Arbitrage pricing, weak beta, strong beta: identification-robust and simultaneous inference [a]

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ABSTRACT

Factor models based on Arbitrage Pricing Theory (APT) characterize key parameters *jointly* and nonlinearly, which complicates identification. We propose simultaneous inference methods which preserve equilibrium relations between all model parameters including *ex-post* sample-dependent ones, without assuming identification. Confidence sets based on inverting joint tests are derived, and tractable analytical solutions are supplied. These allow one to assess whether traded and nontraded factors are priced risk-drivers, and to take account of cross-sectional intercepts. A formal test for traded factor assumptions is proposed. Simulation and empirical analyses are conducted with Fama-French factors. Simulation results underscore the information content of cross-sectional intercept and traded factor restrictions. Three empirical results are especially noteworthy: (1) the Fama-French three factors are priced before 1970; thereafter, we find no evidence favoring any factor relative to the market; (2) heterogeneity is not sufficient to distinguish priced momentum from profitability or investment risk; (3) after the 1970s, factors are rejected or appear to be weak, depending on intercept restrictions or test portfolios.

Key words: capital asset pricing model; CAPM; Arbitrage Pricing Theory; Black; Fama-French factors; mean-variance efficiency; non-normality; weak identification; identification-robust; projection; Fieller; multivariate linear regression; uniform linear hypothesis; exact test; Monte Carlo test; bootstrap; nuisance parameters.
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1 Introduction

Arbitrage Pricing Theory (**APT**) and its concepts are core components of financial economics. Despite enduring disagreements about risk factors and the measurement of risk premiums, related factor models are workhorse tools for asset pricing; for some references which illustrate these debates, see Harvey, Liu and Zhu (2016), Gagliardini, Ossola and Scaillet (2016), Ahmed, Bu and Tsvetanov (2019), Hou, Mo, Xue and Zhang (2018), and Chib and Zeng (2019). This paper addresses an aspect of such models not broadly recognized: weak identification. If identification can be arbitrarily weak, conventional methods deliver tests and confidence intervals that are invalid even asymptotically and thereby yield misleading empirical decisions. Motivated by these considerations and the abundance of available candidate factors, this paper proposes econometric methods that: (i) reveal weak factors when present and deliver valid inference on pricing; (ii) detect misspecification including assumptions on tradable factors; and (iii) preserve APT fundamentals with traded and nontraded factors.

Our analysis is based on equilibrium specifications that characterize the risk premiums *jointly*, along with the zero-beta rate, factor expectations and the unknown factor loadings (the so-called factor *betas*). Formally, the APT stipulates that the unconditional expectation of returns, denoted thereafter as the n-dimensional vector μ_r, is linear in factor loadings:

$$\mu_r = \iota_n \gamma_c + b'\Gamma \tag{1.1}$$

where Γ is the vector of risk premiums, the scalar γ_c is the so-called cross-sectional intercept or the zero-beta rate, $b = [b_1 \cdots b_n]$ is the $q \times n$ matrix of loadings and q is the number of relevant risk factors. All of these parameters including b are unknown.

This explains why factor models based on (1.1) have traditionally been estimated using so-called two-pass methods [as reviewed *e.g.* by Shanken and Zhou (2007)], where: (i) the first pass uses time series regressions of returns on factors, in order to estimate b; and (ii) the second pass involves cross-sectional regressions of returns on the estimated b, in order to identify Γ. Consequently, measurement errors arising from estimated *betas* have long been considered as a major identification threat. A recent research strand also highlights deeper problems resulting from insignificant or homogenous *betas*; see Kan and Zhang (1999), Beaulieu, Dufour and Khalaf (2009), Kleibergen (2009), Beaulieu, Dufour and Khalaf (2013), Kan, Robotti and Shanken (2013), Gospodinov, Kan and Robotti (2014), Kleibergen and Zhan (2015, 2020), and Kleibergen, Lingwei and Zhan (2019).

More broadly, it is clear from (1.1) that Γ is not identified unless the true and unknown b matrix has full rank. Identification problems thus affect multiple parameters and may have several sources. Sorting out these multiple influences may be difficult, due to the nonlinear structure of (1.1). Instead, our aim is to present measures of estimation uncertainty that preserve the APT-based association between all model parameters including *realized* or sample dependent random ones. In particular, an alternative parameter introduced by Shanken (1985) and Shanken (1992) as the *ex-post* risk premium has recently regained interest:

$$\Gamma_* = \Gamma + \bar{\mathscr{R}} - \mu_{\mathscr{R}} \tag{1.2}$$

where $\bar{\mathscr{R}}$ is the empirical factor mean and $\mu_{\mathscr{R}}$ is its expectation; see Khalaf and Schaller (2016), Jegadeesh, Noh, Pukthuanthong, Roll and Wang (2019), and Kim and Skoulakis (2018).

Given the importance of *alphas* and *betas* for assessing the quality of an asset pricing model, we first propose simultaneous confidence intervals for (in turn) the unrestricted components of the time-series intercepts and each one of the loading vectors. Next, we construct level-correct confidence sets for the zero-beta rate and the risk premiums again viewed jointly and using traditional first-pass estimates, yet accounting for estimation error regardless of whether factor *betas* are jointly informative or heterogenous enough. In particular, these confidence sets serve to robustly assess whether candidate factors are priced risk-drivers. This approach extends the single-benchmark identification-robust method proposed by Beaulieu et al. (2013) to multivariate beta-pricing models.

In doing so, a framework is required in the presence of traded and non-traded factors. Despite well-known advantages, restricting focus to traded factors is unduly restrictive; see Shanken and Weinstein (2006), Shanken and Zhou (2007), and the above cited literature on competing risk factors; for a discussion on some advantages of traded factors, see Gospodinov, Kan and Robotti (2019), Barillas and Shanken (2017, 2018), and Pukthuanthong, Roll and Subrahmanyam (2019). Concretely, implications of traded factors have been operationalized by restrictions involving the zero-beta rate [Barone-Adesi, Gagliardini and Urga (2004), Penaranda and Sentana (2016)]. While this principle is well accepted, empirical analysts often sidestep cross-sectional intercepts [Lewellen, Nagel and Shanken (2010)], thereby forfeiting important equilibrium relations. In contrast, we provide simultaneous confidence sets with both traded and nontraded factors. In addition, and crucially, our empirical approach exploits the information content of the cross-sectional intercept to uncover links that would likely be lost when returns are considered in deviation from some asset, as in Kleibergen (2009), Kleibergen and Zhan (2015), Kleibergen et al. (2019), Kleibergen and Zhan (2020). Simultaneous inference ensures that equilibrium restrictions are jointly maintained, which as emphasized, is a fundamental equilibrium requirement. This is however not the whole story, since identification concerns provide compelling statistical rationale for simultaneous methods.

Indeed, to control statistical coverage without assuming identification, we proceed by inverting joint model tests. These include: (i) the joint regression intercept test statistic by Gibbons, Ross and Shanken (1989) and its counterparts pertaining to each factor [see *e.g.* Dufour and Khalaf (2002) and Beaulieu, Dufour and Khalaf (2010)], and (ii) the cross-sectional statistics discussed by Shanken and Zhou (2007) and Lewellen et al. (2010). When underlying parameters are fixed, all these statistics are of the Hotelling form [Hotelling (1947)]. We show that the resulting inversion requires multi-dimensional quadratic inequalities. We provide a unified and tractable analytical solution to these inequalities and supportive finite sample and simulation assessments in non *i.i.d.* and non Gaussian settings, all of which are new to both asset pricing and econometric literatures. Analytical computations rely on the mathematics of *quadrics* [Dufour and Taamouti (2005), Dufour and Taamouti (2007)].[1]

Features of our methodology which are worth emphasizing – as well as illustrated in an extensive simulation study – include the following. The first one is a joint treatment of factors viewed simultaneously rather than individual proxies. The second feature is our reliance on set rather than just point estimates for parameters of interest. In contrast with Kan et al. (2013) and Gospodinov et al. (2014), the statistics we invert to derive these sets are not t-type measures and can be empty or unbounded, reflecting misspecified information or lack thereof. The third notable feature is our analytical solution to both point and set estimates. In contrast with Kleibergen (2009), Kleibergen et al. (2019) and Kleibergen and Zhan (2020) who propose numerical test inversion methods, our analytical solutions cover the zero-beta rate and control for factors that are traded portfolios. In addition, we propose a formal test for traded factors assumptions, which to the best of our knowledge is new to the literature. Our simulation results underscore the information content of cross-sectional intercepts and traded factor restrictions.

Our main empirical finding concerns the potential weakness (from an identification viewpoint) of the Fama-French-Carhart factors [Fama and French (1992), Fama and French (1993), Carhart (1997), Fama and French (2015)]. Using NYSE data from 1961-2010, we find the Fama-French three factors are priced concurrently before 1970 with equally weighted industry portfolios, as well as with size-sorted portfolios in the 1970s (only). Evidence of pricing weakens thereafter, as the model is either rejected or weakly identified, depending on intercept restrictions or test portfolios. Interestingly, we do not find evidence favouring size and book-to-

[1] For further quadric based solutions in different contexts, see Bolduc, Khalaf and Yelou (2010) for inference on multiple ratios, and Khalaf and Urga (2014) for inference on cointegration vectors.

market risk over the market risk. For instance, with size portfolios, in all subperiods except the 1980s and 2000s in which our confidence sets on the market risk are uninformative, the market is significantly priced. The Carhart and the recent Fama and French (2015) factors are affected by weak-identification problems. Finally, the considered models do not fare well when test assets are used jointly, and data is generally less informative after 2000.

The paper is organized as follows. Section 2 sets the asset pricing and statistical framework. Section 3 provides our inference methodology. Our simulation and empirical results are reported in section 4. Section 5 concludes the paper, and proofs are presented in a technical appendix.

2 Model and identification framework

Let r_i, $i = 1, \ldots, n$, be a vector of T returns on n assets, over the period $t = 1, \ldots, T$, and $\mathscr{R} = [\mathscr{R}_1 \cdots \mathscr{R}_q]$ a $T \times q$ matrix of observations on a set of q risk factors that potentially explain returns. It is now generally agreed that candidate models should also attempt to price proposed factors and include both traded and nontraded factors. To describe how to do so, assume that $\mathscr{R}_1$ is a vector of returns on a tradable factor, for example a market benchmark, so that $\mathscr{R} = [\mathscr{R}_1\ \mathscr{F}]$ where $\mathscr{F} = [\mathscr{R}_2 \cdots \mathscr{R}_q]$ is a $T \times (q-1)$ matrix of observations on $(q-1)$ nontraded factors.[2]

The APT equilibrium condition leads one to consider regressions of the form

$$r_i = a_i \iota_T + \mathscr{R}_1 b_{i1} + \mathscr{F} b_{i\mathscr{F}} + u_i\,, \quad i = 1, \ldots, n, \tag{2.1}$$

$$a_i = \gamma_0(1 - b_{i1}) - \gamma'_{\mathscr{F}} b_{i\mathscr{F}}\,, \quad \text{[restricted]} \tag{2.2}$$

$$a_i = \gamma_c - \gamma_0 b_{i1} - \gamma'_{\mathscr{F}} b_{i\mathscr{F}}\,, \quad \text{[unrestricted]} \tag{2.3}$$

where b_{i1} is a scalar, $b_{i\mathscr{F}}$ is a $(q-1)\times 1$ vector, γ_0 and $\gamma_{\mathscr{F}}$ incorporate the risk premiums as follows:

$$\theta = \left(\gamma_0, \gamma'_{\mathscr{F}}\right)' \equiv \mu_{\mathscr{R}} - \Gamma, \tag{2.4}$$

where Γ and γ_c are as in (1.1) and $\mu_{\mathscr{R}} = (\mu_{\mathscr{R}_1}, \ldots, \mu_{\mathscr{R}_q})'$ is the vector of unknown factor means as in (1.2); see Campbell, Lo and MacKinlay (1997, Chapter 6), Shanken and Zhou (2007) and references therein.[3]

2.1 APT conditions, traded and nontraded factors

Condition (2.3) introduces the APT risk premiums as free parameters, hence we denote it as the *unrestricted* APT specification. In contrast, condition (2.2) that we describe as the *restricted* specification further allows the traded factor $\mathscr{R}_1$ *to price itself* [Lewellen et al. (2010, Prescription 4)] if it is added to the set of left-hand side test assets. In other words, since $\mathscr{R}_1$ itself should satisfy (1.1) then it should be that $\Gamma_1 = \mu_{\mathscr{R}_1} - \gamma_c$, which in view of (2.4) implies that $\gamma_0 = \gamma_c$. Clearly, setting $\gamma_0 = \gamma_c$ in (2.3) gives (2.2). This restriction and the information content of γ_c matter importantly for model assessment [Barone-Adesi et al. (2004), Lewellen et al. (2010) and Penaranda and Sentana (2016)].

