0, v) \geqslant F(\alpha;\ v_0, v)] = \alpha$ and $0 < \alpha < 1$; see Dufour (1982a). When the matrices $X_{(1)}, \ldots, X_{(m)}$ all have full column rank, the restricted and unrestricted minimum sum of squares correspond to unique least squares estimates of β and $\bar{\beta}$ respectively, and the degrees of freedom are

$$v = T - km, \qquad v_0 = (m - 1)k. \tag{4.11}$$

The basic difficulty now is that λ is unknown. Let $C_\lambda(\alpha_1)$ be an exact confidence set for λ with level $1 - \alpha_1$ (at least) where $0 \leqslant \alpha_1 < \alpha < 1$:

$$P[\lambda \in C_\lambda(\alpha_1)] \geqslant 1 - \alpha_1. \tag{4.12}$$

We saw in Section 3 that we can in fact construct an exact similar confidence set with size $1 - \alpha_1$, i.e., such that $P[\lambda \in C_\lambda(\alpha_1)] = 1 - \alpha_1$ irrespective of the values of λ, β, σ, and y_0. Consider now the two following statistics:

$$D_L(\alpha_1) = \inf\{D(H_0; \lambda_0): \lambda_0 \in C_\lambda(\alpha_1)\}, \tag{4.13}$$

$$D_U(\alpha_1) = \sup\{D(H_0; \lambda_0): \lambda_0 \in C_\lambda(\alpha_1)\}. \tag{4.14}$$

Taking $\alpha_2 = \alpha - \alpha_1$ and $\alpha'_2 = \alpha + \alpha_1 < 1$, we see easily from Lemma 1 that, under H_0,

$$P[D_L(\alpha_1) \geqslant F(\alpha_2; \nu_0, \nu)] \leqslant \alpha_2 + \alpha_1 = \alpha, \tag{4.15}$$

$$P[D_U(\alpha_1) < F(\alpha'_2; \nu_0, \nu)] \leqslant 1 - \alpha'_2 + \alpha_1 = 1 - \alpha. \tag{4.16}$$

We thus have the following level-α generalized bounds test for H_0:

$$\begin{aligned}&\text{reject } H_0 \text{ when } D_L(\alpha_1) \geqslant F(\alpha_2; \nu_0, \nu),\\ &\text{accept } H_0 \text{ when } D_U(\alpha_1) < F(\alpha'_2; \nu_0, \nu),\\ &\text{consider the test inconclusive otherwise,}\end{aligned} \tag{4.17}$$

where α, α_1, α_2, and α'_2 satisfy (2.8). Defining the tail area function

$$G(x; \nu_0, \nu) = P[F(\nu_0, \nu) \geqslant x], \tag{4.18}$$

the above procedure is also equivalent to:

$$\begin{aligned}&\text{reject } H_0 \text{ when } G[D_L(\alpha_1); \nu_0, \nu] \leqslant \alpha_2,\\ &\text{accept } H_0 \text{ when } G[D_U(\alpha_1); \nu_0, \nu] > \alpha'_2,\\ &\text{consider the test inconclusive otherwise.}\end{aligned} \tag{4.19}$$

The probability $\alpha_L \equiv G[D_L(\alpha_1); \nu_0, \nu]$ can be interpreted as a 'conservative' p-value for testing H_0, while $\alpha_U \equiv G[D_U(\alpha_1); \nu_0, \nu]$ is a 'liberal' p-value for H_0. It is easy to see how equality restrictions between subvectors of β_i, $i = 1, \ldots, m$, can be tested in a comparable way.

The confidence set for λ can be obtained from at least two different models: the restricted model (1.2) or the unrestricted model (4.4). Both, of course, yield valid confidence sets for λ as well as valid bounds tests under H_0. On the other hand, the confidence set based on (1.2) is not generally valid under the alternative (4.4), and this may deflect the power properties of such a procedure. Finding which one is preferable is left to further research.

5. CUSUM tests

Analysis-of-covariance tests are built against specific alternatives where the breakpoints of the changes in β are specified *a priori*. To get tests against less specific structural change alternatives, we now consider generalizations of the well-known CUSUM and CUSUM-of-squares tests proposed by Brown, Durbin, and Evans (1975), henceforth BDE. For this purpose, we will make the additional rank assumption:

$$\operatorname{rank}(X_k) = k, \tag{5.1}$$

where $X_r = [x_1, x_2, \ldots, x_r]'$ is the $r \times k$ matrix of regressors for the first r observations $(1 \leqslant r \leqslant T)$. Note that (5.1) implies

$$\operatorname{rank}(X_r) = k, \quad r = k, k+1, \ldots, T. \tag{5.1'}$$

As we did for analysis-of-covariance tests, let us suppose first that $\lambda = \lambda_0$, with λ_0 known. Then, model (1.2) can be written

$$y(\lambda_0) = X\beta + u, \quad u \sim \mathrm{N}(0, \sigma^2 I_n), \tag{5.2}$$

where all the assumptions of the classical linear model are satisfied. The CUSUM test against the presence of structural change is based on the statistic

$$CS(\lambda_0) = \max\{|\tilde{W}_r(\lambda_0)| : r = k+1, \ldots, T\}, \tag{5.3}$$

where

$$\tilde{W}_r(\lambda_0) = \left\{\sum_{t=k+1}^{r} w_t(\lambda_0)\right\} \Big/ \{\hat{\sigma}(\lambda_0)\Delta(r-k, T-k)\}, \tag{5.4}$$

$$\Delta(r-k, T-k) = \sqrt{T-k}\left\{1 + 2\left(\frac{r-k}{T-k}\right)\right\}, \tag{5.5}$$

$w_t(\lambda_0)$, $t = k+1, \ldots, T$, are the recursive residuals based on model (5.2), and $\hat{\sigma}(\lambda_0)^2 = \sum_{t=k+1}^{T} w_t(\lambda_0)^2/(T-k)$. The recursive residuals are defined by

$$w_t(\lambda_0) = [y_t(\lambda_0) - x_t' b_{t-1}(\lambda_0)]/d_t(\lambda_0), \quad t = k+1, \ldots, T, \tag{5.6}$$

where

$$b_t(\lambda_0) = (X_t' X_t)^{-1} X_t' Y_t(\lambda_0), \qquad t = k, \ldots, T,$$

$$d_t(\lambda_0) = [1 + x_t'(X_{t-1}' X_{t-1})^{-1} x_t]^{1/2}, \qquad t = k+1, \ldots, T,$$

and $Y_t(\lambda_0) = [y_1(\lambda_0), y_2(\lambda_0), \ldots, y_t(\lambda_0)]'$. Under model (5.2), the residuals $w_t(\lambda_0)$, $t = k+1, \ldots, T$ are i.i.d. $\mathrm{N}(0, \sigma^2)$. The CUSUM test rejects the null hypothesis of stability at level α when $CS(\lambda_0) \geqslant c_1(\alpha)$, where $c_1(\alpha)$ is selected so that $P[CS(\lambda_0) \geqslant c_1(\alpha)] = \alpha$ when $w_{k+1}(\lambda_0), \ldots, w_T(\lambda_0)$ are i.i.d. $\mathrm{N}(0, \sigma^2)$.

