Exact and asymptotic tests for possibly non-regular hypotheses on stochastic volatility models [*]

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ABSTRACT

We study the problem of testing hypotheses on the parameters of one- and two-factor stochastic volatility models (SV), allowing for the possible presence of nonregularities such as singular moment conditions and unidentified parameters, which can lead to non-standard asymptotic distributions. We focus on the development of simulation-based exact procedures – whose level can be controlled in finite samples – as well as on large-sample procedures which remain valid under non-regular conditions. We consider Wald-type, score-type and likelihood-ratio-type tests based on a simple moment estimator, which can be easily simulated. We also propose a $C(\alpha)$-type test which is very easy to implement and exhibits relatively good size and power properties. Besides usual linear restrictions on the SV model coefficients, the problems studied include testing homoskedasticity against a SV alternative (which involves singular moment conditions under the null hypothesis) and testing the null hypothesis of one factor driving the dynamics of the volatility process against two factors (which raises identification difficulties). Three ways of implementing the tests based on alternative statistics are compared: asymptotic critical values (when available), a local Monte Carlo test (or parametric bootstrap) procedure, and a maximized Monte Carlo (MMC) procedure. The size and power properties of the proposed tests are examined in a simulation experiment. The results indicate that the $C(\alpha)$-based tests have the best size and power properties, while Monte Carlo tests are much more reliable than those based on asymptotic critical values. Further, in cases where the parametric bootstrap appears to fail (for example, in the presence of identification problems), the MMC procedure easily controls the level of the tests. Moreover, MMC-based tests exhibit good power performance despite the conservative feature of the procedure. Finally, we present an application to a time series of returns on the Standard and Poor's Composite Price Index.

Key words: testing; exact test; Monte Carlo test; maximized Monte Carlo test; Wald test; LR test; LM test; $C(\alpha)$ test; homoskedasticity; stochastic volatility; two-factor volatility; identification; singular moment conditions.

JEL classification: C1, C12, C13, C15, C32, G1.

RÉSUMÉ

Dans ce texte, nous étudions des tests d'hypothèses sur les paramètres de modèles de volatilité stochastique (SV) à un ou deux facteurs, en permettant la présence de non-régularités, tels que la singularité locale des conditions de moment définissant l'estimateur ou encore des paramètres de nuisance non-identifiés, ce qui peut conduire à une théorie distributionnelle non standard. Nous développons des procédures exactes dont la taille peut être contrôlée pour une taille donnée d'échantillon, ainsi que des tests justifiés par des arguments asymptotiques, lesquels sont à la fois simples du point de vue numérique et relativement fiables sur de petits échantillons. Nous considérons des critères de types Wald, score et quotient de vraisemblance fondés sur un estimateur des moments (et non sur le maximum de vraisemblance) qui est simple du point de vue numérique. Nous proposons aussi un test de type $C(\alpha)$ qui est très facile à utiliser et qui affiche de bonnes propriétés de niveau et de puissance. Outre des tests de restrictions linéaires sur les coefficients du modèle de volatilité stochastique, les problèmes étudiés comprennent des tests d'homoscédasticité (contre un modèle de volatilité stochastique) et des tests de l'hypothèse nulle d'une volatilité à un facteur contre une volatilité à deux facteurs, lesquels soulèvent des problèmes de singularité locale et d'identification. Nous comparons trois variantes différentes de chaque test suivant que l'on utilise des points critiques asymptotiques standards, une procédure de test de Monte Carlo (ou bootstrap paramétrique) et une procédure de test de Monte Carlo maximisé (MMC). Le niveau et la puissance des procédures proposées sont étudiées par simulation. Les résultats soulignent la supériorité du test $C(\alpha)$ dans les cas réguliers, à la fois pour le niveau et la puissance, tandis que les tests de Monte Carlo s'avèrent plus fiables que leurs homologues asymptotiques. En outre, dans des situations où le bootstrap paramétrique ne parvient pas à contrôler le niveau (par exemple, en présence de problèmes d'identification), la procédure MMC contrôle facilement le niveau des tests. De plus, les tests fondés sur la procédure MMC affichent une bonne puissance bien que cette méthode soit conservatrice par construction. Finalement, nous présentons une application à une série de rendements quotidiens de l'indice boursier du Standard and Poor's.

Mots clé: test d'hypothèse; test exact; tests de Monte Carlo; test de Monte Carlo maximisé; test de Wald; test du score; test du quotient de vraisemblance; test $C(\alpha)$; volatilité stochastique; volatilité à deux facteurs; identification; conditions de moments singulières.

JEL classification: C1, C12, C13, C15, C32, G1.

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1. Introduction

Modelling conditional heteroskedasticity is one of the central problems of financial econometrics. The two main families of models for that purpose consist of GARCH-type processes, originally introduced by Engle (1982), and stochastic volatility (SV) models proposed by Taylor (1986, 1994). Even though GARCH-type models are more widely used than SV models, the latter may be preferable for several reasons. First, SV models are directly connected to diffusion processes used in theoretical finance and allow for a volatility process that does not depend on observable variables. Second, as pointed out by Carnero, Peña and Ruiz (2004), kurtosis, volatility shock persistence and serial correlation of squared variables differ markedly in GARCH and in autoregressive SV models (ARSV). This difference may explain why the estimated persistence is usually higher in GARCH than in Gaussian ARSV models, and why GARCH models usually require leptokurtic conditional distributions. Third, from a theoretical viewpoint, the SV model naturally arises from appropriate distributional assumptions, such as a distribution mixture assumption on the joint distribution of daily security returns and trading volumes; see Clark (1973), Morgan (1976), Tauchen and Pitts (1983), and Fridman and Harris (1998). Fourth, there is evidence that SV models offer increased flexibility over the GARCH family; see Geweke (1994) and Fridman and Harris (1998).

GARCH models, on the other hand, are relatively easy to estimate and remain much more popular; for reviews, see Gouriéroux (1997) and Palm (1996). In particular, evaluating the likelihood function of GARCH models is simple compared to SV models, for which it is quite difficult to obtain a likelihood in closed form; see Shephard (1996), Mahieu and Schotman (1998) and the review of Ghysels, Harvey and Renault (1996). This is a general feature of most nonlinear latent variable models, because the latent variables must be integrated out of the joint density for the observed and latent processes, leading to an integral of high dimensionality. As a result, maximum likelihood (ML) methods are prohibitively expensive from a computational viewpoint, and alternative methods appear to be required for applying such models. This fundamental difficulty may have prevented the widespread use of SV models and has made GARCH the model of choice in practice.

Much progress has been achieved on the estimation of nonlinear latent variable models in general [see Fiorentini, Sentana and Shephard (2004)] and SV models in particular. Four main types of methods have been proposed for that purpose: quasi maximum likelihood approaches [Nelson (1988), Harvey, Ruiz and Shephard (1994), Ruiz (1994)], generalized method of moments (GMM) procedures [Melino and Turnbull (1990), Andersen and Sørensen (1996)], simulation-based estimation techniques, and Bayesian methods. Simulation-based estimation, which has become more attractive due to increasing computer power, comprises: (1) indirect inference [Gouriéroux, Monfort and Renault (1993)] which has been used to estimate SV models by Monfardini (1998) and Pastorello, Renault and Touzi (2000); (2) the efficient method of moments [Gallant and Tauchen (1996)] applied to SV models by Andersen, Chung and Sørensen (1999) and Chernov, Gallant, Ghysels and Tauchen (2003); (3) simulated maximum likelihood (SML), which can be implemented in SV models using importance sampling [see Danielsson and Richard (1993), Danielsson (1994)]. Bayesian techniques can also be applied in this context through computer-intensive methods, such as Markov Chain Monte Carlo (MCMC) methods [see Chib (2001, 2004)] and Gibbs sampling with the Metropolis-Hastings algorithm [see Chib and Jeliazkov (2005)], and appear to yield relatively good results; see Jacquier, Polson and Rossi (1994), Kim and Shephard (1994), Kim, Shephard and Chib (1998), Chib, Nardari and Shephard (2002) and Wong (2002*b*, 2002*a*).

All these studies focus on the estimation of SV models. Test problems have received much less attention. The available results on hypothesis testing for such models are rather incomplete and scattered. These include: GMM-based t-type tests on individual coefficients [Andersen and Sørensen (1996), Andersen et al. (1999)], and various specification tests such as tests for goodness of fit, diagnostic checking and model comparison [see Andersen and Sørensen (1996), Gallant, Hsieh and Tauchen (1997), Kim et al. (1998), Monfardini (1998), Andersen et al. (1999), and Fleming and Kirby (2003)]. A systematic discussion of hypothesis testing on SV model coefficients does not appear to be available. Further, even in parametric SV models, all the available test procedures are based on large-sample approximations and do not address non-regular problems that show up naturally in this context, such as testing the hypothesis of homoskedasticity against a SV model, or testing the hypothesis of a one-factor SV model against a two-factor SV model.

In this paper, we focus on hypothesis testing in parametric SV models. Our main objective is to develop both exact tests as well as asymptotically justified procedures that are markedly more reliable than those based on usual large-sample approximations, especially in the presence of non-regularities and non-standard asymptotic distributions. The proposed procedures are also designed to be computationally manageable.

Exploiting the fact that many SV models are parametric models involving only a finite number of unknown parameters, our basic outlook is to develop finite-sample simulation-based procedures as opposed to procedures based on establishing asymptotic distributions. For that purpose, we rely on extensions of the basic idea of Monte Carlo (MC) tests originally proposed by Dwass (1957) and Barnard (1963). When the distribution of a test statistic does not depend on (unknown) nuisance parameters, the technique of MC tests yields an exact test provided one can generate a few i.i.d. (or exchangeable) replications of the test statistic under the null hypothesis; for example, 19 replications are sufficient to get a test with level 0.05; see Dufour and Khalaf (2001). This technique can be extended to test statistics which depend on nuisance parameters by considering maximized Monte Carlo (MMC) tests; see Dufour (2005). MMC tests yield exact tests whenever the distribution of the test statistic can be simulated as a function of the nuisance parameters: no additional assumption on its distribution is needed. Further, computationally more tractable versions of this procedure, such as MMC tests on consistent set estimators of model nuisance parameters, provide asymptotically valid tests irrespective of the presence of non-regularities and non-standard asymptotic distributions, such as those associated with identification problems. Parametric bootstrap tests may also be interpreted as degenerate MMC tests, where the simulated p-value function is evaluated at a single nuisance-parameter point estimate. However, the asymptotic validity of the parametric bootstrap method requires stronger assumptions than the MMC procedure and it may fail to control the level of the test even asymptotically, especially in non-regular problems (where the MMC procedure remains valid).

Even though the general approach proposed here can be applied to a wide array of setups and relatively general SV models, we focus here on a relatively simple log-normal SV model of order one with an autoregressive mean, which has been widely studied in the SV literature (usually in a more restricted form); see Harvey et al. (1994), Jacquier et al. (1994), Danielsson (1994), Gallant et al. (1997) and Tauchen (1997). Further, for the sake of numerical tractability, we consider tests based on a simple two-step moment estimator which is available in closed form. This estimator is studied in detail in Dufour and Valéry (2005).

To be more specific, the contributions of the paper can be summarized as follows. *First*, we implement and compare the three standard test statistics, *i.e.* Wald-type, score-type and likelihood-ratio-type tests based on the computationally simple moment estimator available in Dufour and Valéry (2005). Further, we propose a $C(\alpha)$-type test [see Neyman (1959), Smith (1987), Dagenais and Dufour (1991)] which turns out to be relatively easy to implement in our framework and exhibits remarkably good size and power properties. Under standard regularity conditions, these test criteria follow asymptotic chi-square distributions under the null hypothesis. This holds, in particular, for linear hypotheses on the coefficients of the SV models and various (sufficiently smooth) nonlinear hypotheses. However, in view of the fact that the asymptotic distribution may be quite unreliable in finite samples (a fact documented in a simulation study), we suggest that such tests be implemented using MMC techniques (which are provably valid without further regularity conditions) and parametric bootstrapping. We also compare the relative performance of the different test criteria.

Second, we study in greater detail three relatively important special hypotheses in the context of the SV model, namely: (1) the hypothesis of non persistence in volatility (against persistence in volatility); (2) homoskedasticity (against the SV alternative); (3) one-factor SV against a two-factor SV.

The *first problem* is a regular one for which standard asymptotic distributions are applicable. Through simulation evidence, however, we find that asymptotic critical values can lead to sizeable over-rejection rates even with a fairly large sample. In contrast, the bootstrap procedure controls test level relatively well (at least for samples larger than 200), while the MMC method controls test level in all cases.