Estimating and testing this model confront enduring hurdles since the b matrix is unobserved. Indeed, from (2.2) or (2.3), it is clear that the components of θ cannot be identified, *e.g.* when the corresponding components of $b_i = [b_{i1}, b'_{i\mathscr{F}}]'$ do not differ enough over i (*i.e.*, in cross-section), and in particular, are jointly close to one

[2] We consider a single traded factor for notational ease. Extensions to multiple tradable factors follow straightforwardly. Our main empirical analysis considers this restriction for the market benchmark only, hence this notation.

[3] Taking unconditional expectations of the unconstrained (2.1) regression with a time invariant perspective implies $\mu_R = (a_1, \ldots, a_n)' + b'\mu_{\mathscr{R}}$, which equated with (1.1) yields (2.3).

or to zero. Possibly non-informative factors and reliance on portfolios which tends to equalize *betas* imply that identification cannot be taken for granted.

Furthermore, (2.4) evinces the fundamental difficulty of identifying Γ, as $\mu_{\mathscr{R}}$ is unknown. This fact has long been exploited to justify two-pass methodologies [as reviewed in *e.g.* Shanken and Zhou (2007)].[4] Instead, Shanken (1985) provides economic motivation for using the *ex-post* risk premium Γ_* defined in (1.2) as a function of the factors' empirical mean $\bar{\mathscr{R}}$. In the present regression context, $\Gamma_* = \bar{\mathscr{R}} - \theta$. Empirically, it has long been recognized [see *e.g.* Shanken (1992), Campbell et al. (1997, Chapters 5 & 6)] that θ can be estimated even though $\mu_{\mathscr{R}}$ is unknown. From there on, Γ_* can be estimated conditioning on the factors. The gains from using Γ_* are especially notable in finite samples as $\bar{\mathscr{R}}$ can deviate markedly from μ_F in some subperiods. We thus focus on this parameter given our finite sample perspective, to exploit the statistical properties of (2.1).

It is also important to note that (2.2) or (2.3) are jointly determined by the elements of the vector θ, so a change in one element of θ may be "cancelled" by a change on another element of θ. Consequently, it is crucial to make joint inference of the vector θ. Formally, we derive a joint confidence region for θ conditioning on the factors (imposing or relaxing $\gamma_0 = \gamma_c$) and project this region to obtain simultaneous confidence sets for each of the components of θ. We next assess pricing reflecting zero-restrictions on the components of $\bar{\mathscr{R}} - \theta$: each factor is considered not priced if its empirical mean is not covered by the confidence set associated with the corresponding component of θ. Our confidence intervals are simultaneous, which implies that decisions on pricing will also be simultaneous.

As emphasized above, γ_c also holds important information on model fit, and so does the restriction $\gamma_0 = \gamma_c$. Our method will produce a confidence interval for γ_c in addition to each component of $\left(\gamma_0, \gamma'_{\mathscr{F}}\right)$. Testing $\gamma_c = \gamma_0$ can be conducted via the following reparameterization:

$$\gamma_c - \gamma_0 b_{i1} - \gamma'_{\mathscr{F}} b_{i\mathscr{F}} = \gamma^*_c - \gamma_0 \left(b_{i1} - 1\right) - \gamma'_{\mathscr{F}} b_{i\mathscr{F}}\,, \quad \gamma^*_c = \gamma_c - \gamma_0\,. \tag{2.5}$$

The intercept can be "partialled-out" if we rewrite the regressions in deviation from one of them, leading to $n-1$ equations, in which case we will obtain another confidence interval for each component of $\left(\gamma_0, \gamma'_{\mathscr{F}}\right)$, at the expense of foregoing information on γ_c. The statistic we consider to do this is a monotonic transformation of the LR-based criterion [the so-called FAR test] introduced by Kleibergen (2009). Interpretations on pricing are unchanged, and as in Kleibergen (2009), Kleibergen et al. (2019) and Kleibergen and Zhan (2020), the statistic is invariant to the equation chosen as the deviation basis. We formally assess the pros and cons of evacuating γ_c, for estimation and fit purposes.

For further reference, the frameworks we consider are categorized as follows: model (2.1)-(2.2) is denoted **RAPT** where **R** stands for "restricted" which refers to traded factor constraints; model (2.1)-(2.3) which relaxes the latter constraints is denoted **UAPT** where **U** stands for "unrestricted", in which case we refer to partialling γ_c out as the **PAPT** approach, where **P** stands for "partialling-out". We will also refer to the hypothesis

$$H^*_c : \gamma^*_c \equiv \gamma_c - \gamma_0 = 0 \tag{2.6}$$

which can be tested by checking whether the confidence set for γ^*_c in (2.5) covers zero.

2.2 Reduced rank regressions

The above equilibrium models can be defined via rank restrictions on a multivariate regression of the form:

$$Y = XB + U, \quad U = WJ' \Leftrightarrow Y_t = B'X_t + U_t, \quad U_t = JW_t,\ t = 1,\ \ldots\ , T\,, \tag{2.7}$$

[4]"An average return carries no information about the mean of the factor that is not already observed in the sample mean of the factor." [Cochrane (2005, p. 245)]. See also Penaranda and Sentana (2016), on including moment conditions on factors means with GMM.

where Y is a $T \times n$ matrix of observations on n endogenous variables, X is a $T \times k$ full-column rank matrix of exogenous variables, Y_t' and X_t' are, respectively, the t-th row of Y and X so that Y_t and X_t provide the t-th observation on the dependant variables and regressors, J is unknown, non-singular and possibly random, U_t' is the t-th row of U, W is a $T \times n$ matrix of random errors, W_t' is the t-th row of W, and the joint distribution of $W_1, \ldots, W_T$ is either fully specified, or specified up to a nuisance parameter μ. Finite sample results assume we can condition on X for statistical analysis.

Throughout the paper, we maintain the following assumptions and notation. $\mathscr{D}(d_1, \ldots, d_m)$ refers to an m-dimensional diagonal matrix with diagonal elements $d_1, \ldots, d_m$. ι_j refers to a j-dimensional vector of ones. The number of factors is $q = k - 1$. DIAG(A) refers to a column vector from the diagonal of a matrix A. For any $N \times K$ matrix A, vec(A) returns an $NK \times 1$ vector, with the columns of A stacked on top of each other; $\mathscr{M}[A] = I - A(A'A)^{-1}A'$ for any full column rank matrix A. We refer to a $1-\alpha$ level CS for a parameter as CS$_\alpha(.)$. Let

$$\hat{B} = (X'X)^{-1}X'Y , \quad \hat{S} = \hat{U}'\hat{U}, \quad \hat{U} = Y - X\hat{B}. \tag{2.8}$$

For presentation ease, we use the following matrix partitions:

$$(X'X)^{-1} = \begin{bmatrix} x^{11} & x^{12} \\ x^{21} & x^{22} \end{bmatrix} \tag{2.9}$$

where x^{11} is a scalar, $x^{21} = x^{12\prime}$ is $q \times 1$ and x^{22} is $q \times q$, and

$$B = \begin{bmatrix} a' \\ b \end{bmatrix}, \quad \hat{B} = \begin{bmatrix} \hat{a}' \\ \hat{b} \end{bmatrix}, \quad b = \begin{bmatrix} b_1 & \cdots & b_n \end{bmatrix} = \begin{bmatrix} \beta_2' \\ \vdots \\ \beta_k' \end{bmatrix}, \quad \hat{b} = \begin{bmatrix} \hat{\beta}_2' \\ \vdots \\ \hat{\beta}_k' \end{bmatrix} \tag{2.10}$$

where $a = (a_1, \ldots, a_n)'$ is the vector of n intercepts, and b is $q \times n$.

The rank restrictions in question can be written as:

$$H_{\mathbf{RAPT}} : (1, \theta')B = 0, \qquad \text{for some unknown vector } \theta, \tag{2.11}$$

$$H_{\mathbf{UAPT}} : (1, \theta')B = \phi\iota_n', \qquad \text{for some unknown vector } (\theta', \phi)', \tag{2.12}$$

where θ is $q \times 1$ and ϕ is an unknown scalar.[5] Indeed, rewriting (2.1)-(2.2) with left-hand side returns in deviation from $\mathscr{R}_1$ yields, for $i = 1, \ldots, n$:

$$r_i - \mathscr{R}_1 = (\mathscr{R}_1 - \iota_T\gamma_0)[b_{i1} - 1] + \left(\mathscr{F} - \iota_T\gamma_{\mathscr{F}}'\right)b_{i\mathscr{F}} + u_i, \qquad i = 1, \ldots, n,$$

or alternatively

$$r_i - \mathscr{R}_1 = a_i\iota_T + \mathscr{R}_1 d_i + \mathscr{F}b_{i\mathscr{F}} + u_i\,, , \qquad i = 1, \ldots, n \tag{2.13}$$

$$a_i = -\gamma_0 d_i - \gamma_{\mathscr{F}}' b_{i\mathscr{F}}, \quad d_i = b_{i1} - 1\,, \tag{2.14}$$

which is a special case of (2.7) where Y stacks the matrix of returns in deviation from the tradable benchmark, imposing (2.11) with

$$\theta = \left(\gamma_0, \gamma_{\mathscr{F}}'\right)'. \tag{2.15}$$

The non-tradable case (2.1)-(2.3) is the regression

$$r_i = a_i\iota_T + \mathscr{R}_1 b_{i1} + \mathscr{F}b_{i\mathscr{F}} + u_i, \qquad i = 1, \ldots, n, \tag{2.16}$$

$$a_i = \gamma_c - \gamma_0 b_{i1} - \gamma_{\mathscr{F}}' b_{i\mathscr{F}} \tag{2.17}$$

[5]Typically, (2.11) and (2.12) assume that $k \le n$.

which again is a special case of (2.7) where Y stacks the matrix of returns, imposing (2.12) with

$$\theta = \left(\gamma_0, \gamma_{\mathscr{F}}'\right)', \text{ and } \phi = \gamma_c.$$

Regression (2.1)-(2.3) can also be re-expressed as

$$r_i - \mathscr{R}_1 = a_i \iota_T + \mathscr{R}_1 d_i + \mathscr{F} b_{i\mathscr{F}} + u_i, \qquad i = 1, \ldots, n, \tag{2.18}$$

$$a_i = \gamma_c^* - \gamma_0 d_i - \gamma_{\mathscr{F}}' b_{i\mathscr{F}}, \quad d_i = b_{i1} - 1, \quad \gamma_c^* = \gamma_c - \gamma_0, \tag{2.19}$$

in which imposing (2.12) with $\phi = \gamma_c^*$ provides a test of (2.6). Finally, it is also straightforward to see that the model in deviation from one of the returns yields a system of $n-1$ equations conformable with (2.11).