BDE (1975) only provided approximate (although quite accurate) critical values for the CUSUM statistic. However, when $\lambda = \lambda_0$, we have $w_t(\lambda_0)/\sigma \overset{\text{ind}}{\sim} \mathrm{N}(0, 1)$, $t = k+1, \ldots, T$. If we define the random variable

$$CS_n = \max\{|\tilde{W}_r|: r = 1, \ldots, n\}, \tag{5.7}$$

where $\tilde{W}_r = \{\sum_{t=1}^{r} v_t\}/\{(\sum_{t=1}^{n} v_t^2/n)^{1/2}\ \Delta(r, n)\}$, $r = 1, \ldots, n$, and $v_1, \ldots, v_n$ are i.i.d. N(0, 1) variables, then we see immediately that $CS(\lambda_0)$ is distributed like CS_{T-k} when $\lambda = \lambda_0$. Further, it is straightforward to simulate the distribution of CS_{T-k}, so that an exact Monte Carlo CUSUM test can be performed as described at the end of Section 2.

To obtain a procedure valid without knowing λ, we consider again a confidence set for λ that satisfies (4.12) and define the statistics:

$$\begin{aligned} CS_L(\alpha_1) &= \inf\{CS(\lambda_0): \lambda_0 \in C_\lambda(\alpha_1)\}, \\ CS_U(\alpha_1) &= \sup\{CS(\lambda_0): \lambda_0 \in C_\lambda(\alpha_1)\}. \end{aligned} \tag{5.8}$$

Then, provided (2.8) holds, we get from Lemma 1:

$$P[CS_L(\alpha_1) \geqslant c_1(\alpha_2)] \leqslant \alpha, \quad P[CS_U(\alpha_1) \leqslant c_1(\alpha_2')] \leqslant 1 - \alpha, \tag{5.9}$$

which yields the bounds test:

$$\begin{aligned} &\text{reject stability if } CS_L(\alpha_1) \geqslant c_1(\alpha_2), \\ &\text{accept stability if } CS_U(\alpha_1) < c_1(\alpha_2'), \\ &\text{consider the test inconclusive otherwise.} \end{aligned} \tag{5.10}$$

Similarly, for $\lambda = \lambda_0$ given, the CUSUM-of-squares test based on model (5.2) has the form $CQ(\lambda_0) \geqslant c_2(\alpha)$ for a test of level α, where

$$CQ(\lambda_0) = \max\left\{\left|S_{k+t}(\lambda_0) - \frac{t}{T-k}\right|: t = 1, 2, \ldots, T-k\right\}, \tag{5.11}$$

where

$$S_r(\lambda_0) = \sum_{t=k+1}^{r} w_t(\lambda_0)^2 \bigg/ \sum_{t=k+1}^{T} w_t(\lambda_0)^2, \quad r = k+1, \ldots, T;$$

see BDE (1975) and Dufour (1982b, 1986). Again, computation of the distribution of the CUSUM-of-squares statistic is discussed by BDE (1975), who provided an approximation based on the earlier results of Durbin (1969). Further, as for the CUSUM test, it is easy to see that $CQ(\lambda_0)$ is distributed like the variable CQ_{T-k} defined by

$$CQ_n = \max\left\{\left|S_r - \frac{r}{n}\right|: r = 1, 2, \ldots, n\right\}, \tag{5.12}$$

with $n = T - k$, where $S_r = (\sum_{t=1}^{r} v_t^2)/(\sum_{t=1}^{n} v_t^2)$, $r = 1, \dots, n$, and $v_1, \dots, v_n \overset{\text{ind}}{\sim}$ N(0, 1). Clearly, it is quite easy to simulate CQ_{T-k} and thus to perform an exact Monte Carlo version of the test.

By applying Lemma 1, we see that

$$P[CQ_L(\alpha_1) \geqslant c_2(\alpha_2)] \leqslant \alpha, \quad P[CQ_U(\alpha_1) < c_2(\alpha_2')] \leqslant 1 - \alpha, \tag{5.13}$$

where

$$\begin{aligned} CQ_L(\alpha_1) &= \inf\{CQ(\lambda_0)\colon \lambda_0 \in C_\lambda(\alpha_1)\}, \\ CQ_U(\alpha_1) &= \sup\{CQ(\lambda_0)\colon \lambda_0 \in C_\lambda(\alpha_1)\}, \end{aligned} \tag{5.14}$$

and (2.8) holds, which yields a generalized bounds test analogous to the one in (5.10).

6. Tests against changes in λ

The tests described in Sections 4 and 5 are built against alternatives where the dynamic parameter λ is assumed constant. This does not mean that they have no power against alternatives where λ changes: shifts in λ will clearly affect the distributions of the analysis-of-covariance and CUSUM test statistics previously described. However, since these tests do not explicitly allow for the possibility of changes in λ, they can easily be biased against such alternatives. In this section as well as the following one, we describe two tests that consider in a more explicit way the possibility of changes in λ. The first one is a predictive test which looks at 'prediction errors' for the observations in the second sample obtained after estimating regression coefficients from the first sample. The second one is an AOC-type procedure which considers the difference between 'estimators' of λ based on the two samples and chosen so that the null distribution of the difference between the two estimators does not depend on nuisance parameters.

We consider a partition of y, y_{-1}, X, and u into two subvectors or submatrices, giving the following extension of model (1.2):

$$y_{(i)} = \lambda_i y_{-1}^{(i)} + X_{(i)}\beta_i + u_{(i)}, \quad u_{(i)} \overset{\text{ind}}{\sim} \mathrm{N}[0, \sigma_i^2 I_{T_i}], \quad i = 1, 2, \tag{6.1}$$

where $y_{(i)}$, $y_{-1}^{(i)}$, and $X_{(i)}$ are defined as in (4.1) with $m = 2$, $T_1 + T_2 = T$, $T_i \geqslant 1$, $\lambda_i \in \mathscr{D}_\lambda$, $i = 1, 2$, $\mathscr{D}_\lambda$ is the set of admissible values for λ, and

$$1 \leqslant r_1 = \operatorname{rank}(X_{(1)}) = k < T_1; \tag{6.2}$$

no rank condition is imposed on $X_{(2)}$. We want to test

$$\mathrm{H}_0'\colon \quad \beta_1 = \beta_2, \quad \lambda_1 = \lambda_2, \quad \sigma_1 = \sigma_2 \tag{6.3}$$

against an alternative in which all parameters (including λ) may change.