The *second problem* (testing homoskedasticity) is, of course, an important pre-test before trying to include a latent factor to drive the dynamics of the volatility process which makes its estimation much more complicated. However, moment conditions become locally singular in this case so that standard regularity conditions are not anymore applicable. Further, score-type test criteria [LM and $C(\alpha)$] and Wald-type are no longer computable in this case – at least without modification – so that they cannot be used. By contrast, bootstrap and MMC versions of LR-type tests appear to work well in this case.

The *third problem* (testing one factor against two-factor SV) is motivated by the fact that standard SV models do not capture important features of asset returns distribution such as *tail thickness*; see Chernov et al. (2003) and Durham (2004*a*, 2004*b*). As a solution, a second factor in the volatility dynamics may account for tail behavior. Eraker, Johannes and Polson (2003) proposed to model the same feature by introducing a jump component to the SV factor. Testing one factor against two-factor SV introduces an unidentified parameter under the null hypothesis [as in Hansen (1996), Andrews (2001) and Dufour, Khalaf, Bernard and Genest (2004, section 3.2)], so that standard asymptotic regularity conditions do not hold. Here score-type criteria are not applicable because covariance matrices are singular and Wald-type tests become utterly unreliable [see Dufour (1997, 2003)]. Even bootstrapping appears to fail in this case. In contrast, we find that MMC-based LR-type tests work well for that problem. It is also interesting to note that developing and justifying solutions such as those proposed by Hansen (1996) and Andrews (2001) would require considerable additional theoretical work. In contrast, the MMC approach works transparently.[1]

Fourth, we perform a Monte Carlo study to compare the finite-sample properties of the various procedures considered. We make two important observations: (1) in regular test problems, $C(\alpha)$ and LR-type tests exhibit good performance, especially when they are implemented in a simulated approach (bootstrap or MMC); (2) in non-regular problems, the only procedure which is both widely applicable and allows one to control test level is the MMC-based LR-type test.

Fifth, the proposed procedures are applied to the Standard and Poor's Composite Price Index. For this series, we find evidence that stochastic volatility is present through a one-factor specification with strong persistence.

The paper is organized as follows. Section 2 sets the framework underlying the one-factor and two-factor SV models and reviews the estimation procedure used to implement the tests. The test criteria considered and the associated confidence sets are discussed in Section 3. In Section 4, we examine why some basic problems in this setup, such as testing homoskedasticity against SV or testing one-factor SV against two-factor SV, lead to non-regularities. In Section 5 we review the technique of Monte Carlo tests. Simulation results are presented in Section 6, while empirical results on the Standard and Poor's Composite Price Index 500 return series appear in Section 7. We conclude in Section 8.

2. Framework

2.1. One-factor SV model

The basic form of the stochastic volatility model we study here comes from Gallant et al. (1997). Let us denote by y_t the variable of interest. For example, y_t can denote the first difference over a short time interval, a day for instance, of the log-price of a financial asset traded on security markets.

Assumption 2.1 *The process* $\{y_t : t \in \mathbb{N}\}$ *follows a stochastic volatility model of the type:*

$$y_t - \mu_y = \sum_{i=1}^{L_y} c_i(y_{t-i} - \mu_y) + u_t \ , \tag{2.1}$$

$$u_t = \exp(w_t/2) r_y z_t \ , \tag{2.2}$$

$$w_t - \mu_w = \sum_{j=1}^{L_w} a_{wj}(w_{t-j} - \mu_w) + r_w v_t \ , \tag{2.3}$$

[1]Related work on assessing the number of factors in a GARCH-type model may be found in Lanne and Saikkonen (2002) and Quintos (2005). Lanne and Saikkonen (2002) derived rank tests for the number of factors in an orthogonal GARCH system introduced by Alexander (2001) as a generalization of the GARCH factor model [Engle (1984), Engle, Ng and Rothschild (1990)]; see also van der Weide (2002) and Vrontos, Dellaportas and Politis (2003). More recently, Quintos (2005) extended Lanne and Saikkonen's (2002) factor tests by allowing k conditionally heteroskedastic factors and $p - k$ less persistent factors in a p-variate system.

where μ_y, $\{c_j\}_{j=1}^{L_y}$, r_y, μ_w, $\{a_{wj}\}_{j=1}^{L_w}$ and r_w are unknown parameters and $s_t = (y_t, w_t)'$ is initialized from its stationary distribution.

In the above model, (2.1) is the mean equation, while (2.3) is the volatility equation. We shall call the model represented by (2.1)-(2.3) the stochastic volatility model of order L_w with autoregressive mean of order L_y [ARSV(L_y, L_w) for short]. The lag lengths of the autoregressive specifications used in the literature are typically short. Usual configurations are $(L_y, L_w) = (0, 1), (1, 1)$ or $(2, 2)$; see Andersen and Sørensen (1996), Gallant et al. (1997) and Andersen et al. (1999). An important special case of (2.1) - (2.3) consists in setting $\mu_w = 0$, $c_j = a_{wj} = 0$, $\forall j \geq 2$, and $\delta = (c, \theta')'$ with $\theta = \theta_1$, where $\theta_1 = (a_w, r_y, r_w)'$. We then have:

$$y_t - \mu_y = c(y_{t-1} - \mu_y) + u_t \ , \quad |c| < 1 \ , \tag{2.4}$$

$$u_t = [r_y \exp(w_t/2)]z_t \ , \tag{2.5}$$

$$w_t = a_w w_{t-1} + r_w v_t \ , \quad |a_w| < 1 \ . \tag{2.6}$$

Assumption 2.2 *The vectors $(z_t, v_t)'$, $t \in \mathbb{N}$ are i.i.d. according to a $N(0, I_2)$ distribution.*

Assumption 2.3 *The process $s_t = (y_t, w_t)'$ is strictly stationary.*

The ARSV(L_y, L_w) process is Markovian of order $L_s = \max(L_y, L_w)$. Let us denote by

$$\delta = (\mu_y, c_1, \ldots, c_{L_y}, r_y, \mu_w, a_{w1}, \ldots, a_{wL_w}, r_w)' \tag{2.7}$$

the parameter vector of the model. Here $\{y_t\}$ is observed, while $\{w_t\}$ is a latent variable. Accordingly, the joint density of the observation vector $y_{(T)} = (y_1, \ldots, y_T)$ is not available in closed form, for it requires evaluating an integral with dimension equal to the whole path of the latent volatilities. Let

$$F(y_1, \ldots, y_T) = \mathsf{P}[Y_1 \leq y_1, \ldots, Y_T \leq y_T|\delta] \equiv F_0(y_{(T)}|\delta)$$

denote its unknown distribution function.

We shall now focus on the ARSV$(1, 1)$ model. To estimate it, we consider a two-step method whose first step consists in applying ordinary least squares (OLS) to the mean equation which yields a consistent estimate of the autoregressive parameter c and of the mean parameter μ_y, denoted by $\hat{c}$, $\hat{\mu}_y$ and the residuals $\hat{u}_t \equiv u_t(\hat{c}) = y_t - \hat{\mu}_y - \hat{c}(y_{t-1} - \hat{\mu}_y)$. Then, we apply in a second step a method of moments to the residuals $\hat{u}_t$ to get the estimate of the parameter $\theta_1 = (a_w, r_y, r_w)'$ of the mean and volatility equations. Unlike the other estimators proposed in the financial literature for estimating SV models, this two-step moment estimator is easy to implement and available in closed form, an appealing feature for complicated latent variable models. Besides, its simplicity allows for simulation-based inference and will be further exploited to obtain simulated testing procedures. In the sequel we will focus on the particular case where $\mu_y = 0$ but all the results still hold in the general case.

Under the assumptions **2.1** to **2.3**, with $\mu_y = \mu_w = 0$ and $c_i = a_{wi} = 0, \ \forall i \geq 2,$ the perturbation term u_t has the following moments for positive even values of j and k:

$$\mu_k(\theta_1) \equiv \mathsf{E}(u_t^k) = r_y^k \frac{k!}{2^{(k/2)}(k/2)!} \exp\left[\frac{k^2}{8} r_w^2/(1-a_w^2)\right] , \tag{2.8}$$

$$\begin{aligned} \mu_{j,\,k}(l|\theta_1) &\equiv \mathsf{E}(u_t^j u_{t+l}^k) \\ &= r_y^{j+k} \frac{j!}{2^{(j/2)}(j/2)!} \frac{k!}{2^{(k/2)}(k/2)!} \exp\left[\frac{r_w^2}{8(1-a_w^2)}(j^2+k^2+2jka_w^{|l|})\right] . \end{aligned} \tag{2.9}$$

The odd moments are equal to zero. In particular, for $j = 2$, $j = 4$ and $j = k = 2$ and $l = 1$, we have:

$$\mu_2(\theta_1) = \mathsf{E}(u_t^2) = r_y^2 \exp[(1/2) r_w^2/(1-a_w^2)] , \tag{2.10}$$

$$\mu_4(\theta_1) = \mathsf{E}(u_t^4) = 3 r_y^4 \exp[2 r_w^2/(1-a_w^2)] , \tag{2.11}$$

$$\mu_{2,\,2}(1|\theta_1) = \mathsf{E}[u_t^2 u_{t-1}^2] = r_y^4 \exp[r_w^2/(1-a_w)] ; \tag{2.12}$$

see Dufour and Valéry (2005). Let

$$\kappa = \frac{\mu_4(\theta_1)}{\mu_2^2(\theta_1)} \tag{2.13}$$

be the kurtosis coefficient of the process. It is easy to see that $\kappa \geq 3,$ with $\kappa > 3$ as soon as $r_w \neq 0$ (*i.e.*, when the volatility is constant). Solving the above moment equations corresponding to $j = 2$, $j = 4$ and $l = 1$ yields the following expressions: provided $\kappa > 3$,

$$a_w = \frac{\log[\mu_{2,\,2}(1|\theta_1)] + \log\left[\kappa/3\mu_2^2(\theta_1)\right]}{\log(\kappa/3)} - 1 , \tag{2.14}$$

hence

$$r_y = \frac{3^{1/4}\mu_2(\theta_1)}{\mu_4(\theta_1)^{1/4}} = \Big(\frac{3\mu_2^2(\theta_1)}{\kappa}\Big)^{1/4} , \quad r_w = \left[(1-a_w^2)\log\left(\kappa/3\right)\right]^{1/2}, \quad \text{if } \kappa > 3. \tag{2.15}$$

If $\kappa \leq 3,$ the volatility is constant and it is natural to set

$$a_w = r_w = 0 \ \text{ and } \ r_y = \sqrt{\mu_2(\theta_1)} \ \text{ if } \ \kappa \leq 3 . \tag{2.16}$$

Given the latter definitions, it is easy to compute a method-of-moment estimator for $\theta_1 = (a_w,\, r_y,\, r_w)'$ replacing the theoretical moments by sample counterparts based on the residuals $\hat{u}_t$. Let $\hat{\theta}_T$ denote the method-of-moments estimator of θ_1. Typically, $\mathsf{E}(u_t^2)$, $\mathsf{E}(u_t^4)$ and $\mathsf{E}(u_t^2 u_{t-1}^2)$ are approximated by:

$$\hat{\mu}_2 = \frac{1}{T}\sum_{t=1}^{T} \hat{u}_t^2 \quad \hat{\mu}_4 = \frac{1}{T}\sum_{t=1}^{T} \hat{u}_t^4 \ , \quad \hat{\mu}_2(1) = \frac{1}{T}\sum_{t=1}^{T} \hat{u}_t^2 \hat{u}_{t-1}^2$$

respectively. This yields the following estimators of the stochastic volatility coefficients:

$$\hat{a}_w = \begin{cases} \Delta & \text{if } \tilde{a}_w > \Delta\,, \\ \tilde{a}_w & \text{if } |\tilde{a}_w| \le \Delta\,, \\ -\Delta & \text{if } \tilde{a}_w < -\Delta\,, \end{cases} \tag{2.17}$$

$$\begin{aligned} \hat{r}_y &= \left(3\hat{\mu}_2^2/\hat{\kappa}\right)^{1/4} && \text{if } \hat{\kappa} > 3, \\ &= \hat{\mu}_2^{1/2} && \text{if } \hat{\kappa} \le 3, \end{aligned} \tag{2.18}$$

$$\begin{aligned} \hat{r}_w &= \left[(1-\hat{a}_w^2)\log\left(\hat{\kappa}/3\right)\right]^{1/2} && \text{if } \hat{\kappa} > 3, \\ &= 0 && \text{if } \hat{\kappa} \le 3, \end{aligned} \tag{2.19}$$

where $\hat{\kappa} = \hat{\mu}_4/\hat{\mu}_2^2$ and

$$\begin{aligned} \tilde{a}_w &= \left[\log[\hat{\mu}_2(1)] + \log\left(\hat{\kappa}/3\hat{\mu}_2^2\right)\right]/\log\left(\hat{\kappa}/3\right) && \text{if } \hat{\kappa} > 3, \\ &= 0 && \text{if } \hat{\kappa} \le 3. \end{aligned} \tag{2.20}$$

In (2.17), Δ is a number close to one which is used to bound the estimator away from the stationary boundary. This is important to avoid numerical instability. In the simulations and application below, we used $\Delta = 0.99,$ but a value closer to one could be considered. Under the assumptions of the model, the restriction $\hat{\kappa} \ge 3$ must hold with probability converging to one. Provided $|a_w| < \Delta,$ the estimator $\hat{\theta}_T = [\hat{a}_w,\, \hat{r}_y,\, \hat{r}_w]'$ is consistent and asymptotically normally distributed; see Dufour and Valéry (2005) for a detailed presentation of its asymptotic properties.