This paper focuses on estimating and testing θ and ϕ. Furthermore, we provide cross-equation simultaneous confidence sets for the parameters of the unrestricted asset pricing regression. Formally, we invert the test that fixes each row of B, in turn, to a fixed vector; the associated hypotheses takes the form

$$H_j : s_k[j]'B = \bar{\beta}_j', \quad j \in \{1, \ldots, k\}, \quad \bar{\beta}_j \text{ known} \tag{2.20}$$

where $s_k[j]$ denotes a k-dimensional selection vector with all elements equal to zero except for the j-th element which equals 1. To interpret H_j, recall that the classical zero restriction hypothesis underlying the Hotelling statistic which is viewed as the multivariate extension of the Student-t based significance test corresponds to

$$H_{0j} : s_k[j]'B = 0, \quad j \in \{1, \ldots, k\} \tag{2.21}$$

so for example using $s_k[1]$ provides inference on the unrestricted regression intercept, and in the context of an unrestricted regression in deviation from the tradable factor [(2.13) above, ignoring the constraints], $s_k[2]$ allows one to assess the *betas* on the tradable factor in deviation from one. Assembling the $\bar{\beta}_j$ vectors that are not rejected at a given level yields a joint confidence set for the corresponding row of B which contain, in turn for $j = 1, \ldots, k$, the n-dimensional vector of intercepts, and the n-dimensional vector of betas (possibly in deviation from one) on each factor over all considered assets.

In addition to useful information on underlying assets, the unrestricted regression intercepts and *betas* underlie identification of the above defined risk premiums. Formally, for θ to be recoverable with no further data and information (in particular in the absence of other instruments), the *betas* per factor need to vary enough across equation. Concrete identification failure problems discussed in Beaulieu et al. (2013) (and the reference therein) relate to benchmark *betas* jointly [across i] equal to one. Kleibergen (2009) discusses the case of small *betas* in the sense of jointly [across i] equal to zero, which may be traced back to Kan and Zhang (1999). Regardless of the source, identification of θ is driven by the joint cross-equation nature of the information conveyed by each factor. Our simultaneous approach for inference on θ as well as for the underlying reduced form *betas* thus zooms in on the core of the financial problem. Concretely, using portfolios rather than individual assets as test assets (*i.e.*, for r_i in our notation) tends to equalize *betas* across equations; whether moving away from portfolios to individual assets which calls for alternative information reduction technique is an answer to this problem remains an open question which is beyond the scope of the bulk of the this literature as well as the present paper which requires $T - k - n > 0$. Our methodology is presented in the next section for the general (2.7) regression.

3 Confidence sets for factor loadings and risk premiums

Following Beaulieu et al. (2013) and Kleibergen (2009), we focus on inverting identification-robust statistics, *i.e.*, statistics whose null distributions are provably invariant to whether identification holds or not. We focus on Hotelling-type statistics

$$\Lambda(\theta, \phi) = \frac{[(1, \theta')\hat{B} - \phi \iota_n']\hat{S}^{-1}[\hat{B}'(1, \theta')' - \phi \iota_n]}{(1, \theta')(X'X)^{-1}(1, \theta')'} \tag{3.1}$$

$$\Lambda(\theta) = \frac{(1, \theta')\hat{B}\hat{S}^{-1}\hat{B}'(1, \theta')'}{(1, \theta')(X'X)^{-1}(1, \theta')'} \tag{3.2}$$

where θ' and ϕ are given. These statistics serve to assess the special cases of $H_{\mathbf{RAPT}}$ and $H_{\mathbf{UAPT}}$ [in (2.11)) - (2.12] respectively

$$H_{\mathbf{R}} : (1, \theta')B = 0, \qquad \theta \text{ known.} \tag{3.3}$$

$$H_{\mathbf{U}} : (1, \theta')B = \phi \iota_n', \qquad (\theta', \phi)' \text{ known,} \tag{3.4}$$

In addition, we invert the series of statistics associated with each of the H_j (2.20):

$$\Lambda(\bar{\beta}_j) = \frac{(\hat{\beta}_j - \bar{\beta}_j)'\hat{S}^{-1}(\hat{\beta}_j - \bar{\beta}_j)}{s_k[j]'(X'X)^{-1}s_k[j]} \frac{\tau_n}{n} \tag{3.5}$$

where and $\hat{\beta}_j'$ is the jth row of $\hat{B}$. These statistics are also of the Hotelling form [see Dufour and Khalaf (2002)]; note that the classical Hotelling statistics to assess each of H_{0j} (2.21) are

$$\Lambda_{0j} = \frac{s_k[i]'\hat{B}\hat{S}^{-1}\hat{B}'s_k[i]}{s_k[i]'(X'X)^{-1}s_k[i]}. \tag{3.6}$$

When errors are normal then

$$\Lambda(\theta) \frac{\tau_n}{n} \sim F(n, \tau_n), \quad \Lambda(\theta, \phi) \frac{\tau_n}{n} \sim F(n, \tau_n), \quad \Lambda(\bar{\beta}_j) \frac{\tau_n}{n} \sim F(n, \tau_n) \tag{3.7}$$

where $\tau_n = T - k - n + 1$. The latter distributional results do not require any identification restriction.[6] Λ_{0j} also follow the same null distribution. Underlying finite sample theory is discussed in section 3.3. Simulations reported in 4.1 show that for the problem under consideration corresponding to 5 or 10 year subsamples, the normal cut-off controls size whether errors are multivariate Student-*t* or in the presence of GARCH effects. Prior to these analyses, we discuss in the next section how inverting the proposed tests can be performed analytically.

3.1 Analytical solution

Inverting the above tests requires solving in turn, over (θ, ϕ), θ and $\bar{\beta}_j$ respectively, the inequalities

$$\Lambda(\theta, \phi) \frac{\tau_n}{n} \leq f_{n, \tau_n, \alpha}, \quad \Lambda(\theta) \frac{\tau_n}{n} \leq f_{n, \tau_n, \alpha}, \quad \Lambda(\bar{\beta}_j) \frac{\tau_n}{n} \leq f_{n, \tau_n, \alpha}, \tag{3.8}$$

where $f_{n, \tau_n, \alpha}$ denotes the α-level cut off point from the $F(n, \tau_n)$ distribution. The following unified analysis using the mathematics of Quadrics generalizes the Beaulieu et al. (2013) solution to: (ii) the multi-factor context, and (iii) the estimation of factor loadings and Jensen-type *alphas*.

Each inequation in (3.8) is rewritten as

$$(1, \zeta')A(1, \zeta')' \leq 0 \tag{3.9}$$

[6] Other than the usual Least Squares assumptions on $X'X$ and $\hat{S}$ of course.

where ζ is the $m \times 1$ vector of unknown parameters and A is an $(m+1) \times (m+1)$ data dependent matrix. Next, inequality (3.9) is re-expressed as

$$\zeta' A_{22} \zeta + 2A_{12}\zeta + A_{11} \leq 0 \tag{3.10}$$

which leads to the set-up of Dufour and Taamouti (2005) so projections based CSs for any linear transformation of ζ of the form $\omega'\zeta$ can be obtained as described in these papers. The solution is reproduced in the Appendix for completion.

Moving from (3.9) to (3.10) requires partitioning A as follows

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \tag{3.11}$$

where A_{11} is a scalar, A_{22} is $m \times m$, and $A_{12} = A_{21}'$ is $1 \times m$. Simple algebraic manipulations suffice to show that for the test defined by (3.2), we have:

$$A = \hat{B}\hat{S}^{-1}\hat{B}' - (X'X)^{-1}(n/\tau_n) f_{n,\tau_n,\alpha} \tag{3.12}$$

setting $\zeta = \theta$. Using the partitionings (2.10) and (2.9), $A_{11} = \hat{a}'\hat{S}^{-1}\hat{a} - ((n/\tau_n) f_{n,\tau_n,\alpha})x^{11}$, $A_{12} = A_{21}' = \hat{a}'\hat{S}^{-1}\hat{b}' - [(n/\tau_n) f_{n,\tau_n,\alpha}]x^{12}$ and

$$A_{22} = \hat{b}\hat{S}^{-1}\hat{b}' - [(n/\tau_n) f_{n,\tau_n,\alpha}]x^{22}. \tag{3.13}$$

In the case of (3.1), we have $\zeta = \left(\theta', \phi\right)'$ and

$$A = \begin{bmatrix} \hat{B}\hat{S}^{-1}\hat{B}' - (X'X)^{-1} f_{n,\tau_n,\alpha}\frac{n}{\tau_n} & -\hat{B}\hat{S}^{-1}\iota_n \\ -\iota_n'\hat{S}^{-1}\hat{B}' & \iota_n'\hat{S}^{-1}\iota_n \end{bmatrix}. \tag{3.14}$$

Finally, inverting (3.5) yields $\zeta = \bar{\beta}_i'$ and the quadric form (3.10) with

$$A_{22} = \hat{S}^{-1}, \quad A_{12} = -\hat{\beta}_i{}'\hat{S}^{-1}, \quad A_{11} = -n\left(s_k[i]'(X'X)^{-1}s_k[i]\right)/\tau_n. \tag{3.15}$$

The outcome of resulting projections can be empty, bounded, or the union of two unbounded disjoint sets. Dufour and Taamouti (2005) discuss such outcomes depending importantly on the A_{22} matrix. In particular, the confidence set is unbounded if A_{22} is not positive definite. It is thus clear that inverting (3.5) produces bounded sets as $\hat{S}$ is assumed invertible. The following Theorem further shows that if any of the Hotelling tests based on Λ_{0j}, $j = 2, \ldots, k$ is not significant then the A_{22} matrix will not be positive definite and the confidence set will be unbounded. That is, if any of the factors is redundant from a joint significance perspective, then information on risk prices *for all factors* is compromised.

Theorem 3.1 *In the context of* (2.7)*, if*

$$(\tau_n/n)\Lambda_j < f_{n,\tau_n,\alpha}, \quad j \in \{2, \ldots, k\} \tag{3.16}$$

where Λ_i are the Hotelling statistics defined in (3.5)*, then the CS for θ as defined in* (2.11)] *which inverts the statistic* (3.2) *at the α-level is unbounded.*

The above condition is sufficient but not necessary. It follows that although Hotelling tests on each factor are useful, they remain insufficient, and perhaps more importantly, are embedded in our methodology without compounding type-I errors. This characterization also holds when inverting the test defined in (3.1) and (3.7).

3.2 Empty confidence sets and minimum distance statistics

The confidence set for factor loadings cannot be empty. Indeed, Dufour and Taamouti (2005) show that in the context of (3.10) and a positive definite A_{22}, the confidence set is empty $if \quad \tilde{D} = A_{12}A_{22}^{-1}A_{12} - A_{11} < 0$. Here, from (3.15) we have

$$\tilde{D} = \hat{\beta}_i{}'\hat{S}^{-1}\hat{\beta}_i + n\left(s_k[i]'(X'X)^{-1}s_k[i]\right)/\tau_n \geq 0.$$

Moving on to the case of $\Lambda(\theta)$, we proceed by generalizing the single-beta results in Beaulieu et al. (2013). Because the cut-off point underlying test inversion denoted $f_{n,\tau_n,\alpha}$ above is the same for all θ values, an empty set would result when $\min_\theta \Lambda(\theta) \geq f_{n,\tau_n,\alpha}$. It can be shown that minimizing $\Lambda(\theta)$ produces the Gaussian-LR statistic to test the nonlinear restriction which defines θ, namely (2.11); general derivation are available in e.g. Gouriéroux, Monfort and Renault (1996).