Suppose now that $\lambda_1 = \lambda_0$. Then, under H_0', a natural way of testing H_0' against (6.1) consists in testing whether the elements of the vector of prediction errors

$$\tilde{u}_{(2)}(\lambda_0) = y_{(2)}(\lambda_0) - X_{(2)}\hat{\beta}_1(\lambda_0) \tag{6.4}$$

have mean zero, where $y_{(i)}(\lambda_0) = y_{(i)} - \lambda_0 y_{-1}^{(i)}$, $i = 1, 2$, and $\hat{\beta}_1(\lambda_0) = (X_{(1)}'X_{(1)})^{-1} \times X_{(1)}'y_{(1)}(\lambda_0)$ is the least-squares estimate of β_1 obtained from the regression

$$y_{(1)}(\lambda_0) = X_{(1)}\beta_1 + u_{(1)}. \tag{6.5}$$

This yields the well-known predictive Chow statistic (assuming $\lambda_1 = \lambda_2 = \lambda_0$) which can be written in two alternative forms:

$$\begin{aligned} PC(\lambda_0) &= \frac{T_1 - k}{T_2}\left\{\frac{\tilde{u}_{(2)}(\lambda_0)'[I_{T_2} + X_{(2)}(X_{(1)}'X_{(1)})^{-1}X_{(2)}']^{-1}\tilde{u}_{(2)}(\lambda_0)}{\bar{S}_1(\lambda_0)}\right\} \\ &= \frac{T_1 - k}{T_2}\left\{\frac{S_0(\lambda_0) - \bar{S}_1(\lambda_0)}{\bar{S}(\lambda_0)}\right\}, \end{aligned}$$

where $S_0(\lambda_0)$ is the minimum sum of squares (4.8) from the estimation of the complete regression (4.4) with $\lambda = \lambda_0$, while $\bar{S}_1(\lambda_0)$ is the minimum sum of squares from the estimation of the first regression in (6.1) also with $\lambda = \lambda_0$, i.e.,

$$\bar{S}_1(\lambda_0) = \min_{\beta_1}\,[y_{(1)}(\lambda_0) - X_{(1)}\beta_1]'[y_{(1)}(\lambda_0) - X_{(1)}\beta_1]. \tag{6.7}$$

Under H_0' and $\lambda_1 = \lambda_0$, $PC(\lambda_0) \sim F(T_2, T_1 - k)$ so that the appropriate level-α critical region is $PC(\lambda_0) \geqslant F(\alpha;\, T_2, T_1 - k)$.

Let $C_{\lambda_1}(\alpha_1)$ be an exact confidence set for λ_1 with level not smaller than $1 - \alpha_1$, which is valid at least under H_0':

$$P[\lambda_1 \in C_{\lambda_1}(\alpha_1)] \geqslant 1 - \alpha_1. \tag{6.8}$$

Such a confidence set can be constructed by applying the methods of Section 2 to the first sample (y_t, $t = 0, 1, \ldots, T_1$). Let also

$$\begin{aligned} PC_L(\alpha_1) &= \inf\{PC(\lambda_0)\colon \lambda_0 \in C_{\lambda_1}(\alpha_1)\}, \\ PC_U(\alpha_1) &= \sup\{PC(\lambda_0)\colon \lambda_0 \in C_{\lambda_1}(\alpha_1)\}. \end{aligned} \tag{6.9}$$

Then, provided (2.8) holds, we have

$$\begin{aligned} &P[PC_L(\alpha_1) \geqslant F(\alpha_2;\, T_2, T_1 - k)] \leqslant \alpha, \\ &P[PC_U(\alpha_1) < F(\alpha_2';\, T_2, T_1 - k)] \leqslant 1 - \alpha, \end{aligned} \tag{6.10}$$

which yield the bounds test

$$\begin{aligned}&\text{reject } H_0' \text{ if } PC_L(\alpha_1) \geqslant F(\alpha_2;\, T_2,\, T_1 - k),\\ &\text{accept } H_0' \text{ if } PC_U(\alpha_1) < F(\alpha_2';\, T_2,\, T_1 - k),\\ &\text{consider the test inconclusive otherwise.}\end{aligned} \tag{6.13}$$

It is interesting to note that $PC(\lambda_0)$ is the Fisher statistic for testing model (1.2) against an extended model where a dummy variable has been added for each observation in the second period ($t = T_1 + 1, \dots, T$):

$$y_t(\lambda_0) = x_t'\beta + \sum_{s=T_1+1}^{T} \gamma_s D_{ts} + u_t, \quad t = 1, \dots, T, \tag{6.14}$$

where $D_{ts} = 1$ if $t = s$ and $D_{ts} = 0$ if $t \neq s$; see Dufour (1980, 1982c).[1] Further, we can look at the t-statistics for each element of the vector $\gamma = (\gamma_{T_1+1}, \gamma_{T_1+2}, \dots, \gamma_T)'$:

$$\begin{aligned} t_s(\lambda_0) &= \hat{\gamma}_s(\lambda_0)/\hat{\sigma}_s(\lambda_0)\\ &= \frac{y_s - x_s'\hat{\beta}_1(\lambda_0)}{s_1(\lambda_0)[1 + x_s'(X_{(1)}'X_{(1)})^{-1}x_s]^{1/2}}, \end{aligned} \tag{6.15}$$

$s = T_1 + 1, \dots, T$, where $s_1(\lambda_0)^2 = S_1(\lambda_0)/(T_1 - k)$, each of which follows a Student distribution $t(T_1 - k)$ under H_0' (when $\lambda_1 = \lambda_0$). As suggested in Dufour (1980), these statistics provide a simple way of analyzing the form and timing of possible structural changes over the second period. To get tests valid without knowing λ_1, we consider the statistics:

$$F_s^L(\alpha_1) = \inf\{t_s(\lambda_0))^2\colon \lambda_0 \in C_{\lambda_1}(\alpha_1)\}, \qquad s = T_1 + 1, \dots, T, \tag{6.16}$$

$$F_s^U(\alpha_1) = \sup\{(t_s(\lambda_0))^2\colon \lambda_0 \in C_{\lambda_1}(\alpha_1)\}, \quad s = T_1 + 1, \dots, T. \tag{6.17}$$

Clearly again,

$$\begin{aligned}&P[F_s^L(\alpha_1) \geqslant F(\alpha_2;\, 1,\, T_1 - k)] \leqslant \alpha,\\ &P[F_s^U(\alpha_1) \leqslant F(\alpha_2';\, 1,\, T_1 - k)] \leqslant 1 - \alpha,\end{aligned} \tag{6.18}$$

so that we have a sequence of bounds tests for each observation in the second sample:

$$\begin{aligned}&\text{reject } H_0' \text{ when } F_s^L(\alpha_1) \geqslant F(\alpha_2;\, 1,\, T_1 - k),\\ &\text{accept } H_0' \text{ when } F_s^U(\alpha_1) < F(\alpha_2';\, 1,\, T_1 - k).\end{aligned} \tag{6.19}$$

Predictive tests like those just defined may be viewed as portmanteau tests aimed at detecting any form of structural change that could affect the

[1] For the related (asymptotic) generalization to dynamic possibly nonlinear models, see Dufour, Ghysels, and Hall (1994).

coefficients of the model (λ, β, or σ). Since it would also be of interest to have a procedure for detecting change in λ, irrespective of whether β or σ has changed, we now derive an AOC-type test for changes in λ which is insensitive to changes in β and σ of the form allowed in (6.1). More precisely, we again take model (6.1) and we consider the problem of testing

$$\bar{\mathrm{H}}_0\colon \quad \lambda_1 = \lambda_2 \quad \text{against} \quad \bar{\mathrm{H}}_1\colon \quad \lambda_1 \neq \lambda_2, \tag{6.20}$$

where it is not assumed that $\beta_1 = \beta_2$ nor $\sigma_1 = \sigma_2$ under $\bar{\mathrm{H}}_0$. Further to allow one to estimate the model separately on the two subperiods considered, we replace (6.2) by the stronger assumption

$$1 \leqslant r_i = \text{rank}\,(X_{(i)}) = k < T_i, \quad i = 1,2. \tag{6.21}$$