2.2. Two-factor SV model

A simple single-factor SV model appears to be sufficient to capture the salient properties of volatility such as randomness and persistence. It is the shape of the conditional distribution of financial returns which constitutes the problem; see Chernov et al. (2003) and Durham (2004*a*, 2004*b*). Standard SV models cannot match the high conditional kurtosis of returns (tail thickness) documented in the financial literature, for example in the case of equities. Trying to capture nonlinearities in financial returns has important implications for risk management and option pricing.

Consequently, we also consider a two-factor specification driving the dynamics of the volatility process of the following form:

$$y_t - \mu_y = c(y_{t-1} - \mu_y) + u_t\,, \quad |c| < 1\,, \tag{2.21}$$

$$u_t = [r_y \exp(w_t/2 + \eta_t/2)]z_t\,, \tag{2.22}$$

$$w_t = a_w w_{t-1} + r_w v_{1t}\,, \quad |a_w| < 1\,, \tag{2.23}$$

$$\eta_t = a_\eta \eta_{t-1} + r_\eta v_{2t}\,, \quad |a_\eta| < 1\,, \tag{2.24}$$

$(z_t,\, v_{1t},\, v_{2t})$ are i.i.d. Gaussian vectors such that $z_t \sim N(0,\, 1)$ and

$$(v_{1t},\, v_{2t}) \sim N(0,\, \Sigma_v)\,, \quad \Sigma_v = \begin{bmatrix} 1 & \rho_{12} \\ \rho_{12} & 1 \end{bmatrix}\,, \quad \mathsf{E}[(v_{1t},\, v_{2t})z_t] = 0\,. \tag{2.25}$$

We shall call the above model represented by equations (2.21)-(2.25) the autoregressive stochastic volatility model with two factors. Let $\theta_2 = (a_w,\, r_y,\, r_w,\, b_\eta,\, r_\eta,\, \rho_{12})'$ denote the parameter corresponding to the two-factor SV model. We derive the moment conditions used in a just-identified GMM framework which are stated in the proposition below.

Proposition 2.4 MOMENTS OF THE TWO-FACTOR SV PROCESS. *Under the assumptions* (2.21) *to* (2.25)*, we have for positive even values of* j *and* k :

$$\mathsf{E}(u_t^k) = r_y^k \frac{k!}{2^{(k/2)}(k/2)!} \exp\left[\frac{k^2}{8} r_w^2/(1-a_w^2) + \frac{k^2}{8} r_\eta^2/(1-a_\eta^2) + \frac{k^2}{4}\frac{r_w r_\eta \rho_{12}}{1-a_w a_\eta}\right],$$

$$\begin{aligned} \mathsf{E}[u_t^j u_{t+l}^k] \;=\; & r_y^{j+k} \frac{j!}{2^{(j/2)}(j/2)!}\frac{k!}{2^{(k/2)}(k/2)!} \exp\left[\frac{r_w^2}{8(1-a_w^2)}\big(j^2+k^2+2jka_w^{|l|}\big)\right. \\ & + \frac{r_\eta^2}{8(1-a_\eta^2)}\big(j^2+k^2+2jka_\eta^{|l|}\big) \\ & \left. + \frac{1}{8}\big(2j^2+2k^2+2jka_\eta^{|l|}+2jka_w^{|l|}\big)\frac{r_w r_\eta \rho_{12}}{1-a_w a_\eta}\right]. \end{aligned} \tag{2.26}$$

The proof of this proposition is given in Appendix A. In particular, for $j = 2, 4, 6$ and $j = k$, the above formulae yield the following moments:

$$\mathsf{E}(u_t^2) = r_y^2 \exp\left(\frac{1}{2}\frac{r_w^2}{1-a_w^2} + \frac{1}{2}\frac{r_\eta^2}{1-a_\eta^2} + \frac{r_w r_\eta \rho_{12}}{1-a_w a_\eta}\right) \equiv \mu_2(\theta_2)\,, \tag{2.27}$$

$$\mathsf{E}(u_t^4) = 3\, r_y^4 \exp\left(\frac{2r_w^2}{1-a_w^2} + \frac{2r_\eta^2}{1-a_\eta^2} + \frac{4r_w r_\eta \rho_{12}}{1-a_w a_\eta}\right) \equiv \mu_4(\theta_2)\,, \tag{2.28}$$

$$\mathsf{E}(u_t^6) = 15\, r_y^6 \exp\left(\frac{9}{2}\frac{r_w^2}{1-a_w^2} + \frac{9}{2}\frac{r_\eta^2}{1-a_\eta^2} + \frac{9r_w r_\eta \rho_{12}}{1-a_w a_\eta}\right) \equiv \mu_6(\theta_2)\,, \tag{2.29}$$

$$\mathsf{E}[u_t^2 u_{t-1}^2] = r_y^4 \exp\left(\frac{\sigma^2}{2}\right) = \mu_{2,\,2}(1|\theta_2)\,, \tag{2.30}$$

$$\mathsf{E}[u_t^4 u_{t-1}^4] = 9\, r_y^8 \exp(2\sigma^2) = \mu_{4,\,4}(1|\theta_2)\,, \tag{2.31}$$

$$\mathsf{E}[u_t^6 u_{t-1}^6] = 225\, r_y^{12} \exp\left(\frac{9}{2}\sigma^2\right) = \mu_{6,\,6}(1|\theta_2)\,, \tag{2.32}$$

where

$$\sigma^2 \equiv \mathsf{Var}(w_t + \eta_t + w_{t-1} + \eta_{t-1}) = \frac{2r_w^2}{1-a_w^2} + \frac{2r_\eta^2}{1-a_\eta^2} + \frac{4r_w r_\eta \rho_{12}}{1-a_w a_\eta} + \frac{2a_w r_w^2}{1-a_w^2}$$

$$+\frac{2a_w r_w r_\eta \rho_{12}}{1-a_w a_\eta} + \frac{2a_\eta r_w r_\eta \rho_{12}}{1-a_w a_\eta} + \frac{2a_\eta r_\eta^2}{1-a_\eta^2}\,. \tag{2.33}$$

These moment conditions constitute a just-identified GMM setup we shall use below in order to test the number of SV factors in the volatility process. The associated estimators, however, are not available in closed form, in contrast with the one-factor setup. But the moment conditions (2.27) - (2.32) yield a GMM estimator in the usual way through nonlinear optimization techniques.

It is important to note that another set of moment conditions (larger or simply different) could be used to estimate the two-factor model. This might lead to more precise estimates and eventually more powerful tests. Finding a better or "optimal" moment conditions goes beyond the scope of the present paper. But the general testing approach proposed below remains applicable if different sets of moment conditions are employed.

3. Test statistics and confidence sets

In this section, we set the framework for testing general hypotheses as $H_0(\psi_0) : F \in \mathcal{H}_0(\psi_0)$ where $\mathcal{H}_0(\psi_0)$ is a subset of all possible distributions for the SV model (2.4)- (2.6) [or (2.21)- (2.25) for the two-factor SV model], that is,

$$\mathcal{H}_0(\psi_0) \equiv \{F(\cdot) : F(y_{(T)}) = F_0[y_{(T)}|\delta] \text{ with } \psi(\theta) = \psi_0\} \tag{3.1}$$

where $\delta = (c, \theta')'$, $\psi(\theta)$ is a $p \times 1$ continuously differentiable function of θ and ψ_0 is the hypothesized value of $\psi(\theta)$, such as $\psi_0 = 0$. $H_0(\psi_0)$ is usually abbreviated as:

$$H_0(\psi_0) : \psi(\theta) = \psi_0\,.$$

Let us assume that the derivative of the constraints

$$P(\theta) = \frac{\partial \psi}{\partial \theta'}$$

has full row rank, let $\hat{\theta}$ be the unrestricted estimator and $\hat{\theta}^0$ the constrained estimator obtained by minimizing the following criterion

$$M_T^*(\theta) \equiv [\bar{g}_T(\hat{U}_T) - \mu(\theta)]'\hat{\Omega}_*^{-1}[\bar{g}_T(\hat{U}_T) - \mu(\theta)] \tag{3.2}$$

where $\bar{g}_T(\hat{U}_T)$ denotes the vector of empirical moments based on the residual vector $\hat{U}_T$ corresponding to $\mu(\theta)$. $\hat{\Omega}_*$ denotes a consistent estimator of Ω_*,

$$\Omega_* = \lim_{T\to\infty} \mathsf{E}\{T\big[\bar{g}_T(U_T) - \mu(\theta_0)\big]\big[\bar{g}_T(U_T) - \mu(\theta_0)\big]'\}\,, \tag{3.3}$$

with θ_0 denoting the true value of θ. Such an estimator $\hat{\Omega}_*$ can easily be obtained [see Newey and West (1987*b*)] using a Bartlett kernel:

$$\hat{\Omega}_* = \hat{\Gamma}_0 + \sum_{k=1}^{K(T)} \Big(1 - \frac{k}{K(T)+1}\Big)(\hat{\Gamma}_k + \hat{\Gamma}_k') \tag{3.4}$$

where

$$\hat{\Gamma}_k = \frac{1}{T} \sum_{t=k+1}^{T} [g_{t-k}(\hat{U}_T) - \mu(\tilde{\theta})][g_t(\hat{U}_T) - \mu(\tilde{\theta})]' \tag{3.5}$$

where $\tilde{\theta}$ is a consistent estimator of θ, $g_t(\hat{U}_T) = [\hat{u}_t^2,\, \hat{u}_t^4,\, \hat{u}_t^2\hat{u}_{t-1}^2]'$ for the SV model (2.4) - (2.6), and $g_t(\hat{U}_T) = [\hat{u}_t^2,\, \hat{u}_t^4,\, \hat{u}_t^6,\, \hat{u}_t^2\hat{u}_{t-1}^2,\, \hat{u}_t^4\hat{u}_{t-1}^4,\, \hat{u}_t^6\hat{u}_{t-1}^6]'$ for the SV model (2.21) - (2.25), respectively. In a just-identified framework, the choice of weight metric $\hat{\Omega}_*^{-1}$ is irrelevant.