Theorem 3.2 *In the context of* (2.7) *and the nonlinear hypothesis* (2.11) *the confidence set estimate for* θ *which inverts the statistic* $\Lambda(\theta)$ *defined in* (3.2) *at the* α*-level is empty if and only if*

$$\Lambda_{\mathbf{RAPT}} = \min_\theta \Lambda(\theta) = \Lambda(\hat{\theta}_{\mathbf{RAPT}}) = \hat{g}/(1-\hat{g}) \equiv \hat{\rho} \geq f_{n,\tau_n,\alpha} \tag{3.17}$$

where $\hat{g}$ *is the minimum non-zero root of* $C(X,Y) = (X'X)^{-1}X'Y(Y'Y)^{-1}Y'X$ *and*

$$\hat{\theta}_{\mathbf{RAPT}} = -\left(\hat{b}\hat{S}^{-1}\hat{b}' - \hat{\gamma}x^{22}\right)^{-1}\left(\hat{b}\hat{S}^{-1}\hat{a} - \hat{\gamma}x^{21}\right). \tag{3.18}$$

In other words, the confidence set estimate for θ is empty if and only if the minimum distance LR-based Hotelling statistic associated with (2.11) is significant when referred to the $f_{n,\tau_n,\alpha}$ cut-off which can be viewed as a finite-sample bound cut-off point for this test. Observe that $\hat{\rho}$ coincides with the minimum root of both determinantal equations:

$$|\hat{B}\hat{S}^{-1}\hat{B}' - \rho(X'X)^{-1}| = 0\,, \tag{3.19}$$

$$|\hat{B}'\left(X'X\right)\hat{B} - \rho\hat{S}| = 0\,. \tag{3.20}$$

Since the underlying eigenvector solution is not unique, we provide a proof for (3.18) in the Appendix which easily extends to the less restricted (2.12) definition and will allow us to link our results to existing related works namely Kandel (1984) and Kandel (1986), and more recently to Kleibergen (2009). Our approach is general and relates to important applications of reduced rank regression based inference in econometrics; these include limited information simultaneous equation and cointegration models; see Dhrymes (1974, Chapter 7), Davidson and MacKinnon (2004, Chapter 12), or Johansen (1995). We rely on the following matrix algebra result pertaining to an equation of the form

$$\Sigma(1,\,\zeta')' = 0, \qquad |\Sigma| = 0, \qquad \Sigma = \left[\begin{array}{cc} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{array}\right] \tag{3.21}$$

where ζ is the m-dimensional unknown given an $(m+1)\times(m+1)$ matrix Σ, and Σ_{11} is a scalar, $\Sigma_{12} = \Sigma_{21}'$ is $1\times m$ and Σ_{22} is $m\times m$ and is invertible. As summarized in the Appendix,

$$\hat{\zeta} = -\Sigma_{22}^{-1}\Sigma_{21} \tag{3.22}$$

provides a unique solution to this system.

Theorem 3.3 *In the context of* (2.7) *and the nonlinear hypothesis* (2.12), *the minimum distance estimators associated with the criterion* $\Lambda(\theta, \phi)$ *defined in* (3.1) *can be derived as*

$$\Lambda_{\mathbf{UAPT}} = \min_{\theta,\phi} \Lambda(\theta, \phi) = \Lambda(\hat{\theta}_{\mathbf{UAPT}}, \hat{\phi}_{\mathbf{UAPT}}) = \hat{\nu} \tag{3.23}$$

where $\hat{\nu}$ *is the minimum root of*

$$|\hat{B}\hat{R}_n\hat{B}' - \nu(X'X)^{-1}| = 0 \tag{3.24}$$

where $\hat{R}_n := \hat{S}^{-1} - \hat{S}^{-1}\iota_n \left(\iota_n'\hat{S}^{-1}\iota_n\right)^{-1} \iota_n'\hat{S}^{-1}$ *and*

$$\hat{\theta}_{\mathbf{UAPT}} = -\left[\hat{b}\hat{R}_n\hat{b}' - \hat{\nu}x^{22}\right]^{-1}\left[\hat{b}\hat{R}_n\hat{a} - \hat{\nu}x^{21}\right], \tag{3.25}$$

$$\hat{\phi}_{\mathbf{UAPT}} = \frac{(1, \hat{\theta}'_{\mathbf{UAPT}})\hat{B}\hat{S}^{-1}\iota_n}{\iota_n'\hat{S}^{-1}\iota_n}. \tag{3.26}$$

The formulas in (3.25)-(3.26) coincide with the solution obtained (using another method of proof) by Shanken and Zhou (2007).

3.3 Finite-sample distributional theory

For some though not all inferential problems considered, we will assume the following mixture distributional setting:

$$W = VZ \tag{3.27}$$

where V is $T \times T$, unknown and possibly random (in which case it is independent of Z), and Z is a $T \times n$ matrix of *i.i.d.* n-dimensional standard normal variables *i.e.* if we denote the t-th row of Z as Z_t', then

$$Z_t \overset{i.i.d.}{\sim} N[0, I_n]. \tag{3.28}$$

Assumption (3.27) is sufficiently general and includes various n-dimensional elliptically contoured distributions and skew-elliptical distributions. Special cases of (3.27) include the normal distribution

$$W_t = Z_t \overset{i.i.d}{\sim} N[0, I_n] \tag{3.29}$$

and the multivariate Student-t distribution with μ degrees-of freedom [denoted as $\mathtt{t}(\mu)$].

The hypotheses associated with all statistics we aim to invert as introduced in the previous section fall within the uniform linear class [see Dufour and Khalaf (2002), Beaulieu, Dufour and Khalaf (2007) and the references therein] of the form:

$$\tilde{H}[C, G, D] : CBG = D \text{ for known } C, G \text{ and } D \tag{3.30}$$

where C is $c \times k$ with rank c, $0 \le c \le k$, and G is $n \times g$, with rank g. The restricted estimators in this case are:

$$\tilde{B}(C, G, D) = \hat{B} - \left(X'X\right)^{-1}C'[C(X'X)^{-1}C']^{-1}(C\hat{B}G - D)(G'\hat{S}G)^{-1}G'\hat{S}, \tag{3.31}$$

$$\tilde{S}(C, G, D) = \tilde{U}(C, G, D)'\tilde{U}(C, G, D), \tag{3.32}$$

$$\tilde{U}(C, G, D) = Y - X\tilde{B}(C, G, D), \tag{3.33}$$

where

$$\tilde{S}(C, G, D) = \hat{S} + \hat{S}G(G'\hat{S}G)^{-1}(C\hat{B}G - D)'[C(X'X)^{-1}C']^{-1}(C\hat{B}G - D)(G'\hat{S}G)^{-1}G'\hat{S}. \tag{3.34}$$

Commonly used statistics including the LR and Wald criteria [see Berndt and Savin (1977), Gouriéroux, Monfort and Renault (1995), Dufour and Khalaf (2002) and the references therein] to test $\tilde{H}[C, G, D]$ can be expressed as

$$\mathscr{L}(C, G, D) = T \ln\left(|\tilde{S}(C, G, D)|/|\hat{S}|\right) = -T\sum_{i=1}^{l} \ln\left(1-\lambda_i(C, G, D)\right), \tag{3.35}$$

$$\mathscr{W}(C, G, D) = T \operatorname{tr}\left(\hat{S}^{-1}[\tilde{S}(C, G, D) - \hat{S}]\right) = T\sum_{i=1}^{l} \frac{\lambda_i(C, G, D)}{1-\lambda_i(C, G, D)}, \tag{3.36}$$

where $l = \min\{c, g\}$ and $\lambda_1(C, G, D) \geq \cdots \geq \lambda_n(C, G, D)$ are the eigenvalues of $\tilde{S}(C, G, D)^{-1}[\tilde{S}(C, G, D) - \hat{S}]$. Clearly, $\lambda_i(C, G, D),\ i = 1, \ldots, l$ coincide with the roots of $S^{-1}(C, G, D)[S(C, G, D) - G'\hat{S}G]$ where

$$S(C, G, D) = G'\hat{S}G + \left(C\hat{B}G - D\right)' \left[C(X'X)^{-1}C'\right]^{-1} \left(C\hat{B}G - D\right). \tag{3.37}$$

Solving for eigenvalues in question thus requires considering the determinantal equation

$$\left|(S(C, G, D) - G'\hat{S}G) - \lambda S(C, G, D)\right| = 0. \tag{3.38}$$

Theorem 3.4 *In the context of* (2.7) *and the null hypothesis* $\tilde{H}[C, I_n, D]$*, the LR criterion simplifies to following form* $\mathscr{L}(C, I_n, D) = T \ln\left(\left|I_c + \tilde{\Lambda}(C, I_n, D)\right|\right)$ *where*

$$\tilde{\Lambda}(C, I_n, D) = [C(X'X)^{-1}C']^{-1}(C\hat{B} - D)\hat{S}^{-1}(C\hat{B} - D)'. \tag{3.39}$$

The latter Theorem covers the linear hypotheses relevant to the APT introduced above since

$$H_{\mathbf{R}} \equiv \tilde{H}[(1, \theta'), I_n, 0], \quad H_{\mathbf{U}} \equiv \tilde{H}[(1, \theta'), I_n, \phi\iota_n'], \quad H_j \equiv \tilde{H}[s_k[j]', I_n, \bar{\beta}_j], \quad H_{0j} \equiv \tilde{H}[s_k[j]', I_n, 0] \tag{3.40}$$

leading to

$$\Lambda(\theta, \phi) = \tilde{\Lambda}\left((1, \theta'), I_n, \phi\iota_n'\right), \qquad \Lambda(\theta) = \tilde{\Lambda}\left((1, \theta'), I_n, 0\right), \tag{3.41}$$

$$\Lambda(\bar{\beta}_j) = \tilde{\Lambda}\left(s_k[j]', I_n, \bar{\beta}_j\right), \qquad \Lambda_{0j} = \tilde{\Lambda}\left(s_k[j]', I_n, 0\right). \tag{3.42}$$

Furthermore, the **PAPT** approach amounts to a convenient selection of the postmultiplying matrix G in (3.30). Via this matrix, our analysis extends to any non-redundant combination of returns one may wish to consider. The following Theorem establishes the finite sample distribution for the above eigenvalue based statistics with emphasis on the role of G, which was not discussed in (Dufour and Khalaf (2002)).