To derive a test of $\bar{\mathrm{H}}_0$, we consider first the more restrictive hypothesis

$$\bar{\mathrm{H}}_0(\lambda_0)\colon \quad \lambda_1 = \lambda_2 = \lambda_0, \tag{6.22}$$

where $\lambda_0 \in \mathscr{D}_\lambda$. For each $\lambda_0 \in \mathscr{D}_\lambda$, we can compute estimates $\hat{\lambda}_1(\lambda_0)$ and $\hat{\lambda}_2(\lambda_0)$ of λ_1 and λ_2 based on extended regressions such as (3.1), i.e.,

$$y_{(i)} = \lambda_i y_{-1}^{(i)} + X_{(i)}(\lambda_0)\beta_{i*} + u_{(i)}, \quad i = 1,2, \tag{6.23}$$

where $X_{(i)}(\lambda_0)$ has full column rank and spans the same space as that spanned by the columns of the matrix $[X_{(i)} \,\vdots\, \iota_{T_i}(\lambda_0) \,\vdots\, J_{T_i}(\lambda_0)X_{(i)}]$. When $\lambda_i = \lambda_0$,

$$\hat{\lambda}_i(\lambda_0) = \lambda_i + Q_i(\lambda_0, u_{(i)}), \quad i = 1,2, \tag{6.24}$$

where

$$\begin{aligned} Q_i(\lambda_0, u_{(i)}) &= \frac{u_{(i)}' J_{T_i}(\lambda_0)' M[X_{(i)}(\lambda_0)] u_{(i)}}{u_{(i)}' J_{T_i}(\lambda_0)' M[X_{(i)}(\lambda_0)] J_{T_i}(\lambda_0) u_{(i)}} \\ &= \frac{v_{(i)}' J_{T_i}(\lambda_0)' M[X_{(i)}(\lambda_0)] v_{(i)}}{v_{(i)}' J_{T_i}(\lambda_0)' M[X_{(i)}(\lambda_0)] J_{T_i}(\lambda_0) v_{(i)}}, \quad i = 1,2, \end{aligned}$$

where $v_{(i)} = u_{(i)}/\sigma_i \sim \mathrm{N}[0, I_{T_i}]$.

Clearly, the distribution of $Q_i(\lambda_0, u_{(i)})$ does not depend on any nuisance parameter. Now to test $\bar{H}_0(\lambda_0)$, it is natural to consider the difference $\Delta(\lambda_0) = \hat{\lambda}_1(\lambda_0) - \hat{\lambda}_2(\lambda_0)$. When $\lambda_1 = \lambda_2 = \lambda_0$,

$$\Delta(\lambda_0) = Q_1(\lambda_0, v_{(1)}) - Q_2(\lambda_0, v_{(2)}) \equiv \bar{\Delta}(\lambda_0, v), \tag{6.26}$$

where $v = (v_{(1)}', v_{(2)}')' \sim \mathrm{N}[0, I_T]$, a random variable whose distribution involves no nuisance parameter. Let

$$G_\Delta(x; \lambda_0) = P[|\bar{\Delta}(\lambda_0, v)| \geqslant x | \bar{H}_0(\lambda_0)] \tag{6.27}$$

be the probability of the event $|\bar{\Delta}(\lambda_0, v)| \geqslant x$ when $\lambda_1 = \lambda_2 = \lambda_0$, and let $\hat{\Delta}(\lambda_0)$ be the observed value of $\Delta(\lambda_0)$. Then the test which rejects $\bar{H}_0(\lambda_0)$ when

$$G_\Delta(|\hat{\Delta}(\lambda_0)|; \lambda_0) \leqslant \alpha \tag{6.28}$$

has level α. Further, it is easy to obtain a randomized version of the test in (6.28) by using Lemma 2.

Given the function $G_\Delta(x; \lambda_0)$, there are at least two ways of getting exact tests of the less restrictive hypothesis $\bar{H}_0$. First, it is straightforward to see that the test which rejects $\bar{H}_0$ when

$$\sup\{G_\Delta(|\hat{\Delta}(\lambda_0)|; \lambda_0): \lambda_0 \in \mathscr{D}_\lambda\} \leqslant \alpha \tag{6.29}$$

has level α. Second, when a confidence set $C_\lambda(\alpha_1)$ with level $1 - \alpha_1$ (at least under $\bar{H}_0$) is available, i.e.,

$$P[\lambda \in C_\lambda(\alpha_1)] \geqslant 1 - \alpha_1, \tag{6.30}$$

the following procedures is a valid generalized bounds test at level α:

$$\begin{aligned} &\text{reject } \mathrm{H}_0 \text{ when } \sup\{G_\Delta(|\hat{\Delta}(\lambda_0)|; \lambda_0): \lambda_0 \in C_\lambda(\alpha_1)\} \leqslant \alpha_2, \\ &\text{accept } \mathrm{H}_0 \text{ when } \inf\{G_\Delta(|\hat{\Delta}(\lambda_0)|; \lambda_0): \lambda_0 \in C_\lambda(\alpha_1)\} > \alpha_2', \\ &\text{consider the test inconclusive otherwise,} \end{aligned} \tag{6.31}$$

where (2.8) holds. A simple way to get an appropriate confidence set for λ is to build one by the Kiviet–Phillips procedure from either one of the two subsamples considered. The validity of the bounds test follows again from Lemma 1. Correspondingly, randomized bounds tests can be obtained in the same way after replacing the test in (6.28) with its randomized analogue. Note that the same set of artificial replications should be used for all values of λ_0.

7. Illustrations

We will show now that the inference procedures developed here are operational by applying them to an empirical first-order dynamic autoregressive model for the logarithm of U.S. gross private domestic investment in nonresidential structures (1982 dollars, quarterly, 1952:1–1986:4). But before we do that, we will first illustrate the tests on some artificial data sets. In these calculations, we can choose and change the specification and the parameter values of the model in such a way that some interesting features of the various tests can be demonstrated. So, we perform a few controlled experiments. These, of course, cannot (and are not meant to) replace a full scale Monte Carlo analysis from which the power performance of the tests would be assessed (which goes beyond the scope of the present paper). We just examine a few single realizations of particular data-generating processes in order to show that our exact procedures are feasible and behave reasonably well. Note also that for the model studied in the previous sections, no alternative *finite-sample* structural change test appears to be available.