The Wald-type statistic is defined as

$$\xi_T^W = T[\psi(\hat{\theta}) - \psi_0]'[\hat{P}(\hat{J}'\hat{I}^{-1}\hat{J})^{-1}\hat{P}']^{-1}[\psi(\hat{\theta}) - \psi_0] \tag{3.6}$$

where $\hat{P} = P(\hat{\theta})$, $\hat{I} = I(\hat{\theta}) = \Omega_*(\hat{\theta})$, $\hat{J} = J(\hat{\theta}) = \frac{\partial \mu}{\partial \theta'}(\hat{\theta})$. The score-type statistic is based on the gradient of the objective function with respect to θ evaluated at the constrained estimator. This gradient is

$$\mathcal{D}_T = \frac{\partial \mu'}{\partial \theta}(\hat{\theta}^0)\hat{\Omega}_*^{-1}[\mu(\hat{\theta}^0) - \bar{g}_T(\hat{U}_T)] = \hat{J}_0\hat{I}_0^{-1}[\mu(\hat{\theta}^0) - \bar{g}_T(\hat{U}_T)] \tag{3.7}$$

where $\hat{I}_0 = I(\hat{\theta}^0) = \Omega_*(\hat{\theta}^0)$, $\hat{J}_0 = J(\hat{\theta}^0) = \frac{\partial \mu}{\partial \theta'}(\hat{\theta}^0)$, and the test statistic is

$$\xi_T^S = T\mathcal{D}_T'(\hat{J}_0'\hat{I}_0^{-1}\hat{J}_0)^{-1}\mathcal{D}_T = T[\mu(\hat{\theta}^0) - \bar{g}_T(\hat{U}_T)]'\hat{W}_0[\mu(\hat{\theta}^0) - \bar{g}_T(\hat{U}_T)] \,, \tag{3.8}$$

with $\hat{W}_0 = \hat{J}_0\hat{I}_0^{-1}(\hat{J}_0'\hat{I}_0^{-1}\hat{J}_0)^{-1}\hat{I}_0^{-1}\hat{J}_0'$. Finally, the difference between the restricted and unrestricted optimal values of the objective function is called the LR-type statistic:

$$\xi_T^C = T[M_T^*(\hat{\theta}^0) - M_T^*(\hat{\theta})] \,. \tag{3.9}$$

Provided

$$T[M_T^*(\theta) - M_T(\theta)] \underset{T\to\infty}{\longrightarrow} 0 \tag{3.10}$$

where

$$M_T(\theta) \equiv [\bar{g}_T(\hat{U}_T) - \mu(\theta)]'\Omega_*^{-1}[\bar{g}_T(\hat{U}_T) - \mu(\theta)] \,, \tag{3.11}$$

the three test statistics ξ_T^W, ξ_T^S and ξ_T^C follow a $\chi^2(\nu)$ distribution asymptotically under the null hypothesis (with standard regularity conditions), where ν is the number of constraints.

We also consider the $C(\alpha)$-type test statistic defined by

$$PC(\tilde{\theta}_T^0) = T[\mu(\tilde{\theta}^0) - \bar{g}_T(\hat{U}_T)]'\tilde{W}_0[\mu(\tilde{\theta}^0) - \bar{g}_T(\hat{U}_T)] \tag{3.12}$$

where

$$\tilde{W}_0 = \tilde{I}_0^{-1}\tilde{J}_0\big(\tilde{J}_0'\tilde{I}_0^{-1}\tilde{J}_0\big)^{-1}\tilde{P}_0'\big[\tilde{P}_0^{-1}\big(\tilde{J}_0'\tilde{I}_0^{-1}\tilde{J}_0\big)\tilde{P}_0'\big]^{-1}\tilde{P}_0^{-1}\big(\tilde{J}_0'\tilde{I}_0^{-1}\tilde{J}_0\big)\tilde{J}_0'\tilde{I}_0^{-1}$$

with $\tilde{J}_0 = J(\tilde{\theta}^0) = \frac{\partial\mu}{\partial\theta'}(\tilde{\theta}^0)$, $\tilde{I}_0 = I(\tilde{\theta}^0) = \Omega^*(\tilde{\theta}^0)$, and $\tilde{P}_0 = P(\tilde{\theta}^0)$. $\tilde{\theta}^0$ is any root-n consistent estimator of θ that satisfies $\psi(\tilde{\theta}^0) = 0$. Below, for the ARSV$(1, 1)$ model, $\tilde{\theta}^0$ will be obtained by imposing the constraints in the analytic expressions of the unrestricted method-of-moments estimator $\hat{\theta}$ defined by (2.17) - (2.20), yielding a consistent restricted estimator without the need to perform a nonlinear optimization. Again, under standard regularity conditions, the $C(\alpha)$-type test statistic is asymptotically distributed like a $\chi^2(\nu)$ variable under the null hypothesis; see Davidson and MacKinnon (1993, page 619) and Dufour and Trognon (2001, Proposition 3.1).

In the simulations, we will focus on parametric functions of the form

$$\psi(\theta) = (1,\, 0)\begin{pmatrix}\theta_{s1}\\ \theta_{s2}\end{pmatrix} = \theta_{s1}\,,$$

in which case the null hypothesis $H_0(\psi_0) : \psi(\theta) = \psi_0$ simplifies to $H_0(\psi_0) : \theta_{s1} = \theta_{s1}^0$. For example, we may have $\theta_{s1} \equiv a_w$, $\theta_{s1} \equiv (a_w,\, r_w)'$.

Tests may also be used to build confidence sets for model parameters. Let $S_0 = S(\psi_0,\, y_{(T)})$ note one of the four previous tests statistics computed from the sample points $y_{(T)} = (y_1,\, \ldots\,,\, y_T)$ and under the hypothesis $H_0(\psi_0) : \psi(\theta) = \psi_0$. If the acceptance region of the test for $H_0(\psi_0)$: $\psi(\theta) = \psi_0$ has the form

$$A(\psi_0) = \{y_{(T)} = (y_1,\, \ldots\,,\, y_T) \in \mathcal{Y} : S(\psi_0,\, y_{(T)}) \le c(\alpha)\} \tag{3.13}$$

where $c(\alpha)$ is the critical point for a test with level α, the corresponding confidence set is the set of values ψ_0 which are not rejected by such tests:

$$C_\psi(y_{(T)}) = \{\psi_0 : S(\psi_0,\, y_{(T)}) \le c(\alpha)\} = \{\psi_0 : G[S(\psi_0,\, y_{(T)})] \ge \alpha\}\,, \tag{3.14}$$

where $G(\cdot)$ denotes the p-value function. These sets are connected to each other by the equivalence

$$y_{(T)} \in A(\psi_0) \Leftrightarrow \psi_0 \in C(y_{(T)})\,.$$

>From the level condition,

$$\mathsf{P}_F[Y \notin A(\psi_0)] \le \alpha\,, \quad \forall F \in \mathcal{H}_0(\psi_0)\,,$$

it follows that

$$\mathsf{P}_F[Y \in A(\psi_0)] \ge 1-\alpha\,, \quad \forall F \in \mathcal{H}_0(\psi_0)\,,$$
$$\mathsf{P}_F[\psi_0 \in C(Y)] = \mathsf{P}_F[Y \in A(\psi_0)] \ge 1-\alpha\,, \quad \forall F \in \mathcal{H}_0(\psi_0)\,, \quad \forall \psi_0 \in \Psi_0,$$

and

$$\mathsf{P}_F[\psi(\theta) \in C(Y)] \ge 1-\alpha\,, \quad \text{for all } \theta,$$

which means that $C_\psi(Y)$ is a confidence set with level $1-\alpha$ for $\psi(\theta)$.

Following this methodology, we will build confidence sets for any parameter of the volatility process by finding the values of the parameter for which the p-value function is greater than or equal to α, yielding a confidence set with level $1-\alpha$.

4. Nonregular problems

We will investigate in this section two interesting test problems. The first one consists in testing the homoskedasticity hypothesis ($a_w = r_w = 0$) against the SV alternative, and the second one is to test one-factor SV ($a_\eta = r_\eta = 0$) against a two-factor SV. Although both hypotheses are quite relevant in the context of SV models, they raise statistical difficulties. Indeed, under such null hypotheses, standard regularity conditions turn out to be violated, thus making these problems non-regular, although in somewhat different ways, so that the standard distributional theory presented in Section 3 does not apply anymore.

Let us consider first the problem of testing homoskedasticity. In this case, the Jacobian of the moment conditions (*i.e.*, the derivative matrix of the moments with respect to the SV coefficients) does not have full rank when evaluated at a point that satisfies the null hypothesis. On using the analytical expressions for the derivatives of $\mu(\theta)$ with respect to $\theta = (a,\, r_w,\, r_y),$ as given in Appendix B, we see that

$$\frac{\partial\mu}{\partial\theta'} = \begin{bmatrix} 0 & 0 & 2r_y \\ 0 & 0 & 12r_y^3 \\ 0 & 0 & 4r_y^3 \end{bmatrix} \tag{4.1}$$

when $a_w = r_w = 0,$ so that $\partial\mu/\partial\theta'$ has at most rank one (instead of three in the full-rank case). An important regularity condition is violated. This entails that the score-based statistics [the score and $C(\alpha)$-type statistics] involve non-invertible matrices and are not applicable (at least, without modifications). Further, $\partial\mu/\partial\theta'$ typically has full rank when it is evaluated at a point that does not satisfy the null hypothesis, for example at an unrestricted point estimate of $\theta,$ as done when computing a Wald-type statistic. Therefore, the rank of $\partial\mu/\partial\theta'$ when evaluated at an unrestricted point estimate of θ, will generally exceeds that of its population quantity when the null $a_w = r_w = 0,$ is true. Thus, the rank condition between the population quantity ($\partial\mu/\partial\theta'(\theta_0)$) and its estimator ($\partial\mu/\partial\theta'(\hat{\theta})$) fails. This feature is then passed on the covariance matrices of the Wald statistic. Consequently, the Wald-type statistic must be modified [see Andrews (1987) and Lütkepohl and Burda (1997)] and was hence discarded here.

Second, when testing one-factor SV ($a_\eta = r_\eta = 0$) against a two-factor SV, the correlation parameter $\rho_{12} = \mathrm{corr}(v_{1t},\, v_{2t})$ becomes unidentified under the null hypothesis. Again this creates a singularity and standard regularity conditions are violated. In particular, score-type statistics are not applicable (without modification). Further, it is well known that identification failure – or conditions close to identification failure (such as weak instruments) – can make Wald-type statistics fundamentally invalid and require important adjustments to critical values used for other test statistics, such as LR-type statistics; see Dufour (1997, 2003) and Stock, Wright and Yogo (2002).Thus, the Wald-type statistics were also discarded under identification failure. In Section 6, we present simulation evidence which shows that this is indeed the case for problem at hand here, for LR-type statistics. Although adjustments similar to those considered by Hansen (1996) and Andrews (2001) may be feasible here, justifying and applying such methods here would require a considerable theoretical effort.

In this paper, we shall take a different approach based on using a method which is completely immune to possible singularities and identification problems, such as those described above, namely the technique of maximized Monte Carlo tests [Dufour (2005)]. We will now describe succinctly this method.

5. Monte Carlo tests

The technique of Monte Carlo tests was originally been proposed by Dwass (1957) for implementing permutation tests and did not involve nuisance parameters. This technique was also independently proposed by Barnard (1963) and Birnbaum (1974); for a review, see Dufour and Khalaf (2001). It has the great attraction of providing *exact* (randomized) tests based on any statistic whose finite sample distribution may be intractable but can be simulated. We briefly review the methodology of Monte Carlo tests covering both cases, first without nuisance parameters and then with nuisance parameters. The technique of Monte Carlo tests provides a simple method allowing one to replace the unknown or intractable theoretical distribution $F(y|\delta)$, where $\delta = (c, \theta')'$, by its sample analogue based on the statistics $S_1(\delta), \ldots, S_N(\delta)$ simulated under the null hypothesis. We shall now describe how MC tests can be performed in practice.

For the sake of clarity, let us first consider the case where no nuisance parameters are present.

1. Using the observed sample, we calculate the relevant statistic S_0.

2. Using draws under H_0, we generate N simulated samples $S_1, \ldots, S_N$.

3. Then we consider the following simulated survival function

$$\hat{G}_N[y;\, S(N)] = \frac{1}{N}\sum_{i=1}^{N} s(S_i - y)$$

and the associated p-value function

$$\hat{p}_N(y) = \frac{N\hat{G}_N(y) + 1}{N+1}$$

where $s(x) = 1$ if $x \geq 0$, and $s(x) = 0$ if $x < 0$. If the distribution of S is continuous and N is chosen so that $\alpha(N+1)$ is an integer, then

$$\mathsf{P}[\hat{p}_N(S_0) \leq \alpha] = \alpha \ , \ \text{under } H_0,$$

yielding an exact test.

In most econometric models, the relevant case is the one where the distribution of the test statistic depends on nuisance parameters. To deal with this complication, the MC test procedure can be modified as follows, where $\bar{\delta}$ represents the true parameter vector.

1. To test the null hypothesis
$$H_0 : \bar{\delta} \in \Omega_0 \ ,$$
we use first the observed sample to calculate the relevant statistic denoted by S_0.