Theorem 3.5 *In the context of* (2.7) *and under the null hypothesis* $\tilde{H}[C, G, D]$ *in* (3.30)*, the vector of the roots of* (3.38) *is distributed like the vector of the roots of*

$$\left|\mathscr{G}'W'\left(\mathscr{M}_0[X, C]\right)W\mathscr{G} - \lambda\mathscr{G}'W'\left(\mathscr{M}[X] + \mathscr{M}_0[X, C]\right)W\mathscr{G}\right| = 0 \tag{3.43}$$

where G is the orthogonal $n \times g$ matrix which includes the eigenvectors associated with the non-zero eigenvalues of $J'GG'J$ and

$$\mathscr{M}_0[X, C] = X(X'X)^{-1}C'[C(X'X)^{-1}C']^{-1}C(X'X)^{-1}X'. \tag{3.44}$$

Furthermore, under assumption (3.27)*, the distribution in question follows that of the roots of*

$$\left|\mathscr{Z}'V'\left(\mathscr{M}_0[X, C]\right)V\mathscr{Z} - \lambda\mathscr{Z}'V'\left(\mathscr{M}[X] + \mathscr{M}_0[X, C]\right)V\mathscr{Z}\right| = 0 \tag{3.45}$$

where Z is a $T \times g$ matrix of i.i.d. g-dimensional standard normal variables, and is thus invariant to B and J. For the special case where $G = I_n$, i.e. *hypothesis* $\tilde{H}[C, I_n, D]$*, the distribution in question follows that of the roots of*

$$\left|W'\left(\mathscr{M}_0[X, C]\right)W - \lambda W'\left(\mathscr{M}[X] + \mathscr{M}_0[X, C]\right)W\right| = 0 \tag{3.46}$$

so invariance to B and J holds imposing or ignoring assumption (3.27).

Two results emerging from Theorem 3.5 deserve discussion for the problem under consideration.

1. The pivotal characterization (3.45) may be used to obtain finite sample p-values using the Monte Carlo test method [see *e.g.* Dufour and Khalaf (2002) and Dufour (2006)] if the variates underlying V can be simulated.

2. Under assumption (3.27), the distribution of the roots will depend on C but not on D, and depends on G only through its rank.

So for the family of mixture distributions (3.27), the fact that null distributions depend on G only through its rank underlies and generalizes (beyond the deviation form) the invariance property noted by Kleibergen (2009). The fact that null distributions do not depend on D imply that the null distribution of $\Lambda(\bar{\beta}_j)$ does not depend on $\bar{\beta}_j$, so extending the above defined test inversion beyond normality preserves its quadrics-based analytic solution. Indeed, it suffices to obtain a simulation-based cut-off point depending on the assumed disturbance distributions which will be the same for all $\bar{\beta}_j$. Dependence on θ will not be evacuated in the same way, since θ intervenes in the null distributions in questions via $\mathscr{M}_0[X,C]$. These distributions do not depend on ϕ, which again supports partialling this parameter out at least from a statistical perspective. Recall however that ϕ and whether it differs empirically from the hypothesized zero-beta rate γ_0 [which in our notation is the first component of θ] is an important empirical question in finance. Simultaneous inference on θ and ϕ remains relevant. Our empirical analysis sheds more light on this matter using a well-known prototypical data set.

To conclude, we review two useful approximations to the above finite sample distributions. Given normal errors and if $\min(c,g) \leq 2$ [Rao (1973, Chapter 8), McKeon (1974)] then

$$\frac{\varkappa_1\varkappa_3 - 2\varkappa_2}{cg}\,\frac{1-\left(|\hat{S}|/|\tilde{S}(C,G,D)|\right)^{1/\varkappa_3}}{\left(|\hat{S}|/|\tilde{S}(C,G,D)|\right)^{1/\varkappa_3}} \sim F(cg\,,\,\varkappa_1\varkappa_3-2\varkappa_2) \tag{3.47}$$

$$\varkappa_1 = T-k-((g-c+1)/2), \quad \varkappa_2 = (cg-2)/4, \tag{3.48}$$

$$\varkappa_3 = \begin{cases} [(c^2g^2-4)/(c^2+g^2-5)]^{1/2} & \text{if } c^2+g^2-5>0 \\ 1 & \text{otherwise} \end{cases}. \tag{3.49}$$

The latter result holds as a reliable approximation when $\min(c,g) > 2$. The cutoffs we use to invert the statistics considered in section 3 follow from these approximations. We also verify that deviations from the *i.i.d.* or normal errors assumption do not lead to notable size distortions in empirically relevant multifactor simulation designs.

4 Empirical analysis: Fama-French and momentum factors

In our empirical analysis of a multifactor asset pricing model, we conduct: (i) a simulation study calibrated to observed returns and factors, and (ii) a data-based assessment of factor pricing.

We first produce results for industry portfolios for the US, as in Beaulieu et al. (2013). Following recommendations of Lewellen et al. (2010), we also produce results for size portfolios, based on Fama and French's data base. We consider monthly returns of 25 value weighted and equally weighted portfolios from 1961 to 2010. The benchmark factors are: 1) **MKT**, the excess return on the market defined as the value-weighted return on all NYSE, AMEX, and NASDAQ stocks (from CRSP) minus the one-month Treasury bill rate (from Ibbotson Associates); 2) **SMB** (small minus big) defined as the average return on three small portfolios minus the average return on three big portfolios; 3) **HML** (high minus low) defined as the average return on two value portfolios minus the average return on two growth portfolios; 4) **MOM** (momentum), the average return on the two high prior return portfolios minus the average return on the two low prior return portfolios; 5) **RMW** (robust minus weak) operating profitability; and 6) **CMA** (conservative minus aggressive) investment, constructed from conservative minus aggressive growth of assets for the fiscal year. Further information on this data is provided in the supplementary Appendix.

4.1 Simulation evidence

The experiment reported in this section is designed to assess three issues. First, the above proposed CSs can be conservative when k is large, so we aim to document their coverage properties. Second, we study the performance of the inverted proposed tests when errors are fat-tailed. Third, we assess the implications of imposing and relaxing tradability restrictions; see Lewellen et al. (2010) and Penaranda and Sentana (2016) for theoretical and practical discussions in this regard. To the best of our knowledge, an identification-robust assessment of this important equilibrium-based restriction is as yet unavailable.

We consider an empirically relevant design based on the above data set and the Fama-French three factor model [with MKT, SMB and HML] in which returns are generated by (2.1) with (2.2) or (2.3). We calibrate designs using observed data on value-weighted portfolios covering the full sample [$n = 12$ and $T = 624$], and the last sub-period [$n = 12$, $T = 120$]. The full sample exercise allows us to analyze our results relative to the literature, whereas the shorter sample documents performance as it applies to a standard sub-period analysis.

The factors are drawn as normal with means and variance/covariance calibrated to match the considered sample. We set the factor loadings and the variance/covariance of disturbances to their OLS estimated counterparts for the observed sample. J is obtained as the Cholesky root of this variance/covariance matrix. For conformity, we also compute the cross-sectional two-pass OLS estimates of the zero-beta rate and risk price [denoted θ_{*0}], and their companion standard errors [denoted $SE(\theta_{*0})$]. We use these cross-section estimates to calculate θ as in Shanken and Zhou (2007) and from there on, to initialize the simulations underlying the size study. To assess power, we set the parameter under the alternative as

$$\theta_* = \theta_{*0} + \mathsf{step} \times SE(\theta_{*0}) \tag{4.1}$$

where step measures departure from the null hypotheses; the intercept term γ_c is calibrated in the same way. Of course, for our empirical analysis, we do not compute confidence intervals using OLS estimates nor Wald-based MLEs for that matter. All reported intervals invert the Hotelling-tests we proposed above. The cross-sectional OLS estimates from the training samples, despite their imperfections, are used as well-understood prototypical metric to initialize our data generating processes. For space considerations, the simulation values are not reported here but are available from the authors upon request.

These settings are maintained for all analyzed tests, except in one case in which we provoke under-identification by fixing the MKT betas jointly to zero or one. We report test sizes as a worst scenario check to confirm that no over-rejections occur despite identification failure. The disturbances W_t are generated, in turn, as *i.i.d.* normal, multivariate Student with 5 degrees of freedom, and multivariate GARCH using in this case the data generating process

$$W_t = \mathbf{G}_t^{1/2} Z_t, \quad \mathbf{G}_t = (1-\mu_1-\mu_2)I_n + \mu_1 W_{t-1}W_{t-1}' + \mu_2 \mathbf{G}_{t-1} \tag{4.2}$$

where Z_t are uncorrelated n-dimensional standard normal variables, so in this case the conditional variance of JW_t is given by Σ_t with

$$\begin{aligned} \Sigma_t = J\mathbf{G}_t J' &= (1-\mu_1-\mu_2)JJ' + \mu_1 JW_{t-1}W_{t-1}'J' + \mu_2 J\mathbf{G}_{t-1}J' \\ &= (1-\mu_1-\mu_2)JJ' + \mu_1 JW_{t-1}W_{t-1}'J' + \mu_2 \Sigma_{t-1} \end{aligned} \tag{4.3}$$

Table 1: Designs underlying reported figures

Empirical Model	True Data Generating Process					
	$\mathscr{R}_1$ Tradable			$\mathscr{R}_1$ Non-Tradable		
	Inverted Statistic			Inverted Statistic		
	RAPT	**PAPT**	**UAPT**	**RAPT**	**PAPT**	**UAPT**
$\mathscr{R}_1$	$\Lambda(\theta)$	$\Lambda_D(\theta)$	$\Lambda(\theta, \phi)$	$\Lambda(\theta)$	$\Lambda_D(\theta)$	$\Lambda(\theta, \phi)$
Tradable	Fig. 1 & 7	NA	NA	Fig. 6	NA	NA
	Inverted Statistic			Inverted Statistic		
	RAPT	**PAPT**	**UAPT**	**RAPT**	**PAPT**	**UAPT**
$\mathscr{R}_1$	$\Lambda(\theta)$	$\Lambda_D(\theta)$	$\Lambda(\theta, \phi)$	$\Lambda(\theta)$	$\Lambda_D(\theta)$	$\Lambda(\theta, \phi)$
Non-Tradable	NA	Fig. 3	Fig. 2 & 8	NA	Fig. 5	Fig. 4

Note – This table summarizes designs and methods in reported figures below. The **RAPT** $\Lambda(\theta)$ [in (3.2)] and **UAPT** $\Lambda(\theta, \phi)$ [in (3.1)] statistics, where **R** and **U** stands for "restricted" and "restricted", impose and relax the assumption that $\mathscr{R}_1$ is tradable, respectively. The **PAPT** $\Lambda_D(\theta)$ statistic, where **P** stands for "partialling-out", denotes $\Lambda(\theta)$ applied to a system on $n-1$ returns in deviation from r_n. In this case, all factors are assumed non-tradable in estimating and testing the model but the resulting unrestricted constant is evacuated from the statistical objective function as it is applied to $r_i - r_n$, $i = 1, \ldots, n-1$.

which corresponds to a special case proposed by Engle and Kroner (1995). We use $(\mu_1, \mu_2) = (.15, .80)$. The considered inference methods are *not* corrected for departures from the *i.i.d.* assumption nor from normality. Tests and confidence intervals in what follows are at the 5% and 95% level. We report empirical rejections over 10000 replications for each parameter. In the results below, $\mathscr{R}_1$ is the MKT factor.

The design of the simulation experiment is outlined in Table 1, whereas the results are summarized in Figures 1-8. In each figure, we report: (i) under the heading "True Model", the specification used to generate data, and (ii) under the heading "Empirical Model", the specification that was considered for estimation and inference. The true and empirical specifications differ only regarding the treatment of $\mathscr{R}_1$, as summarized in table 1. Figure 7 replicates the design in 1 with a smaller sample size; we do not replicate all designs for space considerations, so figure 7 aims to broadly illustrates sample size issues. The design underlying figure 8 differs from the rest in that it assesses our proposed tradability test; further discussions below will clarify the design and its implications. In all figures, the parameters corresponding to the null hypothesis are identified via a dashed vertical line. Results can be summarized as follows. In the Supplementary Appendix, we report the proportion of empty joint confidence sets in the experiments underlying each figure, as well as the unidentified experiment results.