We focus on a model with one lagged dependent variable, an intercept, and possibly a linear trend with various forms of structural breaks and shocks to the system. First, we examine a first-order autoregressive process with nonzero mean and a unit root (i.e., a random walk with drift). The data were generated according to

$$\begin{aligned} y_t &= \lambda_1 y_{t-1} + \beta_1 + u_t, \qquad & t = 1, \dots, T_1, \\ y_t &= \lambda_2 y_{t-1} + \beta_2 + \delta d_t + u_t, \qquad & t = T_1 + 1, \dots, T, \end{aligned} \tag{7.1}$$

where $y_0 = 1$ and $u_1, \dots, u_T$ independent and normally distributed with mean zero. For the first T_1 observations of the sample, we invariably have

$$\begin{aligned} &T_1 = 30, \quad \lambda_1 = 1, \quad \beta_1 = 0.02, \quad \sigma_1 = 0.01, \\ &\mathrm{var}(u_t) = \sigma_1^2 \quad \text{for} \quad t = 1, \dots, T_1. \end{aligned} \tag{7.2}$$

For the second part of the sample, consisting of $T_2 = T - T_1$ observations, we have

$$T_2 = 20,\ d_t \text{ is a dummy variable, } \mathrm{var}(u_t) = \sigma_2^2 \text{ for } t = T_1 + 1, \dots, T; \tag{7.3}$$

moreover, we consider six distinct cases:

A (no structural change): $\lambda_2 = 1.0$, $\beta_2 = 0.02$, $\delta = 0$, $\sigma_2 = 0.01$;

B (change in β only): $\lambda_2 = 1.0$, $\beta_2 = 0.03$, $\delta = 0$, $\sigma_2 = 0.01$;

C (change in λ only): $\lambda_2 = 0.95$, $\beta_2 = 0.02$, $\delta = 0$, $\sigma_2 = 0.01$;

D (change in σ only): $\lambda_2 = 1.0$, $\beta_2 = 0.02$, $\delta = 0$, $\sigma_2 = 0.02$;

E (isolated shocks): $\lambda_2 = 1.0$, $\beta_2 = 0.02$, $\delta = 0.05$, $\sigma_2 = 0.01$, with $d_t = 1$ for $t = 35{,}45$ and $d_t = 0$ otherwise;

F (jump in drift): $\lambda_2 = 1.0$, $\beta_2 = 0.02$, $\delta = 0.02$, $\sigma_2 = 0.01$, with $d_t = 1$ for $t \geqslant 40$ and $d_t = 0$ otherwise.

For the six data sets so generated, we estimated the model

$$y_t = \lambda y_{t-1} + \beta_1 + u_t, \tag{7.4}$$

and applied the various exact structural change tests described in the previous sections of this paper.[2] The least-squares estimates of the parameters for

[2]The six data sets are obtained from one and the same realization of $u_1/\sigma_1, \dots, u_{T_1}/\sigma_1, u_{T_1+1}/\sigma_2, \dots, u_T/\sigma_2$. The samples used in these illustrations are available from the authors upon request.

Table 1
Simple AR(1) models: OLS estimates and exact λ confidence sets

Sample	$\hat{\lambda}$	$\hat{\beta}_1$	s
First subsample ($T_1 = 30$)	0.980 (0.010)	0.048 (0.014)	0.011
Second subsample ($T_2 = 20$)			
A	0.972 (0.018)	0.071 (0.032)	0.008
B	0.983 (0.012)	0.063 (0.022)	0.008
C	0.961 (0.008)	0.006 (0.010)	0.009
D	0.929 (0.037)	0.149 (0.067)	0.016
E	0.969 (0.031)	0.082 (0.058)	0.019
F	1.034 (0.013)	− 0.034 (0.024)	0.010
Complete sample ($T = 50$)			
A	0.991 (0.005)	0.034 (0.008)	0.010
B	1.005 (0.005)	0.016 (0.007)	0.011
C	1.016 (0.021)	− 0.023 (0.027)	0.033
D	0.989 (0.007)	0.037 (0.010)	0.014
E	0.998 (0.007)	0.025 (0.011)	0.015
F	1.012 (0.005)	0.006 (0.008)	0.011

Exact confidence sets for λ (from first subsample)

Level	0.99	0.975	0.95	0.925
Based on $\mathscr{L}_{\lambda}^{*}$	[0.8660, 1]	[0.8713, 1]	[0.8745, 1]	[0.8785, 1]
Based on $\mathscr{L}_{\lambda}^{**}$	[0.9370, 1]	[0.9414, 1]	[0.9445, 1]	[0.9473, 1]

OLS standard errors appear in parentheses. The exact confidence sets for λ are randomized with $N - 1 = 499$ Monte Carlo samples. s is the estimated disturbance standard error.

both the complete sample ($1 \leqslant t \leqslant T$) and the relevant subsamples ($1 \leqslant t \leqslant T_1$, $T_1 + 1 \leqslant t \leqslant T$) appear in Table 1, while the p-values of the structural change tests are presented in Table 2. We also present exact (randomized) confidence sets for λ at various levels ($1 - \alpha_1 = 0.999, 0.975, 0.95, 0.925$) based on the LR-type statistics $\mathscr{L}^{*}(\lambda_0)$ and $\mathscr{L}^{**}(\lambda_0)$. Each of these confidence sets is obtained from the first sample ($1 \leqslant t \leqslant 30$) under the restriction $|\lambda| \leqslant 1$, and the randomization uses $N - 1 = 499$ artificial replications of $\mathscr{L}^{*}(\lambda_0)$ or $\mathscr{L}^{**}(\lambda_0)$ under the null hypothesis [see (3.16)]. We see from these results that the confidence sets based on $\mathscr{L}^{**}(\lambda_0)$ are typically shorter than those based on $\mathscr{L}^{*}(\lambda_0)$, which illustrates the fact that the statistic $\mathscr{L}^{**}(\lambda_0)$ takes into account all the restrictions implied by $\lambda = \lambda_0$ on the extended model (3.1) (see the discussions in Section 3). Furthermore, the confidence sets appear to be rather insensitive to the level selected (at least for this data set). For the structural change tests, we will use the confidence set [0.9445, 1] based on $\mathscr{L}^{**}(\lambda_0)$ with $\alpha_1 = 0.05$.

Table 2
Simple AR(1) models: Exact tests for structural change (p-values) based on the $\mathscr{L}_\lambda^{**}$ confidence set from the first subsample; $\alpha = 0.10, \alpha_1 = 0.05, \alpha_2 = 0.05, \alpha_2' = 0.15$ ('r' indicates rejection at level 0.10; 'a' indicates acceptance at level 0.10)

(A) *Global tests*

Test	A	B	C	D	E	F
AOC-β						
D_L	0.927	0.006 r	0.000 r	0.949	0.425	0.005 r
D_U	0.000	0.000	0.000	0.000	0.000	0.000
CUSUM[a]						
CS_L	0.952	0.664	0.006 r	0.998	0.966	0.660
CS_U	0.002	0.002	0.002	0.002	0.002	0.002
CUSUM-of-squares[a]						
CQ_L	0.762	0.602	0.002 r	0.050 r	0.012 r	0.194
CQ_U	0.008	0.002	0.002	0.004	0.002	0.002
Predictive test						
PC_L	0.873	0.530	0.000 r	0.014 r	0.004 r	0.191
PC_U	0.003	0.000	0.000	0.000	0.000	0.000
AOC-λ[a]						
P_Δ^L	0.922	0.956	0.608	0.058	0.292	0.714
P_Δ^U	0.006	0.004	0.002	0.002	0.064	0.168 a