2. For each $\delta \in \Omega_0$, we generate N replications of S: $S_1(\delta), \ldots , S_N(\delta)$.

3. Using these simulations we compute the corresponding simulated p-value function:
$$\hat{p}_N[y|\delta] = \frac{N\hat{G}_N[y|\delta] + 1}{N+1} \ .$$

4. The p-value function $\hat{p}_N[S_0|\delta]$ as a function of δ is maximized over the parameter values compatible with the null hypothesis (Ω_0), and H_0 is rejected if
$$\sup\{\hat{p}_N(S_0|\delta) : \delta \in \Omega_0\} \leq \alpha \, . \tag{5.1}$$
If the number of simulated statistics N is chosen so that $\alpha(N+1)$ is an integer, then we have under H_0:
$$\mathsf{P}[\sup\{\hat{p}_N(S_0|\delta) : \delta \in \Omega_0\} \leq \alpha] \leq \alpha \, , \tag{5.2}$$
that is we control for the level of the test [for a proof, see Dufour (2005)].

Because of the maximization, the critical region in (5.1) is called a *maximized Monte Carlo* (MMC) test. MMC tests provide valid inference under general regularity conditions such as almost-unidentified models or time series processes involving unit roots. In particular, even though the moment conditions defining the estimator are derived under the stationarity assumption, this does not question in any way the validity of *maximized* MC tests, unlike the parametric bootstrap whose distributional theory is based on strong regularity conditions. Only the power of MMC tests may be affected.

A simplified approximate version of the MMC procedure can alleviate the computational load of the MMC procedure whenever a consistent point or set estimate of δ is available. To do this, we shall need to reformulate the setup in order to allow for an increasing sample size.

1. To test the null hypothesis
$$H_0 : \bar{\delta} \in \Omega_0 \ , \quad \text{with } \ \Omega_0 \in \Omega, \quad \Omega_0 \neq \emptyset \, ,$$
we use first the observed sample to calculate the relevant statistic denoted by S_{T0}.

2. We consider C_T, $T \geq I_0$ a sequence of (possibly random) subsets of Ω instead of Ω_0, such that
$$\lim_{T\to\infty} \mathsf{P}[\bar{\delta} \in C_T] = 1 \text{ under } H_0. \tag{5.3}$$

3. For each $\delta \in C_T$, we generate N replications of S: $S_{T1}(\delta), \ldots, S_{TN}(\delta)$, with $T \geq I_0$.

4. Using these simulations we compute the corresponding simulated p-value function:
$$\hat{p}_{TN}[y|\delta] = \frac{N\hat{G}_{TN}[y|\delta] + 1}{N+1} \ .$$

5. The p-value function $\hat{p}_{TN}[S_{T0}|\delta]$ is maximized with respect to δ in C_T, and H_0 is rejected if
$$\sup\{\hat{p}_{TN}(S_{T0}|\delta) : \delta \in C_T\} \leq \alpha \,. \tag{5.4}$$
If the number of simulated statistics N being chosen so that $\alpha(N+1)$ is an integer, we have under H_0:
$$\lim_{T\to\infty} \mathsf{P}[\sup\{\hat{p}_{TN}(S_{T0}|\delta) : \delta \in C_T\} \leq \alpha] \leq \alpha \ , \tag{5.5}$$
i.e., we control for the level asymptotically.

In practice, it is easy to find a consistent set estimate of $\bar{\delta}$, whenever a consistent point estimate $\hat{\delta}_T$ of $\bar{\delta}$ is available. For instance, any set of the form
$$C_T = \{\delta \in \Omega \ : \|\hat{\delta}_T - \delta\| < d\} \tag{5.6}$$
with d a fixed positive constant independent of T, satisfies (5.3). It is worth noting that there is no need to maximize the p-value function with respect to the unidentified parameters under the null hypothesis (which corresponds to ρ_{12} in the two-factor SV framework). Thus, parameters which are unidentified under the null hypothesis can be set to any fixed value and the maximization be performed only over the remaining identified nuisance parameters. When there are several nuisance parameters, one can use simulated annealing [see Goffe, Ferrier and Rogers (1994)], an optimization algorithm which does not require differentiability. Indeed $\hat{G}_N[S_0|\delta]$ is step-type function which typically has zero derivatives almost everywhere, except on isolated points where it is not differentiable. For an example, where this is done using in a VAR model involving a large number of nuisance parameters, see Dufour and Jouini (2005).

Finally, if the set C_T in (5.4) is reduced to a single point estimate $\hat{\delta}_T$, *i.e.* $C_T = \{\hat{\delta}_T\}$, we get a *local MC* (LMC) test
$$\hat{p}_{TN}(S_{T0}|\hat{\delta}_T) \leq \alpha \,, \tag{5.7}$$
which can be interpreted as a *parametric bootstrap* test. Even if $\hat{\delta}_T$ is a consistent estimate of δ (under the null hypothesis), the condition (5.3) is not usually satisfied in this case, so that additional assumptions are needed to show that the parametric bootstrap procedure yields an asymptotically valid test. It is computationally less costly but clearly less robust to violations of regularity conditions than the MMC procedure; for further discussion, see Dufour (2005).

6. Simulation results

In this section, we present some simulation evidence on the finite-sample properties of the procedures described in the previous sections. In particular, we provide results on the actual level of the Wald, score, LR and $C(\alpha)$-type tests for the three main hypotheses discussed: (1) the hypothesis of non-persistence in volatility (against persistence in volatility); (2) homoskedasticity (against the SV alternative); (3) one-factor SV against a two-factor SV. Three ways of implementing the tests are considered: asymptotic critical values, parametric bootstrap, and MMC. We also present results on power for the three types of hypotheses described above.

The Wald-type statistic [defined in equation (3.6)] is evaluated at the unrestricted method-of-moments estimator $\hat{\theta}$. The score-type statistic [defined in (3.8)] is evaluated at the restricted estimator $\hat{\theta}^0$ which minimizes the criterion $M_T^*(\theta)$ in (3.2) under the constraint $a_w = 0$. The $C(\alpha)$-type statistic [defined in (3.12)] is evaluated at the restricted estimator $\tilde{\theta}^0$ of θ, where $\tilde{\theta}^0$ is obtained by setting $a_w = 0$ in the analytical expressions of the unrestricted method-of-moments estimator $\hat{\theta}$ in (2.17) - (2.20). Further, the LR-type test statistic $LR(\hat{\Omega}) \equiv \xi_T^C$ corresponds to the difference between the restricted and the unrestricted optimal values of the objective function, with the restricted objective function evaluated at $\hat{\theta}^0$ and $\hat{\Omega} \equiv \Omega(\hat{\theta})$. The weighting matrix $\hat{\Omega}$ is estimated by a kernel estimator with fixed-bandwith Bartlett kernel, where the lag truncation is set at $K = 2$ [see Newey and West (1987*a*)].

Let S denote the test statistic which alternately takes the form of one of the four test statistics mentioned, and S_0 the statistic computed from the "pseudo-true" data obtained by simulation under the true data generating process. The critical regions have the following forms:

$$\mathcal{R}_a = \{S_0 > \chi^2_\alpha(\nu)\}$$

for the standard asymptotic tests, where $\mathsf{P}[\chi^2(\nu) \geq \chi^2_\alpha(\nu)] = \alpha$ and ν is the number of constraints tested,

$$\mathcal{R}_B = \{\hat{p}_N[S_0|\hat{\delta}^0] \leq \alpha\}$$

for the bootstrap test, and

$$\mathcal{R}_{MMC} = \left\{ \sup\{\hat{p}_{TN}(S_{T0}|\delta) : \delta \in C_T\} \leq \alpha \right\},$$

where

$$\begin{aligned} \hat{p}_N[x|\delta] &= \frac{N\hat{G}_N[x|\delta] + 1}{N+1}, \\ \hat{G}_N[x;\, S(N,\, \delta)] &= \frac{1}{N}\sum_{i=1}^{N} s\big(S_i(\delta) - x\big), \end{aligned}$$

$\hat{\delta}^0$ is a consistent point restricted estimate $\delta = (c,\, \theta')'$, θ is the vector of the SV parameters [*e.g.*, $\theta = (a_w,\, r_y,\, r_w)'$ for the one-factor SV model, $\theta = (a_w,\, r_y,\, r_w,\, a_\eta,\, r_\eta,\, \rho_{12})'$ for the two-factor SV model], and C_T is a restricted consistent set estimator of δ.

For MMC tests of the non-persistence hypothesis in the single-factor SV model ($a_w = 0$), the set C_T over which we maximize the simulated p-value is:

$$C_T^{(1)} = \{(\, c,\, r_y,\, r_w) : |c - \hat{c}| \leq 0.15,\, |c| \leq 0.99,\, |r_y - \hat{r}_y^{(1)}| \leq 0.3,\, |r_w - \hat{r}_w^{(1)}| \leq 0.3\}\,. \tag{6.1}$$

where $\hat{c}$ is the least squares estimates of c [based on fitting the AR(1) model (2.1) with no drift] and $(\hat{r}_y^{(1)}, \hat{r}_w^{(1)})$ is the restricted GMM estimate of (r_y, r_w) in the one-factor model [based on minimizing $M_T^*(\theta)$ subject to the restriction $a_w = 0$]. For the homoskedasticity hypothesis ($a_w = r_w = 0$), the corresponding set is

$$C_T^{(2)} = \{(c,\, r_y) : |c - \hat{c}| \leq 0.15,\, |c| \leq 0.99,\, |r_y - \hat{r}_y^{(2)}| \leq 0.3\} \tag{6.2}$$

where $\hat{r}_y^{(2)}$ is the corresponding restricted GMM estimate of r_y [based on minimizing $M_T^*(\theta)$ subject to the restriction $a_w = r_w = 0$]. Finally, for testing the one-factor model against the two-factor model ($a_\eta = r_\eta = 0$), C_T is

$$\begin{aligned} C_T^{(3)} = \{(c,\, a_w,\, r_y,\, r_w) : |c - \hat{c}| \leq 0.15,\, |c| \leq 0.99,\, |a_w - \hat{a}_w^{(3)}| \leq 0.15,\, |a_w| \leq 0.99, \\ |r_y - \hat{r}_y^{(3)}| \leq 0.3,\, |r_w - \hat{r}_w^{(3)}| \leq 0.3\} \end{aligned} \tag{6.3}$$

where $\hat{r}_y^{(3)}$, $\hat{r}_w^{(3)}$ and $\hat{a}_w^{(3)}$ are restricted moment estimates of the two-factor model [based of the moment equations in (2.27)-(2.32)]. Since the number of nuisance parameters is relatively small, maximization was achieved through a grid search (with points separated by a distance of 0.03 for each coefficient). Note that many other restricted consistent estimates of the relevant nuisance parameters could be used to build the sets C_T.

The nominal level is $\alpha = 0.05$. The number of replications used for Monte Carlo tests is $N = 99$, while the rejection frequencies are estimated with $M = 1000$. T is the sample size of the series y_t whose data generating process is assumed to be specified as in equations (2.4)-(2.6) for the one-factor SV model and as in equations (2.21)-(2.24) for the two-factor SV model. Calculation were performed with the GAUSS software (version 3.2.37). The autoregressive parameters a_w and a_η in the autoregressive specifications for the volatility process are restricted to an interval inside $(-1,\, 1)$ to ensure stationarity.

In the power study (Section 6.2), the asymptotic critical points are *locally level-corrected*, *i.e.* the critical points are modified to ensure that the rejection frequency under the null hypothesis (for the specific nuisance parameter values considered) is equal to 0.05; the corrected critical value is obtained by simulating the test statistic under the null hypothesis with a large number of replications.[2] Corrected asymptotic critical values are estimated from a separate simulation (with 10000 replications). Bootstrap tests are level-corrected by decreasing the threshold under which the boot-

[2]We use the term "locally level-corrected" instead of "size-corrected" because a true size correction would require one to ensure that the probability of rejecting the null hypothesis under *all distributions* compatible with null hypothsesis (i.e., for *all values of the nuisance parameters*) be less than or equal to the level α. Theoretically, a complete size-correction would be the most satisfactory correction to perform for a fair comparison of all the test procedures. However, finding the appropriate size-corrected critical values requires a numerical search that could not performed in the context of the present experiment.

strap p-value must fall to ensure the that bootstrap test rejects with frequency of 0.05 under the null hypothesis; the corrected threshold is estimated from a separate simulation (with 1000 replications).

6.1. Level

We will now examine the empirical levels of the tests. The results on testing volatility non-persistence ($H_0 : a_w = 0$) are reported in Table 1. When the mean process has low persistence ($c = 0.3$), we observe few over-rejections (above the nominal level of 0.05), except for the bootstrap procedure with a low sample size. Indeed, the asymptotic critical values appear to be conservative in this case. In contrast, when mean persistence is high ($c = 0.95$), several asymptotic and bootstrap procedures exhibit notable over-rejection rates even with a sample of $T = 500$. The only procedures which do not exhibit over-rejections in the cases considered are the asymptotic $C(\alpha)$-type test and the MMC versions of all the tests. As expected from theory, the latter may be conservative.