1. *Deviations from normality.* In all designs, deviations from normality are not distortive in the following sense: no over-rejections occur under the null hypothesis when the *i.i.d.* normal assumption is violated. Recall that tests rely on the above defined F critical points regardless of the distributions we use to draw simulated samples. This result is noteworthy given the prevalence of multivariate GARCH or Student-*t* type assumptions on disturbances in theoretical and empirical asset pricing work.

We also find that power results with GARCH-based designs dominate the Gaussian based ones which in turn dominate the Student-*t* case. On balance, a maximum of around 10% difference in power is observed between power curves, respectively. Power costs resulting from Student-*t* errors are expected, since our tests rely on least-squares. Power results with GARCH deserve discussion. Recall that test sizes are controlled even though GARCH was not accounted for. We do not advocate hasty conclusions suggesting that GARCH enhances test performance. Instead, we find that the GARCH case underscores the practical usefulness of our tests in realistic settings: most likely, GARCH adjusts the scale of left-hand side simulated variates relative to the model's covariates, to better match the considered initializing parameters which rely on observed data here. For further insight on single equation least-squares based inference in the presence of ARCH, see (Hamilton (2010)). Both Student-t and GARCH errors do not seem to affect power ranking relative to factor informativeness.

2. *Size of tests, and identification.* Empirical rejections under the null hypothesis do not exceed 5% in all cases, including the under-identified case (reported in the supplementary Appendix). To quantify identification in the baseline design, refer to table 2 [reported in our empirical section below] which summarizes joint factor significance tests[7] and confidence sets for factor loadings, using the last sub-period of our dataset. This is relevant because these loadings drive the simulated design for our short sample (with $T = 120$) which we view as a "stress test" for our methodology. The main point we aim to underscore from this table as it relates to our simulation design, is the following. While the (unrestricted) intercept is not significant at 5% using the Hotelling test, all factors are significant at 5%. Nevertheless, all confidence intervals for the SMB beta cover zero, whereas a small proportion of the intervals for the MKT or HML loadings do not cover zero. Thus factors are not necessarily redundant, yet joint information may not be strong across portfolios with shorter samples. This turns out to matter as we will see below.

Although not obvious on first sight, figure 8 contains a size study on our unrestricted test, when applied in deviation from $\mathscr{R}_1$. In this case, the intercept will measure deviations between the risk price of $\mathscr{R}_1$ [$\gamma_0 = \theta_{\text{MKT}}$] and the zero beta rate [γ_c], which will assess the tradable factor assumption. In this design, the null hypothesis corresponds to $\gamma_0 = \theta_{\text{MKT}} = \gamma_c$ *i.e.*, $\mathscr{R}_1$ is tradable. Departures from the null hypotheses vary γ_c keeping γ_0 constant. Since $\gamma_0 = \theta_{\text{MKT}}$ is kept constant throughout, the curves corresponding to θ_{MKT} describe size and not power. We thus see that size is well controlled again, a point worth emphasizing since this test is new to the (identification-robust) literature.

3. *Power of tests, general findings.* Except with the under-identified case [refer to the Supplementary Appendix], all tests display good and empirically relevant power. Recall that a joint three (or four) parameter test is inverted here, which confirms that the benefits of simultaneous inference is not offset by power losses unless identification fails completely.

Comparing figures 1 and 7, we see that tests are powerful on all parameters even when the sample size drops from 624 to 120. Strikingly, power curves do not differ much between the large and small samples. In particular, and though information on other factors suffers to some extent, power on the HML price with $T = 120$ almost matches the $T = 624$ experiment. This leads us to analyze with further detail how power differs across considered factors; refer to point 4, below.

The unrestricted and partialled-out tests perform exactly the same for inference on risk price, a result which lines up with the above theory [refer to (3.11)-(3.14)]. The main advantage of our test compared to the partialled-out one which relates to an asymptotic test proposed by (Kleibergen (2009)) is the information we provide on the model's intercept. See in particular figure 4: whereas partialling-out evacuates this parameter, our test provides tremendous power on this fundamental coefficient without sacrificing any information on the model's risk price. Further discussion of restricted versus unrestricted testing is discussed in point 5, below.

4. *Power of tests, across factors.* Broadly, tests are more informative on one of the three factors relative to the others. Given the historical debate on MKT beta, comparing figures 4 and 5 to figures 1 and 7 is particularly enlightening. The former imply that the MKT risk is harder to test than the remaining factors. In contrast, tests on the zero-beta rate as depicted in the latter seem more powerful than those on SMB and HML.

In all four figures true and empirical assumptions on $\mathscr{R}_1$ coincide and conformable tests are applied. Thus, both of these alternative findings may initially appear plausible. However, whereas figures 4 and 5 relax tradability of $\mathscr{R}_1$, figures 1 and 7 replicate empirical consensus on $\mathscr{R}_1$, i.e. that it is a tradable factor. This under-

[7] Industry portfolios are used although (unreported) results with size factors convey qualitatively similar information. With reference to market betas, we assess joint deviations from one since the market factor is assumed tradable in this design, in which case bunching up at one is more relevant to gauge identification.

scores the usefulness of our restricted test which is depicted in figures 1 and 7, which in turn leads us to further analyze the effects of restricting versus assessing the tradable factor assumption; refer to point 5 below.

Figures 1 and 7 also suggest that SMB is the least informative factor. The above comments warning that confidence sets on the SMB loading all cover zero apply in this case. Relative power ranking are thus clearly driven by the relative identification strength of SMB in this design, which we emphasize is based on observable factors.

5. *Power of tests, restricted versus unrestricted.* Figure 8 (with the exception of results on θ_{MKT} as noted above) depicts the power of our proposed test which assesses that $\mathscr{R}_1$ is tradable, or formally whether $\gamma_0 = \theta_{\text{MKT}} = \gamma_c$. Departures from this null hypotheses vary γ_c keeping γ_0 constant. The power on non-market risk prices is unaffected relative to the previous designs, whereas we find very good power on $\gamma_c - \gamma_0$. Though ex-ante decisions regarding some factors is possible, most cross-sectional based works in asset pricing tend to leave the intercept unrestricted. Our approach provides an identification-robust assessment of tradable factor restrictions, which, to the best of our knowledge, is a new and useful contribution.

Figure 2 documents the consequences of neglecting this assumption when it holds. Comparing power curves between figure 1 and figure 2, we find that the risk price of $\mathscr{R}_1$ bares all the cost as power is much weaker in figure 2 than in figure 1, which results of course from disregarding a relevant restriction. Our design suggests that resulting power losses are sizable: the risk price of a tradable factor is harder to pin-down even in identified contexts when an unrestricted cross-sectional constant is maintained, which quantifies the consequences of an important "pitfall" raised in particular by Lewellen et al. (2010) and more recently though in a GMM context by Penaranda and Sentana (2016). In contrast to the traditional literature, our findings are based on methods that are robust to the identification of all factors which provides new insights into the historical debate surrounding the role of the MKT factor in multi-factor models.

To further interpret the evidence on intercept tests, note that in the design underlying figure 2, as γ_c is taken away from its value under the null hypothesis, γ_0 follows conformably since the true model throughout imposes $\gamma_0 = \gamma_c$ (hence the need for figure 8, to assess the test for inference on this discrepancy). So results for $\gamma_0 = \gamma_c$ can be interpreted as size [despite the misspecification], whereas tests on γ_c confirms the excellent power properties we noted in commenting on figure 4. Figure 3 illustrates the limitations of partialled-out tests: power on all coefficients, again, coincides exactly with that of our unrestricted test as depicted in figure 2, which we noted to be way lower that in figure 1 for inference regarding $\mathscr{R}_1$. As an added major cost, partialling-out takes away all sources of information on the validity of tradability assumptions, whose usefulness we quantified via figure 8.

Figure 6 illustrates the consequences of imposing the traded factor assumption when it does not hold. Here, what is indicated as a parameter value under the null is in fact a false null, since the model falsely imposes a restriction that does not hold. An important contrast with figure 2 in which case we found that inference regarding $\mathscr{R}_1$ is only affected, here results show that spurious inference *on all* model parameters results, with notable size distortions as empirical rejections exceed 60%. A important cautionary remark about this figure (refer to the Supplementary Appendix for further details), the rejections we depict actually correspond to empty confidence sets in almost all simulations. This means that our companion model checks are conveying evidence of *misspecification*, with very good power. This finding leads to clear prescriptions for empirical work: despite the importance of imposing traded factor assumption, their empirical validation remains a must as serious distortions will result otherwise. This reinforced the usefulness of our proposed intercept test, and more importantly, the usefulness of our built-in specification checks which will return empty sets when the model deviates importantly from asset pricing equilibrium relations.

6. Size and power, and empirical results. Taken collectively, our simulation results suggest that the unbounded confidence sets we observe empirically as reported in section 4.2 are most likely driven by weak factors. Inference problems are thus highly likely even outside the high dimensional settings analyzed for example by Harvey et al. (2016) where k is by far larger than 5, the maximum number we consider. Our results with a small number of commonly used factors also suggest that our confidence sets in which an identification check is "hard-wired" are extremely valuable in practice, since they will allow the researcher to qualify non-rejections. An unbounded confidence set guards the researcher from misreading nonsignificant tests as evidence in favour of models on which data is not informative. Our analytical projection method thus provides an invaluable tool because it easily and surely confirms an unbounded solution, in contrast to e.g. numerical searches that are typically subject to precision, convergence and tractability constraints. On balance, results with both sample sizes illustrate the worth of our analytical F-based motivation for relying on our proposed analytical test inversion formula.

4.2 Empirical results

In the following discussion, significance refers to the 5% level and the restricted test refers to treating MKT as a tradable factor. Our empirical analysis builds on the prescriptions of Lewellen et al. (2010). From a general standpoint, our results can be summarized as follows.

Risk premiums are better identified with industry portfolios than with size portfolios. This result is not driven by the number of the portfolios in question. In fact, when the whole set of portfolios is used jointly following Lewellen et al. (2010, Prescription 1), all considered unconditional models are rejected. Although noteworthy and consistent with the discussion in Lewellen et al. (2010), the latter test may pose an unconventionally high hurdle for goodness of fit. We thus do not aim to overemphasize these rejections. Instead, we view these results as confirming the power of our tests as n increases relative to T.

An alternative and more fundamental argument is that stacking portfolios increases dispersion of factor sensitivities; in contrast, size sorted portfolios yield much more clustered betas than their industry counterparts, which ill-conditions the rank of the associated *beta* matrix thus compromising identification of risk price. Similar distortive clustering results with value weighted portfolios whether we use industry or size sorting, whereas size sorted value-weighted portfolios are the least informative in our considered tests. This is illustrated in table 2 which reports confidence sets for factor loading based on inverting $\Lambda(\bar{\beta}_j)$ in (3.8), as well as zero-parameter test based on Λ_{0j} (3.6) under the heading Hotelling, corresponding to the last sub-period of our dataset with industry value-weighted portfolios.[8] Though all factors seem relevant via significant Hotelling test, a small number of confidence intervals for MKT excludes one, one of the intervals for HML excludes zero and all intervals for SMB cover zero, which suggests severe clustering. For this purpose, we base the bulk of our analysis on equally weighted industry portfolios.

The Fama-French five factor model is severely under-identified even via our most informative checks, namely our restricted test, industry portfolios, and over the whole sample. Since Fama and French (2015) argue that HML in this model is unidentified, we repeat our analysis excluding this factor.

Analyzed results are reported in six tables; the appendix includes results with value weighted portfolios and further results with weaker identification evidence for completion.