(B) *Individual predictive tests (conservative p-values)*

Predicted observation	A	B	C	D	E	F
31	0.115	0.018 r	0.001 r	0.002 r	0.115	0.115
32	0.988	0.476	0.000 r	0.976	0.988	0.988
33	0.532	0.138	0.000 r	0.187	0.532	0.532
34	0.997	0.981	0.000 r	0.897	0.997	0.997
35	0.573	0.154	0.000 r	0.229	0.000 r	0.573
36	0.989	0.703	0.000 r	0.992	0.996	0.989
37	0.973	0.564	0.000 r	0.975	0.999	0.973
38	0.945	0.343	0.000 r	0.822	0.945	0.945
39	0.706	0.212	0.000 r	0.402	0.706	0.706
40	0.969	0.983	0.000 r	0.974	0.990	0.536
41	0.989	0.386	0.000 r	0.951	0.989	0.086
42	0.737	0.227	0.000 r	0.449	0.737	0.042 r
43	0.976	0.956	0.000 r	0.980	0.969	0.472
44	0.915	0.325	0.000 r	0.763	0.915	0.068
45	0.869	0.298	0.000 r	0.677	0.000 r	0.060
46	0.975	0.970	0.000 r	0.989	0.988	0.500
47	0.974	0.974	0.000 r	0.959	0.961	0.548
48	0.529	0.137	0.002 r	0.183	0.529	0.022 r
49	0.964	0.965	0.000 r	0.980	0.983	0.481
50	0.691	0.205	0.002 r	0.380	0.691	0.037 r

(C) *Some tests with λ known*

Test	A	B	C	D	E	F
AOC-β	0.600	0.006 r	—	0.604	0.425	0.005 r
CUSUM	0.344	0.436	—	0.556	0.744	0.540
CUSUM-of-squares	0.416	0.602	—	0.050 r	0.012 r	0.194
Predictive test	0.868	0.530	—	0.014 r	0.004 r	0.191

[a]Randomized p-values based on $N - 1 = 499$ Monte Carlo samples.

To assess the significance of the structural change tests, we shall use a 0.10 level ($\alpha = 0.10$). As expected, none of the tests is significant under the null hypothesis (data set A), while the alternative tests appear differently sensitive to various alternatives. The analysis-of-covariance test for β(AOC-β) easily detects permanent changes in β_1 (sets B and F), a type of alternative against which it is designated, as well as λ (set C). The change in λ is also detected by the CUSUM, the CUSUM-of-squares and the predictive tests. The variance shift (set D) and the two isolated shocks (set E) are detected by the CUSUM-of-squares and the predictive test, which illustrates the fact that these tests are quite sensitive to heteroskedasticity. It is also of interest to note that the individual predictive tests both detect and allow one to date the isolated shocks (E). The jump in the intercept (F) leads to a series of low p-values. The AOC test for λ (AOC-λ) does not detect any of the structural shifts considered here, which suggests that this test has rather low power. For the sake of comparison, we also present in Table 2 (part C) tests obtained under the assumption that λ is known and constant (i.e., the autoregressive part of each equation is eliminated by subtracting it from the dependent variable, under each model considered). We see from these results that the inferences are not affected by the estimation of λ. In fact, we note that the infima for the CUSUM-of-squares and predictive test statistics are often obtained when λ is equal to one.

Next, we examine an AR(1) model with intercept and linear trend term for cases where the lagged dependent variable coefficient is high, but smaller than one. The data here were generated according to the model:

$$\begin{aligned} y_t &= \lambda_1 y_{t-1} + \beta_{11} + \beta_{21}(t/100) + u_t, \qquad && t = 1, \dots, T_1, \\ y_t &= \lambda_2 y_{t-1} + \beta_{12} + \beta_{22}(t/100) + \delta d_t + u_t, \qquad && t = T_1 + 1, \dots, T, \end{aligned} \tag{7.5}$$

where $u_1, \dots, u_T$ are defined as in (7.1)–(7.3), and y_0 was generated independently of $u_1, \dots, u_T$ according to the normal distribution $\mathrm{N}(\mu_0, \sigma_1^2/(1-\lambda_1^2))$ with $\mu_0 = [\beta_{11}/(1-\lambda_1)] - [(\beta_{21}/100)\,\lambda_1/(1-\lambda_1)^2]$ (this choice avoids 'warming-up' problems in the simulation). For the first period, we have

$$T_1 = 50, \quad \lambda_1 = 0.9, \quad \beta_{11} = 1.0, \quad \beta_{21} = 1.0, \quad \sigma_1 = 0.02, \tag{7.6}$$

while the second period, we have $T_2 = 30$ and $\sigma_2 = \sigma_1 = 0.02$, and we consider again six cases:

A (no change): $\lambda_2 = 0.9$, $\beta_{12} = 1.0$, $\beta_{22} = 1.0$, $\delta = 0$;

B (change in trend): $\lambda_2 = 0.9$, $\beta_{12} = 1.0$, $\beta_{22} = 0.9$, $\delta = 0$;

C (change in intercept): $\lambda_2 = 0.9$, $\beta_{12} = 1.2$, $\beta_{22} = 1.0$, $\delta = 0$;

D (change in λ): $\lambda_2 = 0.92$, $\beta_{12} = 1.0$, $\beta_{22} = 1.0$, $\delta = 0$;

Table 3
AR(1) with linear trend models: OLS estimates and exact λ confidence sets

Sample	$\hat{\lambda}$	$\hat{\beta}_{11}$	$\hat{\beta}_{12}$	s
First subsample ($T_1 = 50$)	0.871 (0.029)	1.286 (0.276)	1.236 (0.255)	0.0220
Second subsample ($T_2 = 30$)				
A	0.664 (0.146)	3.105 (1.306)	3.371 (1.457)	0.0198
B	0.881 (0.071)	1.175 (0.704)	1.057 (0.552)	0.0207
C	0.884 (0.019)	1.293 (0.130)	1.265 (0.310)	0.0205
D	0.908 (0.015)	1.031 (0.061)	1.267 (0.333)	0.0205
E	0.653 (0.149)	3.168 (1.311)	3.555 (1.540)	0.0333
F	0.932 (0.039)	0.257 (0.228)	1.498 (0.642)	0.0538
Complete sample ($T = 80$)				
A	0.888 (0.021)	1.114 (0.193)	1.095 (0.192)	0.0213
B	0.923 (0.028)	0.808 (0.272)	0.671 (0.240)	0.0263
C	0.971 (0.011)	0.315 (0.098)	0.490 (0.135)	0.0470
D	0.985 (0.009)	0.169 (0.077)	0.450 (0.131)	0.0640
E	0.892 (0.023)	1.082 (0.220)	1.078 (0.221)	0.0271
F	1.018 (0.012)	− 0.119 (0.107)	− 0.011 (0.124)	0.0415

Exact confidence sets for λ (from first subsample)

Level	0.99	0.975	0.95	0.925
Based on $\mathscr{L}_\lambda^*$	[0.6490, 1]	[0.6665, 1]	[0.6810, 1]	[0.6875, 1]
Based on $\mathscr{L}_\lambda^{**}$	[0.7610, 1]	[0.7736, 1]	[0.7866, 1]	[0.7922, 1]

E (isolated shocks): $\lambda_2 = 0.9$, $\beta_{12} = 1.0$, $\beta_{22} = 1.0$, $\delta = 0.1$, with $d_t = 1$ for $t = 60, 70$ and $d_t = 0$ otherwise;

F (jump in intercept): $\lambda_2 = 0.9$, $\beta_{12} = 1.0$, $\beta_{22} = 1.0$, $\delta = 0.2$, with $d_t = 1$ for $t > 65$ and $d_t = 0$ otherwise.