Results on testing homoskedasticity and the one-factor hypothesis appear in Table 2. Because these hypotheses lead to locally singular moment conditions, the score and $C(\alpha)$-type tests are not applicable here, while Wald tests often depend on covariance matrices which are almost singular (generating numerically unstable behavior). So only LR-type tests are considered. We see from the results that asymptotic LR-type tests are quite conservative for the homoskedasticity hypothesis but can severely over-reject for the one-factor hypothesis. Indeed, size distortions increase with the sample size, indicating that standard critical values are not asymptotically valid. Bootstrapping appears to correct the situation for the first hypothesis, but leaves notable over-rejection rates in the second case. Of course, one cannot exclude the possibility of larger bootstrap failures for different parameter configurations. Clearly, the two types of non-regular problems studied are qualitatively different from the statistical viewpoint. Again, in all cases studied, the MMC-based tests do not exhibit over-rejection rates.

6.2. Power

We will now study the empirical powers of the tests. In Table 3, we report empirical powers for tests of $H_0 : a_w = 0$. We can see from the results that the $C(\alpha)$ and the LR-type tests have more power than the other tests. Further, the $C(\alpha)$-type test is easy to implement in this context since it does not require any optimization procedure unlike the LR and the score-type tests. Further, although the MMC-based tests may be conservative, their power is in fact quite close to the one of the other tests and even perform better, in some cases, than the level-corrected bootstrap and asymptotic tests in small samples (*e.g.* for $T = 50, 100, 200$ in Table 3). In the present situation, MMC-based tests are essentially equivalent to (infeasible) level-corrected bootstrap tests, which suggest that they may dominate size-corrected bootstrap tests (whose level would be controlled over the whole nuisance-parameter space).

We also examine in Table 4 (panel A) the power of homoskedasticity tests (against one-factor SV). All versions of the LR-type test exhibit good power – which increases with the sample size – and are very close of each other. Note that the locally level-corrected asymptotic tests are not feasible in practice (because critical values are computed using unknown parameter values under the null hypothesis).

Table 1. Empirical levels of asymptotic, bootstrap and MMC tests for non-persistence in volatility

$H_0 : a_w = 0$ (non-persistence)									
One-factor SV: $c = 0.3,\ r_y = r_w = 0.5$									
	$T = 50$			$T = 100$			$T = 200$		
	Asy	Boot	MMC	Asy	Boot	MMC	Asy	Boot	MMC
Wald	0.1	6	0.2	0.8	5.6	0.3	0.9	4.3	0.3
Score	0.5	9.1	0.2	1	9.2	0.1	2.2	8.2	0.4
LR	8.4	8.6	0.2	5.6	4.5	0.6	3.9	5.1	1.3
$C(\alpha)$	0.4	9.1	0.3	0.6	8.6	0.3	2.3	7.8	0.8
	$T = 500$			$T = 1000$			$T = 2000$		
	Asy	Boot	MMC	Asy	Boot	MMC	Asy	Boot	MMC
Wald	1.7	4.5	0.3	2.2	5.4	0.7	3.1	5.5	0.5
Score	2.8	7.6	0.8	3	5.1	1	3	2.9	0.9
LR	2.5	6.3	0.8	2.9	5.5	1	3.6	4.8	1.1
$C(\alpha)$	2.9	8.1	1.4	2.9	5.7	1.6	2.9	4	1.1
One-factor SV: $c = 0.95,\ r_y = 0.5,\ r_w = 0.9$									
	$T = 50$			$T = 100$			$T = 200$		
	Asy	Boot	MMC	Asy	Boot	MMC	Asy	Boot	MMC
Wald	2.3	11.6	2.8	4.1	10.6	1.7	7.2	10.3	3.2
Score	3.3	13.5	0.8	6.1	9.5	0.1	5.3	4.5	0.3
LR	15.3	9.7	3.2	12.9	6.9	1.2	9.2	6.5	2.7
$C(\alpha)$	3.3	14.8	3.9	3.9	11.2	2.8	4.2	6.2	2
	$T = 500$			$T = 1000$			$T = 2000$		
	Asy	Boot	MMC	Asy	Boot	MMC	Asy	Boot	MMC
Wald	9.4	9.4	3.9	10.1	5.9	2.8	8.4	5.7	2.9
Score	6.4	5.2	1.5	6.4	4.8	2	6	4	2.2
LR	9.1	8	3.4	7.7	6.1	2.8	6.9	6.3	3.2
$C(\alpha)$	5.5	7.3	3.4	5.5	5.7	2.6	5.7	4.7	2.9

Note - In this table as well as in the other tables, frequencies are reported in percentages.

Table 2. Empirical levels of asymptotic, bootstrap and MMC tests for the number of factors.

(A) $H_0 : a_w = r_w = 0$ (homoskedasticity)									
One-factor SV: $c = 0.3,\ r_y = 0.5$									
	$T = 50$			$T = 100$			$T = 500$		
	Asy	Boot	MMC	Asy	Boot	MMC	Asy	Boot	MMC
LR	8.2	7.6	0.5	6.9	7.7	2.9	1	5.2	4.8
	$T = 1000$			$T = 2000$			$T = 5000$		
	Asy	Boot	MMC	Asy	Boot	MMC	Asy	Boot	MMC
LR	0.2	5.3	4.7	0.4	4.8	3	0.3	4.6	3.6
One-factor SV: $c = 0.95,\ r_y = 0.5$									
	$T = 50$			$T = 100$			$T = 500$		
	Asy	Boot	MMC	Asy	Boot	MMC	Asy	Boot	MMC
LR	8	10.7	4.6	7	9.7	4.7	0.9	6.4	5.2
	$T = 1000$			$T = 2000$			$T = 5000$		
	Asy	Boot	MMC	Asy	Boot	MMC	Asy	Boot	MMC
LR	0.2	5.4	4.5	0.4	5.1	3.4	0.3	4.8	3.1

(B) $H_0 : a_\eta = r_\eta = 0$ (one factor)									
One-factor SV: $c = 0.95,\ r_y = 0.5,\ a_w = 0.7,\ r_w = 0.5$									
	$T = 50$			$T = 100$			$T = 500$		
	Asy	Boot	MMC	Asy	Boot	MMC	Asy	Boot	MMC
LR	3	5	1	5	3	1	9	4	0
	$T = 1000$			$T = 2000$			$T = 5000$		
	Asy	Boot	MMC	Asy	Boot	MMC	Asy	Boot	MMC
LR	21	6	1	35	15	1	63	13	1

Table 3. Powers of MMC, (level-corrected) bootstrap and (level-corrected) asymptotic tests. Hypothesis of non-persistence in volatility

One-factor SV: $c = 0.3,\ r_y = r_w = 0.5$									
$H_0 : a_w = 0$									
$H_1 : a_w = 0.8$									
	$T = 50$			$T = 100$			$T = 200$		
	Asy	Boot	MMC	Asy	Boot	MMC	Asy	Boot	MMC
Wald	12	11.6	5.8	18	12	8.6	28.6	21.8	15.6
Score	14.2	4	2.4	23	6	4.8	47.8	23	19.4
LR	10	6.4	6.2	16.2	8.6	6	35.6	24.6	17.4
$C(\alpha)$	20.6	12	13.4	31.2	18	20.4	54.2	39	40.6
	$T = 500$			$T = 1000$			$T = 2000$		
	Asy	Boot	MMC	Asy	Boot	MMC	Asy	Boot	MMC
Wald	60.6	45.6	36.4	84.4	62	54.8	95.7	84.9	83.6
Score	77	51	54	83.2	81	76.2	92.4	92.1	90
LR	69.4	52.6	49.8	87.3	80.6	73	96.5	95	92.2
$C(\alpha)$	80.8	63.4	65.8	95.3	94	92.1	98.5	98.2	97.4
$H_1 : a_w = 0.99$									
	$T = 50$			$T = 100$			$T = 200$		
	Asy	Boot	MMC	Asy	Boot	MMC	Asy	Boot	MMC
Wald	26.3	25.8	16.50	53.6	42.2	37.3	80.8	78.4	65.1
Score	21	10.4	25.2	32.6	20.2	30.3	50	37.4	41.8
LR	33	29	40	55.6	47	53.4	87.6	81.2	88
$C(\alpha)$	37.2	33.8	40	60.8	52.4	60	86	84.4	83
	$T = 500$			$T = 1000$			$T = 2000$		
	Asy	Boot	MMC	Asy	Boot	MMC	Asy	Boot	MMC
Wald	96.4	93.3	84	98.8	96.7	87.2	99	98	91
Score	58.1	50.8	51.4	78.7	91.9	85.7	93	94	91
LR	97.4	94.6	95.1	99.4	99.8	99.7	99.8	99.6	99
$C(\alpha)$	98	97.8	97.1	99.5	99	98.5	99.8	99.3	99

Table 3 (continued)

One-factor SV: $c = 0.95,\ r_y = 0.5,\ r_w = 0.9$									
$H_0 : a_w = 0$									
$H_1 : a_w = 0.8$									
	$T = 50$			$T = 100$			$T = 200$		
	Asy	Boot	MMC	Asy	Boot	MMC	Asy	Boot	MMC
Wald	23.8	26	21.2	41.8	37.8	35.8	60.4	50.8	55.8
Score	11.2	1.6	4	19.8	6	7.1	36	23	14.8
LR	20.8	31.8	32.6	42.4	38	38.6	65.4	50.4	56.1
$C(\alpha)$	37.8	28.8	33	55.4	43.2	47.8	74.8	62.4	66
	$T = 500$			$T = 1000$			$T = 2000$		
	Asy	Boot	MMC	Asy	Boot	MMC	Asy	Boot	MMC
Wald	78.6	63	70	92	83	81.8	95.8	86.6	86
Score	61.4	41.6	40	68	62.6	58.8	72.6	70.6	66
LR	86.8	70.4	75	96	91.4	91.2	98.4	95.2	95.2
$C(\alpha)$	95.8	88.8	90	98.4	97.2	97.2	98.8	98.2	98.2
$H_1 : a_w = 0.99$									
	$T = 50$			$T = 100$			$T = 200$		
	Asy	Boot	MMC	Asy	Boot	MMC	Asy	Boot	MMC
Wald	37.8	43.2	36.5	66.4	68.2	63	86.6	86.6	82.5
Score	34.2	35.4	33	49	49.5	47.5	61.7	62.3	60
LR	37.6	63	60.5	69.5	81.8	78.9	88.4	94	95
$C(\alpha)$	48.8	62.4	62.6	81.9	81.6	81	90.1	94.8	89
	$T = 500$			$T = 1000$			$T = 2000$		
	Asy	Boot	MMC	Asy	Boot	MMC	Asy	Boot	MMC
Wald	97	96.6	93.4	99.4	99.5	94.5	100	96.8	96.8
Score	80.1	81	80.2	94	93.8	93.8	99.8	99.6	99.6
LR	95.7	97.6	98.5	97	99.6	98.6	97.8	99.8	99.8
$C(\alpha)$	99	98.7	98.4	99.5	99.6	99.1	100	100	100

Note – All asymptotic tests are locally level-corrected. Bootstrap tests are locally level-corrected when the probability of type I error exceeds 0.05. Locally-level corrected tests are not feasible in practice and constitute benchmarks for assessing the performance of MMC tests.