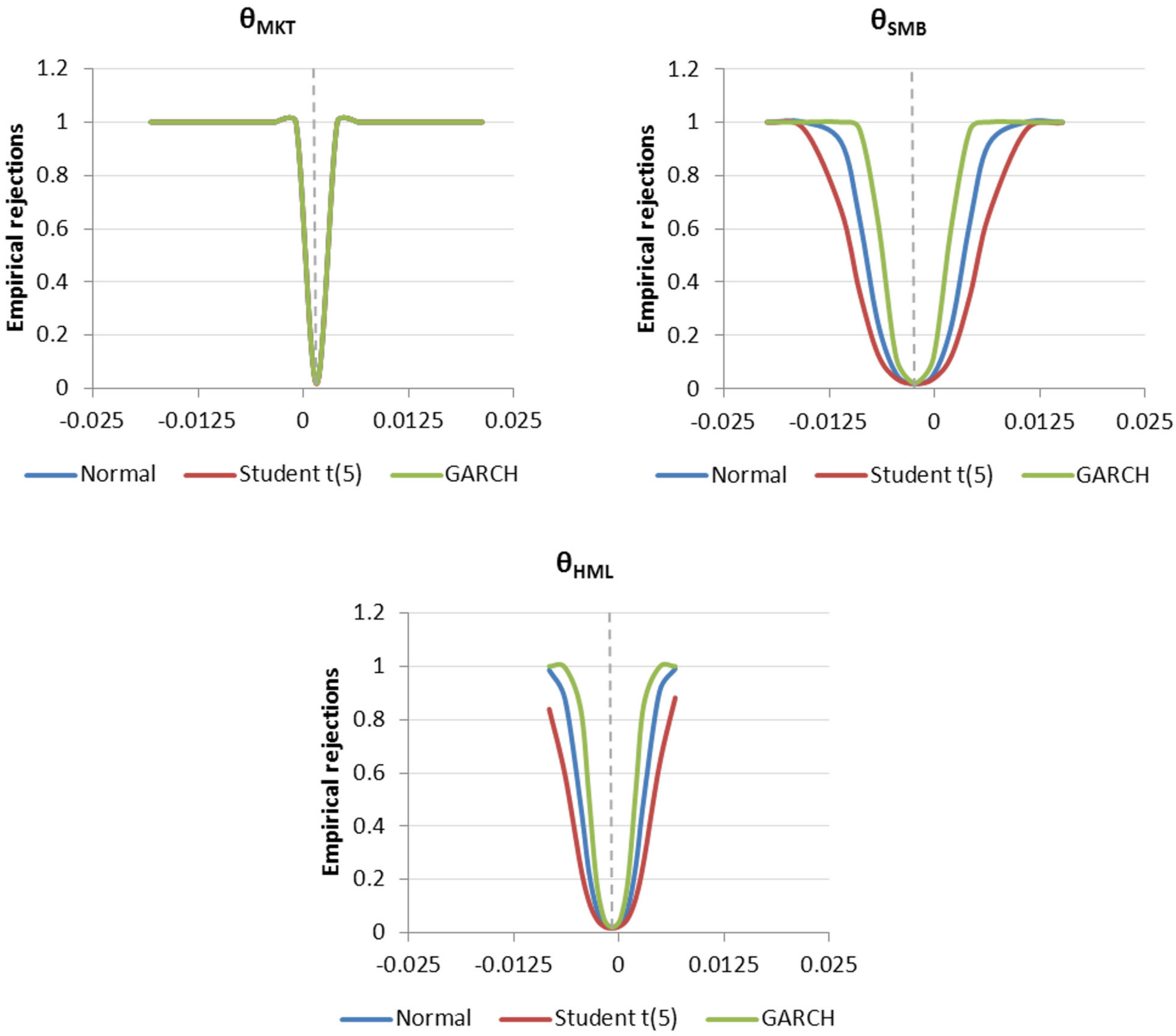
Results imposing tradable MKT differ importantly from their unrestricted counterparts. Compare for example Panel A to Panel B of table 3, and consider first the 1971-1980 and 1991-2000 subperiods in which we reject the three-factor model via our restricted test (of Panel A). For further reference, these subperiods will be denoted as the *atypical* ones, to underscore this rejection. In contrast to the restricted test, our unrestricted inference (in Panel B) fails to reject the model in these subperiods and confirms that: (i) MKT is priced in both cases, (2) HML is not priced whereas SMB is priced only in the 90s.

Does it seem reasonable to retain a model that ignores a key property of the market factor? With reference to Lewellen et al. (2010, Prescription 2), the unrestricted test seems a low hurdle to meet unless (among other

[8] We discussed these results above as they relate to our simulation design.

Figure 1: Monte Carlo study: tests imposing tradable market factor

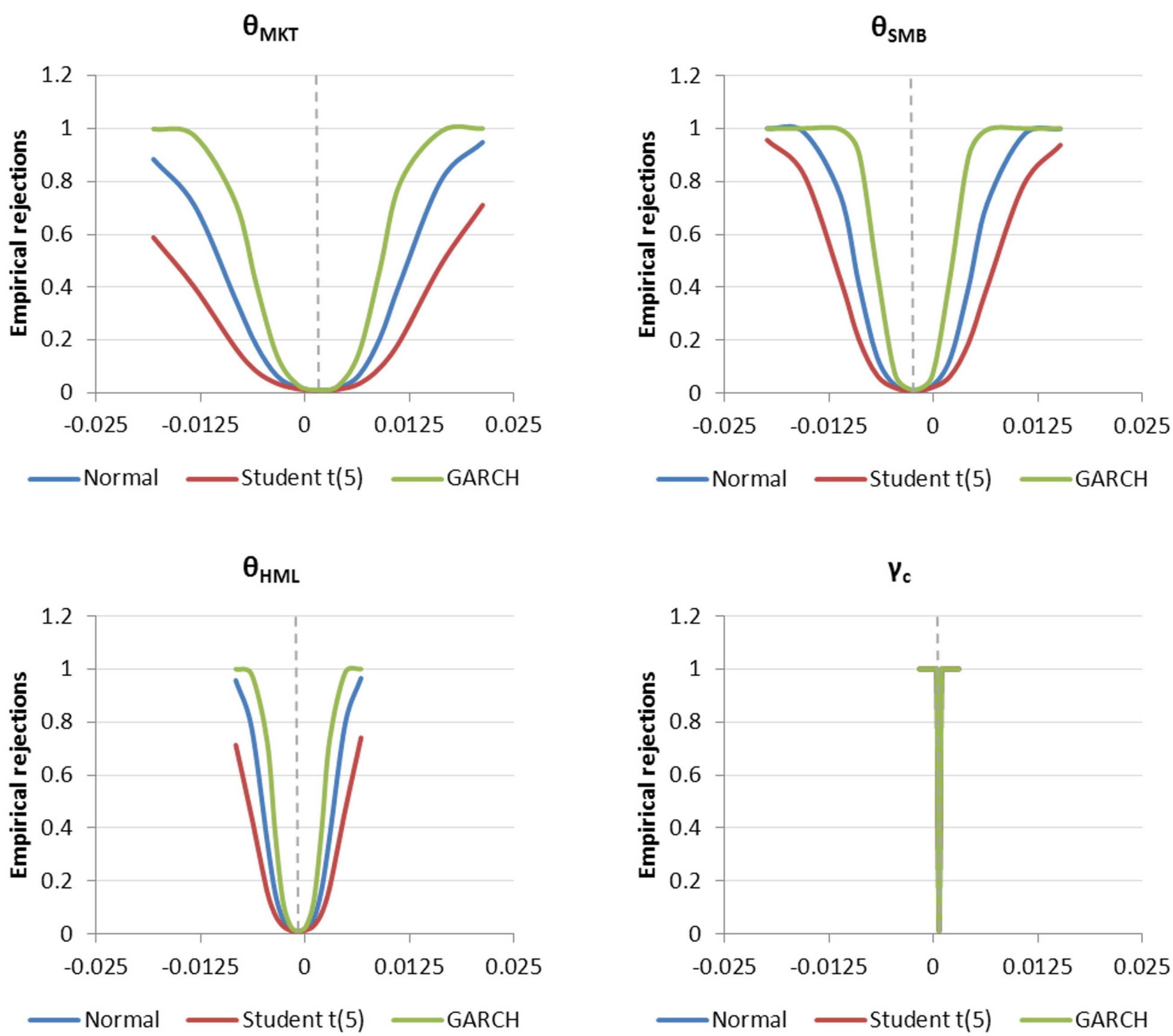
$n=12,\ T=624$	$r_i = a_i \iota_T + \mathscr{R}_1 b_{i1} + \mathscr{F} b_{i\mathscr{F}} + u_i$
True model	$a_i = \gamma_0(1-b_{i1}) - \gamma'_{\mathscr{F}} b_{i\mathscr{F}}, \quad \gamma_0 = \theta_{\text{MKT}}, \quad \gamma_{\mathscr{F}} = (\theta_{\text{SMB}}, \theta_{\text{HML}})'$
Empirical model	$a_i = \gamma_0(1-b_{i1}) - \gamma'_{\mathscr{F}} b_{i\mathscr{F}}$
RAPT: Inverted Statistic $\Lambda(\theta), \quad \theta = (\gamma_0, \gamma'_{\mathscr{F}})'$	

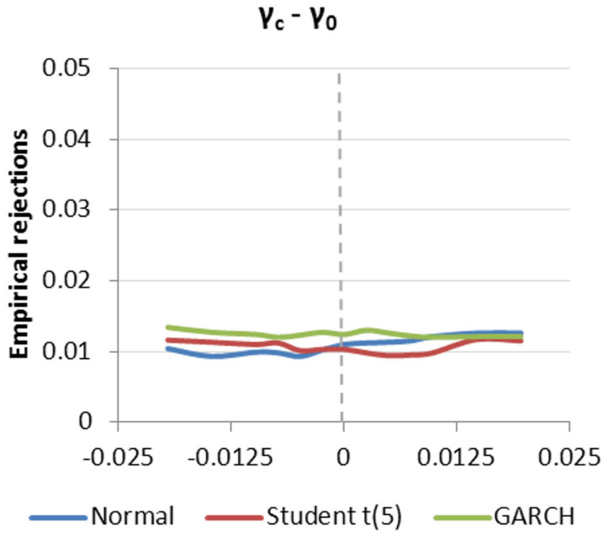


Note – Dashed vertical lines denotes null parameter values. Empirical rejections pertain to 5% tests associated with 95% confidence sets: if the 95% set does not cover the null value or is empty, the reported test is considered significant. Parameters are calibrated to OLS cross-sectional two-pass estimates from a training sample based on industry portfolios, Fama-French factors and monthly data, 1961-2010 hence $T = 624$. See table 1 for further details on design and inverted tests. When one unicolor curve is depicted whereas the legend refers to three cases, this implies that all visually coincide.