Now, the results are based on estimating the equation

$$y_t = \lambda y_{t-1} + \beta_1 + \beta_2(t/100) + u_t, \tag{7.7}$$

instead of (7.4). The parameter estimates appear in Table 3 and the p-values associated with the various structural change tests in Table 4. We see again that the AOC-β test detects the permanent changes in the trend coefficient (data set B), the intercept (C and F), and λ (D), but does not find the isolated shocks (E). The intercept and λ shifts are also detected by the CUSUM-of-squares and predictive tests. While the two latter tests do not detect the shift in trend (B), they do find the two isolated shocks (E). Further, the individual predictive tests again provide useful information on the timing and form of the structural shifts (especially in case D and F). The CUSUM and AOC-λ tests do not detect any of the structural shifts considered. Finally, from part C of Table 4, we see that knowing the true value of λ has little effect for the inferences based on the AOC-β and CUSUM-of-squares tests. However, for these data, the CUSUM test appears to be more powerful for the cases B, C, and F, and the predictive test detects the change in trend (B) when λ is known.

Finally, we examine U.S. gross private domestic investment in nonresidential structure (1982 dollars): 140 quarterly observations from 1952:1 to 1986:4 [source: Berndt (1991, p. 278)], which we divided into two roughly equal subperiods (1952:1–1969:4 and 1970:1–1986:4). Taking the logarithm of this variable, we find that the first half of the sample (1952:1–1969:4) can be described reasonably well by the just examined AR(1) model with intercept and trend:

$$I_t = \lambda I_{t-1} + \beta_1 + \beta_2(t/100) + u_t, \quad |\lambda| \leqslant 1, \tag{7.8}$$

where I_t is the logarithm of real gross domestic investment in nonresidential structures. Although the intercept and trend coefficients are not significant according to the usual asymptotic standards, we find that such a specification does not suffer manifestly from omitted higher-order lags/serial correlation, heteroskedasticity, or nonnormal errors. The OLS estimator for the two subsamples and the complete sample, as well as some standard diagnostics, appear in part A of Table 5.

Table 4
AR(1) with linear trend models: Exact tests for structural change (p-values) based on the $\mathscr{L}_{\lambda}^{**}$ confidence set from the first subsample; $\alpha = 0.10$, $\alpha_1 = 0.05$, $\alpha_2 = 0.05$, $\alpha_2' = 0.15$ ('r' indicates rejection at level 0.10; 'a' indicates acceptance at level 0.10)

(A) *Global tests*

Test	A	B	C	D	E	F
AOC-β						
D_L	0.953	0.000 r	0.000 r	0.000 r	0.956	0.006 r
D_U	0.001	0.000	0.000	0.000	0.000	0.000
CUSUM						
CS_L	1.000	0.256	0.686	0.532	1.000	0.584
CS_U	0.002	0.002	0.002	0.002	0.002	0.002
CUSUM-of-squares						
CQ_L	0.982	0.406	0.002 r	0.002 r	0.028 r	0.002 r
CQ_U	0.380 a	0.022	0.002	0.002	0.004	0.002
Predictive test						
PC_L	0.778	0.204	0.000 r	0.000 r	0.016 r	0.000 r
PC_U	0.327 a	0.002	0.000	0.000	0.001	0.000
AOC-λ						
P_Δ^L	0.822	0.864	1.000	1.000	0.916	0.766
P_Δ^U	0.326 a	0.326 a	0.684 a	0.596 a	0.416 a	0.406 a

Table 4 (continued)

(B) *Individual predictive tests (conservative p-values)*

Predicted observation	A	B	C	D	E	F
51	0.792	0.338 r	0.000 r	0.000 r	0.792	0.792
52	0.945	0.019 r	0.000 r	0.000 r	0.945	0.945
53	0.999	0.058	0.000 r	0.000 r	0.999	1.000
54	0.694	0.289	0.000 r	0.000 r	0.694	0.694
55	0.997	0.019 r	0.001 r	0.000 r	0.997	0.997
56	0.997	0.099	0.000 r	0.000 r	0.997	0.997
57	0.489	0.002 r	0.071	0.000 r	0.489	0.489
58	0.997	0.117	0.005 r	0.000 r	0.997	0.997
59	0.997	0.293	0.004 r	0.000 r	0.997	0.997
60	0.999	0.122	0.035 r	0.000 r	0.004 r	0.999
61	0.549	0.785	0.003 r	0.000 r	0.810	0.549
62	0.997	0.365	0.035 r	0.000 r	0.997	0.997
63	0.944	0.004 r	0.989	0.047 r	0.993	0.944
64	0.906	0.543	0.059	0.000 r	0.998	0.906
65	0.996	0.012 r	0.969	0.072	0.996	0.996
66	0.941	0.006 r	0.958	0.201	0.995	0.000 r
67	0.970	0.013 r	0.979	0.208	0.995	0.000 r
68	1.000	0.352	0.409	0.010 r	0.998	0.000 r
69	0.996	0.452	0.390	0.010 r	0.994	0.000 r
70	0.996	0.344	0.594	0.026 r	0.005 r	0.000 r
71	0.999	0.127	0.965	0.132	0.996	0.003 r
72	0.997	0.124	0.982	0.181	0.995	0.011 r
73	0.995	0.078	0.982	0.336	0.993	0.051 r
74	0.994	0.157	0.992	0.237	0.998	0.050 r
75	0.999	0.468	0.784	0.081	0.994	0.021 r
76	0.993	0.134	0.961	0.378	0.992	0.195
77	0.997	0.198	0.964	0.321	0.998	0.209
78	0.993	0.049 r	0.983	0.842	0.990	0.744
79	0.997	0.042 r	0.982	0.981	0.992	0.966
80	0.999	0.119	0.965	0.662	0.993	0.719

(C) *Some tests with λ known*

Test	A	B	C	D	E	F
AOC-β	0.799	0.000 r	0.000 r	—	0.706	0.000 r
CUSUM	1.000	0.018 r	0.002 r	—	1.000	0.004 r
CUSUM-of-squares	0.416	0.084	0.002 r	—	0.004 r	0.002 r
Predictive test	0.721	0.010 r	0.000 r	—	0.005 r	0.000 r

To perform structural change tests, we first obtained a confidence set for λ (with size 0.95) based on the statistic $\mathscr{L}_\lambda^{**}$ and the first subsample, which yielded the interval [0.8744, 1.0].[3] The p-values for the various test statistics are reported in parts B and C of Table 5 (for the individual predictive tests, in order

[3] Because of the relatively large sample size ($T = 139$), we only used $N - 1 = 199$ Monte Carlo samples in building the confidence set for λ and for assessing the significance of the relevant structural change tests.