Table 4. Empirical powers of MMC, bootstrap and (locally level-corrected) asymptotic tests. Tests for zero (homokedasticity) and one factor SV (against two-factor SV)

(A) $H_0 : a_w = r_w = 0$ (homoskedasticity)									
$H_1 : a_w = r_w = 0.5$									
One-factor SV: $c = 0.3,\ r_y = 0.5$									
	$T = 50$			$T = 100$			$T = 500$		
	Asy	Boot	MMC	Asy	Boot	MMC	Asy	Boot	MMC
LR	11.8	11.2	9.4	15.4	13.8	12	88	87	86
	$T = 1000$			$T = 2000$			$T = 5000$		
	Asy	Boot	MMC	Asy	Boot	MMC	Asy	Boot	MMC
LR	99	99.5	99.1	100	100	100	100	100	100
One-factor SV: $c = 0.95,\ r_y = 0.5$									
	$T = 50$			$T = 100$			$T = 500$		
	Asy	Boot	MMC	Asy	Boot	MMC	Asy	Boot	MMC
LR	11.8	11	9.4	16.2	14	13	87.8	84.8	85.4
	$T = 1000$			$T = 2000$			$T = 5000$		
	Asy	Boot	MMC	Asy	Boot	MMC	Asy	Boot	MMC
LR	99.5	99.4	99	100	100	100	100	100	100

(B) $H_0 : a_\eta = r_\eta = 0$ (one factor)									
Two-factor SV: $c = 0.95,\ r_y = 0.5,\ a_w = 0.7,\ r_w = 0.5,\ \rho_{12} = 0.3$									
$H_1 : a_\eta = r_\eta = 0.9$									
	$T = 50$			$T = 100$			$T = 500$		
	Asy	Boot	MMC	Asy	Boot	MMC	Asy	Boot	MMC
LR	5	8	5	10	23	8	22	30	20
	$T = 1000$			$T = 2000$			$T = 5000$		
	Asy	Boot	MMC	Asy	Boot	MMC	Asy	Boot	MMC
LR	31	41	33	39	50	36	70	76	60

Note – All asymptotic tests are locally level-corrected. Bootstrap tests are locally level-corrected when the probability of type I error exceeds 0.05. Locally-level corrected tests are not feasible in practice.

For tests of the one factor hypothesis (panel B of Table 4), the powers of LR-type tests – though low in comparison with the other hypotheses studied previously – is sizeable and increases with the sample size. For average sample sizes, the MMC-LR procedure has as much power and sometimes more power than the corresponding (infeasible) locally level-corrected asymptotic test. Thus, the only valid feasible test which guarantees to control for the level – unlike the asymptotic and bootstrap procedures – provides reasonable power.

Finally, we provide plots of power functions for $H_0 : a_w = 0$ in the one-factor SV model, where the performance of the locally level-corrected asymptotic tests (dashed line) and bootstrap tests (dotted line) are compared. The Wald, LR, score and $C(\alpha)$-type tests appear in figures 1 to 4 respectively. Once again, we observe that the $C(\alpha)$ and the LR-type tests display higher and smoother power than the Wald and the score-type tests. The score-type tests (Figure 3) behave quite poorly at the boundaries of the domain for the autoregressive parameter a_w. This bad performance may be linked to the instability of the restricted GMM estimator when the autoregressive parameter a_w is close to the boundary of its domain. In contrast, the $C(\alpha)$-type tests behave much better near the boundary, when a_w is close to 1 and to -1. The LR-type test does not seem to suffer from this drawback and displays more robustness at the boundaries of the domain of a_w than the other tests. Further, the bootstrap power functions tend to be dominated by their asymptotic counterparts. However, the asymptotic power functions should be viewed only as a infeasible benchmark which is useful for comparison purposes, since it requires values of nuisance parameters that are not available in practice.

7. Empirical application

In this section, we test the three null hypotheses studied in the simulation experiments from real data on the Standard and Poor's Composite Price Index (1928-87). We actually proceed in three steps in order to select the more suitable specification for this specific data set. First, we test for the null of homoskedasticity against an alternative of stochastic volatility. Second, we perform the test of one factor against two factors in the volatility process. And finally, we implement the test of no-persistence in the one-factor volatility process if the one-factor specification is appropriate.

7.1. Data

The data have been provided by Georges Tauchen where the efficient method of moments (EMM) have been used by Gallant et al. (1997) to fit a SV model. The data to which we fit the stochastic volatility models is a time series comprised of 16,127 daily observations, $\{\tilde{y}_t\}_{t=1}^{16,127}$, on adjusted movements of the Standard and poor's Composite Price Index, 1928-87. The raw series is the Standard and Poor's Composite Price Index (SP), 1928-87 (daily). The raw series is converted to growth rates by the transformation $100[\log(SP_t) - \log(SP_{t-1})]$, and then adjusted for systematic calendar effects, that is, systematic shifts in location and scale due to different trading patterns across days of the week, holidays, and year-end tax trading.

Figure 1. Power functions of level-corrected asymptotic parametric and bootstrap Wald-type tests of $H_0 : a_w = 0$

Model with: $r_w = 0.5,\ r_y = 0.5,\ c = 0.3,\ T = 2000$. Level $= 0.05$

The dashed line represents the asymptotic test and the dotted line the bootstrap test.

Level is 0.05. All tests are locally level-corrected.

a_w

Figure 2. Power functions of level-corrected asymptotic parametric and bootstrap LR-type tests of
$H_0 : a_w = 0$
Model with: $r_w = 0.5,\ r_y = 0.5,\ c = 0.3,\ T = 2000$
The dashed line represents the asymptotic test and the dotted line the bootstrap test.
Level is 0.05. All tests are locally level-corrected.

a_w

Figure 3. Power functions of level-corrected asymptotic parametric and bootstrap score-type tests of $H_0 : a_w = 0$

Model with: $r_w = 0.5,\ r_y = 0.5,\ c = 0.3,\ T = 2000$

The dashed line represents the asymptotic test and the dotted line the bootstrap test.

Level is 0.05. All tests are locally level-corrected.

a_w

Figure 4. Power functions of level-corrected asymptotic parametric and bootstrap $C(\alpha)$-type tests of $H_0 : a_w = 0$

Model with: $r_w = 0.5,\ r_y = 0.5,\ c = 0.3,\ T = 2000$

The dashed line represents the asymptotic test and the dotted line the bootstrap test.

Level is 0.05. All tests are locally level-corrected.

a_w

7.2. Results

The unrestricted estimated value of $(c,\,\theta)$ for the one-factor model obtained from the data is:

$$\begin{array}{rl}\left[\hat{c},\,\hat{a}_w,\,\hat{r}_y,\,\hat{r}_w\right] = & \begin{bmatrix} 0.129, & 0.926, & 0.829, & 0.427 \\ (0.007) & (8.10) & (0.829) & (8.13) \end{bmatrix}\end{array} \tag{7.1}$$

where standard errors are given in parentheses. We may conjecture that there is some persistence in the data during the period 1928-87 what is statistically checked by performing the tests below. The restricted estimated values of $(c,\,\theta)$ from the data are:

$$\begin{array}{rl}\left[\hat{c},\,\hat{a}^0_w,\,\hat{r}^0_y,\,\hat{r}^0_w\right] = & \begin{bmatrix} 0.129, & 0\,, & 0.785\,, & 1.152 \\ (0.007) & (0) & (1.95) & (1.77) \end{bmatrix}\end{array} \tag{7.2}$$

and the consistent restricted estimator derived from the closed-form expression of the unrestricted moment estimator is equal to

$$\begin{array}{rl}\left[\hat{c},\,\tilde{a}^0_w,\,\tilde{r}^0_y,\,\tilde{r}^0_w\right] = & \begin{bmatrix} 0.129, & 0\,, & 0.829\,, & 1.133 \\ (0.007) & (0) & (1.91) & (1.66) \end{bmatrix}\end{array}. \tag{7.3}$$

Note the large discrepancy between the unrestricted and restricted estimates of r_w where the restricted estimates may not be consistent if the null hypothesis $H_0 : a_w = 0$ is false.

In Table 5, we report tests based on the whole sample (16,127 daily observations on the S&P 500), which covers the market crashes of the *Black Thursday* of October 1929 and of October 1987, the Cuban Missile Crisis (October 1962) and the Arab Oil Embargo (October 1973). We then repeat the tests on there sub-samples: 1928 - 1949, 1950 - 1969 and 1970 - 1987. The *long sloped* arrow displayed on Figure 5 shows the Cuban Missile Crisis of October 1962. The outlier appearing at the end of the sample indicates the market crash of October 1987.

We can see from the results in the top panel of Table 5, that the three versions (asymptotic, bootstrap and MMC) of the LR test do reject the null hypothesis of homoskedasticity in favor of a stochastic volatility specification for the volatility process of the S&P 500 index, except for the third sub-period 1970-1987 but with p-values however very close to 0.05 (p-value=0.07).

More recently, Chernov et al. (2003) and Durham (2004*a*, 2004*b*) provide evidence that standard single-factor SV models have some difficulties to model the shape of the conditional distribution of financial returns. In particular, Chernov et al. (2003) show that two-factor SV models better accommodate richer dynamics such as the tail behavior of (conditional) return distributions and possibly capture some rapid moves in the dynamics of volatility during extreme market conditions. The first factor may act as a long-memory component, while the second factor is expected to model tail behavior. To check for that, we test the null of one factor against two factors $(H_0 : a_\eta = r_\eta = 0)$ in the bottom panel of Table 5. All versions (asymptotic, bootstrap and MMC) of the LR test do not indicate that a two-factor specification is needed for the S&P 500 index volatility. Consequently, we chose a one-factor specification for modelling the S&P 500 index volatility.

We will now study in greater detail the volatility parameter in the one-factor SV model. We first test for the null hypothesis of no-persistence in the volatility process (Table 5). All tests, asymptotic, bootstrap and MMC, reject the null hypothesis of no-persistence in the volatility for all the periods considered. Indeed, it is well known in the financial literature that financial returns

Figure 5. Daily observations on the S&P500 index.
The *long sloped* arrow shows the Cuban Missile Crisis of October 1962.

Table 5. Empirical application

S&P 500 index							
Test of homoskedasticity							
$H_0 : a_w = r_w = 0$							
Sample 1928-1987, $T = 16127$							
	Asymptotic tests	Bootstrap tests			MMC tests		
	S_0	$N = 19$	$N = 99$	$N = 999$	$N = 19$	$N = 99$	$N = 999$
LR	9.71	0.05	0.01	0.001	0.05	0.01	0.001
Sample 1928-1949, $T = 6491$							
	Asymptotic tests	Bootstrap tests			MMC tests		
	S_0	$N = 19$	$N = 99$	$N = 999$	$N = 19$	$N = 99$	$N = 999$
LR	42.58	0.05	0.01	0.001	0.05	0.01	0.001
Sample 1950-1969, $T = 5087$							
	Asymptotic tests	Bootstrap tests			MMC tests		
	S_0	$N = 19$	$N = 99$	$N = 999$	$N = 19$	$N = 99$	$N = 999$
LR	6.28	0.05	0.02	0.003	0.05	0.02	0.003
Sample 1970-1987, $T = 4549$							
	Asymptotic tests	Bootstrap tests			MMC tests		
	S_0	$N = 19$	$N = 99$	$N = 999$	$N = 19$	$N = 99$	$N = 999$
LR	2.09	0.10	0.07	0.077	0.10	0.07	0.078

S&P 500 index							
Test of one against two SV factors							
$H_0 : a_\eta = r_\eta = 0$							
Sample 1928-1987, $T = 16127$							
	Asymptotic tests	Bootstrap tests			MMC tests		
	S_0	$N = 19$	$N = 99$	$N = 999$	$N = 19$	$N = 99$	$N = 999$
LR	4.150	0.20	0.14	0.131	0.40	0.30	0.242
Sample 1928-1949, $T = 6491$							
	Asymptotic tests	Bootstrap tests			MMC tests		
	S_0	$N = 19$	$N = 99$	$N = 999$	$N = 19$	$N = 99$	$N = 999$
LR	0.436	0.95	0.89	0.882	0.95	0.92	0.883
Sample 1950-1969, $T = 5087$							
	Asymptotic tests	Bootstrap tests			MMC tests		
	S_0	$N = 19$	$N = 99$	$N = 999$	$N = 19$	$N = 99$	$N = 999$
LR	3.131	0.20	0.11	0.046	0.50	0.41	0.326
Sample 1970-1987, $T = 4549$							
	Asymptotic tests	Bootstrap tests			MMC tests		
	S_0	$N = 19$	$N = 99$	$N = 999$	$N = 19$	$N = 99$	$N = 999$
LR	1.142	0.35	0.26	0.254	0.40	0.36	0.308

Table 5 (continued)