Table 5
AR(1) with linear trend model for logarithm of US gross domestic investment in nonresidential structures in 1982 dollars (I_t); quarterly data, 1952:1–1986:4; Source: Berndt (1991, p. 278; series IS of file Kopcke)

$I_t = \lambda I_{t-1} + \beta_1 + \beta_2(t/100) + u_t, \quad |\lambda| \leq 1$

(A) *OLS estimates*

Sample		$\hat{\lambda}$	$\hat{\beta}_1$	$\hat{\beta}_2$	s	R^2
First subsample	($T_1 = 71$)	0.921 (0.046)	0.872 (0.506)	0.070 (0.044)	0.022	0.9878
Second subsample	($T_2 = 68$)	0.930 (0.049)	0.788 (0.543)	0.033 (0.032)	0.031	0.9443
Complete sample	($T = 139$)	0.956 (0.028)	0.503 (0.309)	0.018 (0.019)	0.027	0.9899

Diagnostics for first subsample		p-value
	First-order serial correlation: $F(1, 67) = 2.54$	0.12
	Fourth-order serial correlation: $F(4, 64) = 1.72$	0.16
	Heteroskedasticity: Breusch–Pagan: $\chi^2(2) = 2.50$	0.20
	Koenker: $\chi^2(2) = 1.92$	0.38
	Nonnormal disturbances: $\chi^2(2) = 1.16$	0.56

(B) *Global test for structural change (exact)*

Confidence set for λ based on $\mathscr{L}_\lambda^{**}$ ($\alpha_1 = 0.05$): [0.8744, 1.00]

Test	Conservative p-value (L)	Liberal p-value (U)
AOC-β	0.788	0.015
CUSUM[a]	0.785	0.580 a
CUSUM-of-squares[a]	0.005 r	0.005
Predictive	0.004 r	0.001
AOC-λ[a]	0.840	0.585 a

(C) *Individual predictive tests (exact)*

Quarter	Conservative p-value
1974:3	0.006 r
1975:1	0.014 r
1982:2	0.046 r
1983:1	0.002 r
1986:2	0.000 r

[a] Randomized p-values based on $N - 1 = 199$ Monte Carlo samples.

to economize space, we only report the conservative p-values which are lower than 0.05); the complete series of individual predictive tests (p-values) is also graphed in Fig. 1. Using $\alpha = 0.10$ as the critical level (with $\alpha_1 = \alpha_2 = 0.05$), we see that the AOC and CUSUM tests do not show evidence of structural change,

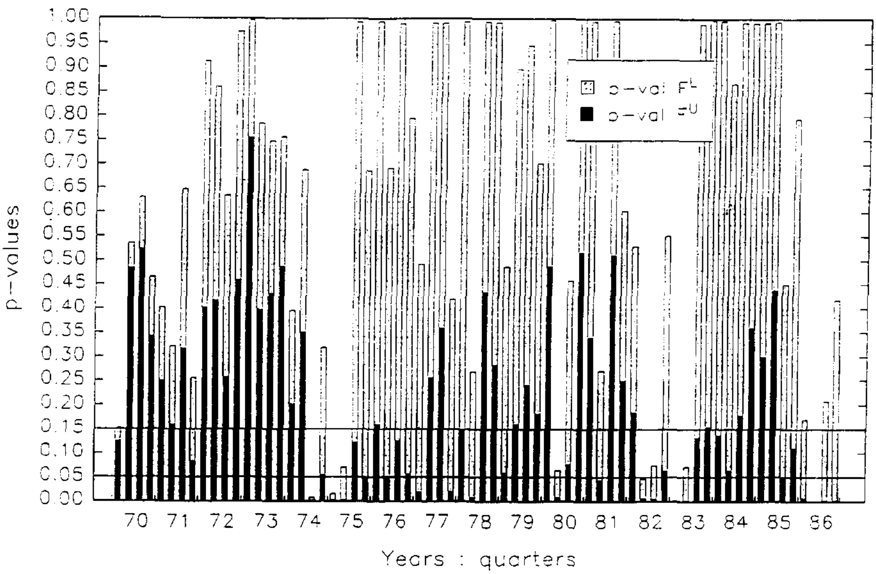

Fig. 1. Individual predictive tests (exact); U.S. investment data, 1970:1–1986:4.

while the CUSUM-of-squares and predictive tests provide rather strong evidence against it. Further, there are five individual predictive test statistics (out of 68) with p-values of F_L lower than 0.05 (1974:3, 1975:1, 1982:2, 1983:1, 1986:2), indicating clusters of low p-values near the end of 1974, in 1982–83, and in 1986. These results suggest relatively short-lived deviations from the model although not permanent structural changes.

8. Conclusion

In this paper, we have described how finite-sample structural change tests can be obtained for a linear regression model with one lagged dependent variable and normal disturbances. The latter are based on combining three distinct techniques: first, using an extended regression, we build an exact confidence set for the autoregressive parameter λ, which is valid at least under the null hypothesis of no structural change; second, after selecting a structural change test whose level can be established for any given $\lambda = \lambda_0$ (which can typically be done by adapting a structural change test designed for static linear regression), we use a 'union-intersection' technique to combine these 'conditional tests' (for given $\lambda = \lambda_0$) with the confidence set for λ, and so produce valid 'unconditional' tests; thirdly, when it is difficult to evaluate analytically the null distribution of a test statistic (for given $\lambda = \lambda_0$) but the latter can be simulated, the test is replaced by a 'randomized' (or Monte Carlo) analogue which remains exact irrespective of the number N of replications and becomes equivalent to the original nonrandomized test as $N \to \infty$. The tests considered above include extensions of analysis-of-covariance (for β and λ), predictive, CUSUM and CUSUM-of-squares tests. The feasibility of the approach suggested was also illustrated with both artificial data and a dynamic trend model for real gross domestic investment in the U.S. The artificial data suggest that alternative tests react differently to various structural-change alternatives, the AOC-β test and (to a lesser extent) the CUSUM test being more sensitive to permanent shifts in coefficients, while the predictive and CUSUM-of-squares tests can detect more easily transitory shifts. Individual predictive tests also provide useful information on the form and timing of structural changes. The empirical results on the investment equation indicate the presence of some form of structural change but of a transitory nature.

It is important to note that the general approach used here to obtain structural change tests for a dynamic linear regression is not limited to the particular tests described above: about any test designed for a static linear regression can be extended to the case of a first-order dynamic linear regression in the same way. Further, in addition to being exact for any full-column rank regressor matrix X (provided it is independent of the disturbance vector), the procedures proposed in this paper remain 'asymptotically valid' in the usual sense (i.e., the probability of type I error does not exceed the stated level as the sample size goes to infinity) under various assumptions of stochastic regressors and nonnormal disturbances, provided the structural change tests are themselves asymptotically valid for given $\lambda = \lambda_0$ under such assumptions. This can be shown easily by an argument similar to the one in Dufour and Kiviet (1993).

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