S&P 500 index						
$H_0 : a_w = 0$						
Sample 1928-1987, $T = 16127$						
	Asymptotic tests	Bootstrap tests			MMC tests	
	S_0	$N = 19$	$N = 99$	$N = 999$	$N = 19$	$N = 99$
Wald	210.85	0.05	0.01	0.001	0.05	0.01
Score	1039.04	0.05	0.01	0.001	0.05	0.01
LR	25.49	0.05	0.01	0.001	0.05	0.01
$C(\alpha)$	854.55	0.05	0.01	0.001	0.05	0.01
Sample 1928-1949, $T = 6491$						
	Asymptotic tests	Bootstrap tests			MMC tests	
	S_0	$N = 19$	$N = 99$	$N = 999$	$N = 19$	$N = 99$
Wald	112.95	0.05	0.01	0.001	0.05	0.01
Score	269.72	0.05	0.01	0.001	0.05	0.01
LR	52.73	0.05	0.01	0.001	0.05	0.01
$C(\alpha)$	185.47	0.05	0.01	0.001	0.05	0.01
Sample 1950-1969, $T = 5087$						
	Asymptotic tests	Bootstrap tests			MMC tests	
	S_0	$N = 19$	$N = 99$	$N = 999$	$N = 19$	$N = 99$
Wald	93.01	0.05	0.01	0.001	0.05	0.01
Score	607.92	0.05	0.01	0.001	0.05	0.01
LR	11.95	0.05	0.01	0.001	0.05	0.01
$C(\alpha)$	304.66	0.05	0.01	0.001	0.05	0.01
Sample 1970-1987, $T = 4549$						
	Asymptotic tests	Bootstrap tests			MMC tests	
	S_0	$N = 19$	$N = 99$	$N = 999$	$N = 19$	$N = 99$
Wald	30.50	0.05	0.01	0.001	0.10	0.03
Score	391.87	0.05	0.01	0.001	0.05	0.01
LR	40.90	0.05	0.01	0.001	0.05	0.01
$C(\alpha)$	165.03	0.05	0.01	0.001	0.05	0.01

Table 6. Confidence sets

Confidence sets for a_w, $1-\alpha=0.95$		
	Asymptotic	Bootstrap
Wald	$[0.800,\ 0.999]$	$[0.748,\ 0.999]$
Score	$[-0.950,\ -0.561] \cup [0.405,\ 0.474]$	$[-0.950,\ -0.561] \cup [0.405,\ 0.476]$
	$\cup[0.713,\ 0.993]$	$\cup[0.640,\ 0.997]$
LR	$[0.810,\ 0.996]$	$[0.750,\ 0.997]$
$C(\alpha)$	$[0.713,\ 0.994]$	$[0.726,\ 0.995]$

display serial dependence in volatility. In Table 6, we also report confidence sets for the one-factor-model persistence parameter a_w obtained by testing the values of a_w between -0.999 and 0.999 [see Section 3]. The nominal coverage probability for the confidence is $1-\alpha = 0.95$. In practice, to implement the confidence sets based on the asymptotic critical point $\chi^2_{0.05}(1) = 3.84$, we keep all the values of the parameter which are not rejected at level of $\alpha = 0.05$. For the bootstrap confidence sets, we proceed as exposed in Section 5 with the observed statistic computed from the actual data. Then, we compute the rank of the observed statistic among the statistics simulated under $H_0 : a_w = a_0$ where the nuisance parameters are evaluated at the consistent estimates obtained from the real data set. For each value of a_0 varying from -0.999 to 0.999, we keep the values a_0 whose p-value is greater than 0.05. The results reported in Table 6, indicate that all confidence sets do cover the estimated value $\hat{a}_w = 0.926$ drawn from the real data. Nevertheless, the confidence sets based on the asymptotic critical point are globally shorter than the bootstrap ones which reveal more conservative in this context with the exception of the one built on the $C(\alpha)$-type test statistic whose length is nearly identical for the bootstrap and the asymptotic confidence sets. Note that the confidence sets built from the score-type test statistic consist of the union of three confidence intervals and again do not perform as well as the ones obtained through the other test statistics as already observed in the simulations. Further, the loss of precision of confidence sets deduced from the tests statistics involving only on the restricted estimates may be due to the fact the restricted estimates may lead to inconsistent estimates when the null being tested is false. At the opposite, the most precise confidence sets are those based on the LR-type statistic.

To summarize, the results presented here indicate that a one-factor model with strong volatility persistence may be appropriate for the S&P 500 index data studied here.

8. Conclusion

In this paper, we have provided finite-sample procedures for testing hypotheses on the parameters of SV models, allowing for the possible presence of non-regular testing problems (underidentification, singularity issues) that can lead to non-standard asymptotic distributional theory. Besides usual linear restrictions on SV coefficients, the problems studied include testing homoskedasticity against a SV alternative and testing the one-factor SV against two-factor SV, which raises singularity and identification difficulties. In addition to the three standard tests, we proposed to use $C(\alpha)$-type tests which are relatively easy to apply and displays good size and power properties (when applicable).

In order to deal with the potential unreliability of asymptotic critical values and bootstrapping, especially in cases where standard regularity conditions fail, we showed that the MMC test approach provides a transparent way of dealing with such difficulties, yielding both exact or asymptotically valid tests without the need to establish a specific distributional theory. In some cases, the MMC method is the only one that yields provably valid tests. Further, in simulations, we observed that the MMC method can indeed be implemented to produce inference on SV models, works very well from the viewpoint of controlling test levels, and does not entail a considerable power loss with respect to alternative (usually infeasible) level-corrected asymptotic or bootstrap approaches.

These testing procedures can easily be extended to accommodate richer dynamics such as fat-tailed and/or correlated errors [see Harvey and Shephard (1996), Jacquier, Polson and Rossi (2004), Omori, Chib, Shephard and Nakajima (2004)], or multivariate stochastic volatility structures [see Harvey et al. (1994), Chib, Nardari and Shephard (2002b), Jacquier, Polson and Rossi (1999)]. They could also be extended to a continuous-time specification of the SV model since all the moments are already derived in Meddahi (2002).

A. Appendix: Proofs

PROOF OF PROPOSITION **2.4** If $U \sim \mathrm{N}(0,\,1)$ then $E(U^{2p+1}) = 0, \forall\, p \in \mathbb{N}$ and $E(U^{2p}) = (2p)!/[2^p p!]\ \forall\, p \in \mathbb{N}$ [see Gouriéroux and Monfort (1995, Volume 2, page 518)]. Hence:

$$\begin{aligned}
\mathsf{E}(u_t^k) &= r_y^k \mathsf{E}(z_t^k)\mathsf{E}\exp[k(w_t/2+\eta_t/2)] \\
&= r_y^k \frac{k!}{2^{(k/2)}(k/2)!}\exp\left[\frac{k^2}{8}\Big(\mathsf{Var}(w_t)+\mathsf{Var}(\eta_t)+2\mathsf{Cov}(w_t,\eta_t)\Big)\right] \\
&= r_y^k \frac{k!}{2^{(k/2)}(k/2)!}\exp\left[\frac{k^2}{8}r_w^2/(1-a_w^2)+\frac{k^2}{8}r_\eta^2/(1-a_\eta^2)+\frac{k^2}{4}\frac{r_w r_\eta \rho_{12}}{1-a_w a_\eta}\right] \quad \text{(A.1)}
\end{aligned}$$

where the second equality uses the definition of the Gaussian Laplace transform of $w_t \sim \mathrm{N}[0,\, r_w^2/(1-a_w^2)]$ (of η respectively) and of the moments of z_t. Further, using

$$\mathsf{E}(w_t)=0, \quad \mathsf{Var}(w_t)=r_w^2/(1-a_w^2)\,, \tag{A.2}$$

$$\mathsf{E}(\eta_t)=0,\ \mathsf{Var}(\eta_t)=r_\eta^2/(1-a_\eta^2)\,, \tag{A.3}$$

$$\mathsf{Cov}(w_t,w_{t+l})=a_w^{|l|}r_w^2/(1-a_w^2)\ \mathsf{Cov}(\eta_t,\eta_{t+l})=a_\eta r_\eta^2/(1-a_\eta^2)\,, \tag{A.4}$$

$$\mathsf{Cov}(w_t,\eta_t)=\frac{r_w r_\eta \rho_{12}}{1-a_w a_\eta}\,, \tag{A.5}$$

$$\mathsf{Cov}(w_t,\eta_{t+l})=a_\eta^{|l|}\mathsf{Cov}(w_t,\eta_t),\ \mathsf{Cov}(w_{t+l},\eta_t)=a_w^{|l|}\mathsf{Cov}(w_t,\eta_t) \tag{A.6}$$

we obtain the cross-moments:

$$\begin{aligned}
\mathsf{E}[u_t^j u_{t+l}^k] &= \mathsf{E}\{r_y^{j+k} z_t^j z_{t+l}^k \exp[j(\frac{w_t}{2}+\frac{\eta_t}{2})+k(\frac{w_{t+l}}{2}+\frac{\eta_{t+l}}{2})]\} \\
&= r_y^{j+k}\mathsf{E}(z_t^j)\mathsf{E}(z_{t+l}^k)\mathsf{E}\left\{\exp[j(\frac{w_t}{2}+\frac{\eta_t}{2})+k(\frac{w_{t+l}}{2}+\frac{\eta_{t+l}}{2})]\right\} \\
&= r_y^{j+k}\frac{j!}{2^{(j/2)}(j/2)!}\frac{k!}{2^{(k/2)}(k/2)!}\exp\left\{\frac{1}{2}\mathsf{Var}\left[\frac{j}{2}(w_t+\eta_t)+\frac{k}{2}(w_{t+l}+\eta_{t+l})\right]\right\} \\
&= r_y^{j+k}\frac{j!}{2^{(j/2)}(j/2)!}\frac{k!}{2^{(k/2)}(k/2)!}\exp\left\{\frac{1}{2}\left[\frac{j^2}{4}\mathsf{Var}(w_t)+\frac{j^2}{4}\mathsf{Var}(\eta_t)+\frac{k^2}{4}\mathsf{Var}(w_{t+l})\right.\right. \\
&\quad +\frac{k^2}{4}\mathsf{Var}(\eta_{t+l})+\frac{2j^2}{4}Cov(w_t,\eta_t)+\frac{2jk}{4}Cov(w_t,w_{t+l}) \\
&\quad +\frac{2jk}{4}Cov(w_t,\eta_{t+l})+\frac{2jk}{4}Cov(\eta_t,w_{t+l})+\frac{2jk}{4}Cov(\eta_t,\eta_{t+l}) \\
&\quad \left.\left.+\frac{2k^2}{4}Cov(w_{t+l},\eta_{t+l})\right]\right\} \\
&= r_y^{j+k}\frac{j!}{2^{(j/2)}(j/2)!}\frac{k!}{2^{(k/2)}(k/2)!}\exp\left[\frac{r_w^2}{8(1-a_w^2)}(j^2+k^2+2jka_w^{|l|})\right. \\
&\quad +\frac{r_\eta^2}{8(1-a_\eta^2)}(j^2+k^2+2jka_\eta^{|l|})
\end{aligned}$$

$$+\frac{1}{8}[2j^2+2k^2+2jka_\eta^{|l|}+2jka_w^{|l|}]\frac{r_w r_\eta \rho_{12}}{1-a_w a_\eta}\Bigg] . \tag{A.7}$$

□

B. Appendix: Analytical derivatives of moment conditions

The analytical expressions of the derivatives of the moment conditions are given by:

$$\frac{\partial \mu_2}{\partial a_w} = \frac{a_w}{(1-a_w^2)^2} r_w^2 r_y^2 \exp[\frac{r_w^2}{2(1-a_w^2)}] ,$$

$$\frac{\partial \mu_2}{\partial r_w} = \frac{r_w}{(1-a_w^2)} r_y^2 \exp[\frac{r_w^2}{2(1-a_w^2)}] ,$$

$$\frac{\partial \mu_2}{\partial r_y} = 2r_y \exp[\frac{r_w^2}{2(1-a_w^2)}]$$

$$\frac{\partial \mu_4}{\partial a_w} = 12\frac{a_w}{(1-a_w^2)^2} r_w^2 r_y^4 \exp[\frac{2r_w^2}{(1-a_w^2)}] ,$$

$$\frac{\partial \mu_4}{\partial r_w} = 12\frac{r_w}{(1-a_w^2)} r_y^4 \exp[\frac{2r_w^2}{(1-a_w^2)}] ,$$

$$\frac{\partial \mu_4}{\partial r_y} = 12r_y^3 \exp[\frac{2r_w^2}{(1-a_w^2)}] ,$$

$$\frac{\partial \mu_{2,2}}{\partial a_w} = \frac{r_w^2}{(1-a_w)^2} r_y^4 \exp[\frac{r_w^2}{(1-a_w)}] ,$$

$$\frac{\partial \mu_{2,2}}{\partial r_w} = \frac{2r_w}{1-a_w} r_y^4 \exp[\frac{r_w^2}{(1-a_w)}] ,$$

$$\frac{\partial \mu_{2,2}}{\partial r_y} = 4r_y^3 \exp[\frac{r_w^2}{(1-a_w)}] .$$

All these derivatives evaluated at $a_w = 0$, $r_w = 0$ gives the results stated in equation (4.1).

□

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