Figure 2: Monte Carlo study: joint tests, market factor tradable assumed non-tradable

$n = 12,\ T = 624$	$r_i = a_i \iota_T + \mathscr{R}_1 b_{i1} + \mathscr{F} b_{i\mathscr{F}} + u_i$
True model	$a_i = \gamma_0(1 - b_{i1}) - \gamma'_{\mathscr{F}} b_{i\mathscr{F}}, \quad \gamma_0 = \theta_{\text{MKT}}, \quad \gamma_{\mathscr{F}} = (\theta_{\text{SMB}}, \theta_{\text{HML}})'$
Empirical model	$a_i = \gamma_c - \gamma_0 b_{i1} - \gamma'_{\mathscr{F}} b_{i\mathscr{F}}$
UAPT: Inverted Statistic $\Lambda(\theta, \phi)$, $\quad \theta = \left(\gamma_0, \gamma'_{\mathscr{F}}\right)', \quad \phi = \gamma_c$ or $(\gamma_c - \gamma_0)$	





Note – See notes to figure 1 and table 1. Results under the heading $\gamma_c - \gamma_0$ are obtained by executing $\Lambda(\theta, \phi)$ on returns in deviation from $\mathscr{R}_1$ in which case tests on risk θ are unchanged and tests on ϕ provide inference on $\gamma_c - \gamma_0$, which is zero throughout this design for all "steps" given the considered true model.

explanations) the zero beta rate differs anomalously from the risk-free rate. Our inference on the cross-sectional intercept can inform in this regard, in contrast to the unrestricted test [adopted in particular by Kleibergen (2009)] that subtracts this intercept away. Let us thus refer to the upper Panel of table 5. For the subperiods in question, we find that despite notable estimation uncertain, the confidence set on γ_c^* does not cover zero which confirms that γ_0 significantly differs from γ_c. This discrepancy seems to be driving the rejection of the tradable MKT model. Interestingly, we are not able to refute a zero γ_c^* outside these subperiods.

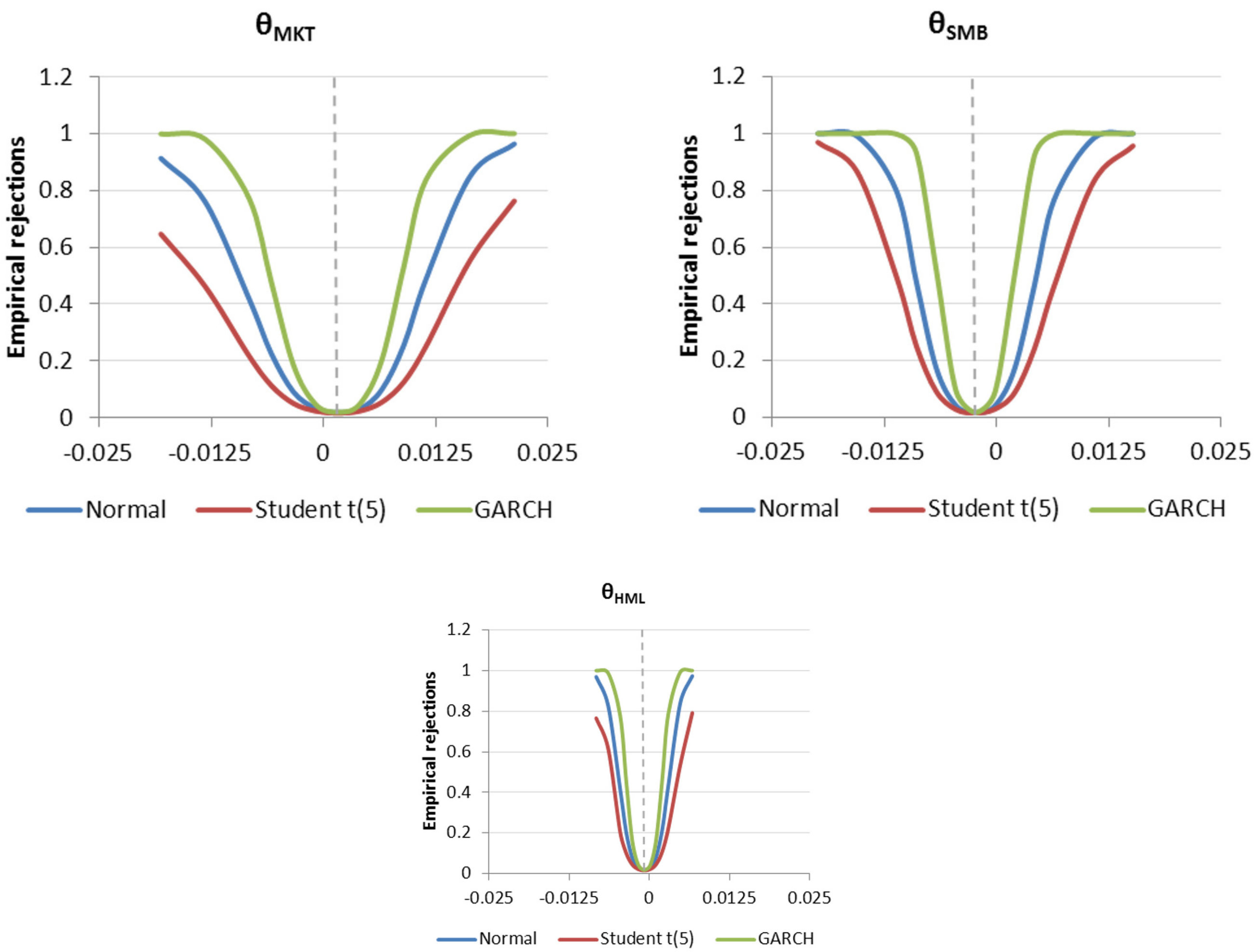
Results with the Carhart model provide further insights on the above anomalous interpretation. Comparing the upper to the lower panels of table 4, we find that the unrestricted test is completely uninformative as the confidence sets are utterly wide. In contrast, with a tradable MKT and again, despite evidence of estimation uncertainty, we find that MOM is priced only in the 1971-1980 and 1991-2000 sub-periods and these are the only subperiods in which the restricted three-factors model is rejected whereas γ_0 significantly differs from γ_c. The MKT risk itself is no longer priced in these subperiods, which stands in sharp contrast with our unrestricted three-factor based evidence. In addition, SMB is priced in the presence of MOM in both subperiods, whereas it is priced in our unrestricted three-factor model only in the nineties.

Divergences between restricted and unrestricted inference also arise when the restricted three-factors model is not rejected. In particular, in the three factors model, the restricted tests confirm that SMB and HML are both priced in the 1960s whereas the unrestricted tests cover zero. Similarly, HML appears priced via the restricted test in 1981-90 and not priced using the unrestricted counterpart, and the same holds for SMB in 2000-2010. Overall, aside from the market and unless the restricted model is rejected, all factors that are priced via the restricted test are no longer priced when the tradable MKT restriction is relaxed. Referring to the upper Panel of table 5 reveals no basis to refute $\gamma_0 = \gamma_c$ when the restricted model is not rejected, and as emphasized above, MOM is not priced in these subperiods as may be checked again from the upper Panel of table 4.

The above interpretation of the momentum effect may be qualified as we interpret results of the Fama-French model with SMB, RMW and CMA over and above the MKT factor. As with the Carhart model, the unrestricted test is completely uninformative yet the model passes our restricted test over all subperiods. The

Figure 3: Monte Carlo study: partialled-out tests, market factor tradable assumed non-tradable

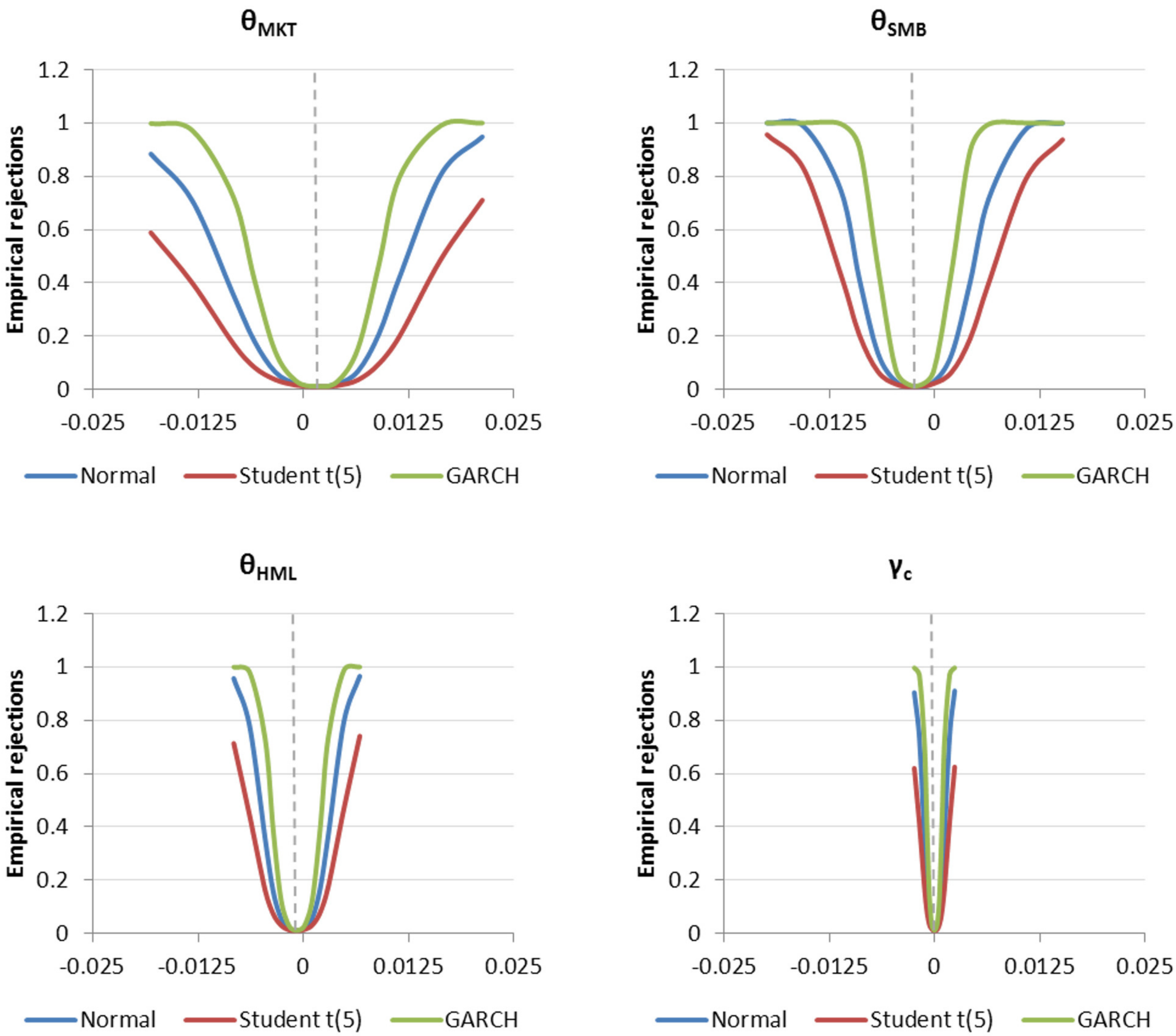
$n = 12,\ T = 624$	$r_i = a_i \iota_T + \mathscr{R}_1 b_{i1} + \mathscr{F} b_{i\mathscr{F}} + u_i$
True model	$a_i = \gamma_0(1 - b_{i1}) - \gamma'_{\mathscr{F}} b_{i\mathscr{F}}, \quad \gamma_0 = \theta_{\text{MKT}}, \quad \gamma_{\mathscr{F}} = (\theta_{\text{SMB}}, \theta_{\text{HML}})'$
Empirical model	$a_i = \gamma_c - \gamma_0 b_{i1} - \gamma'_{\mathscr{F}} b_{i\mathscr{F}}$
PAPT: Inverted Statistic $\Lambda(\theta)$ on $r_i - r_n$ with $\theta = (\gamma_0, \gamma'_{\mathscr{F}})'$	



Note – See notes to figure 1 and table 1. The dashed vertical line denotes the value of the given parameter under the null.

Figure 4: Monte Carlo study: joints tests, relaxing tradable market factor

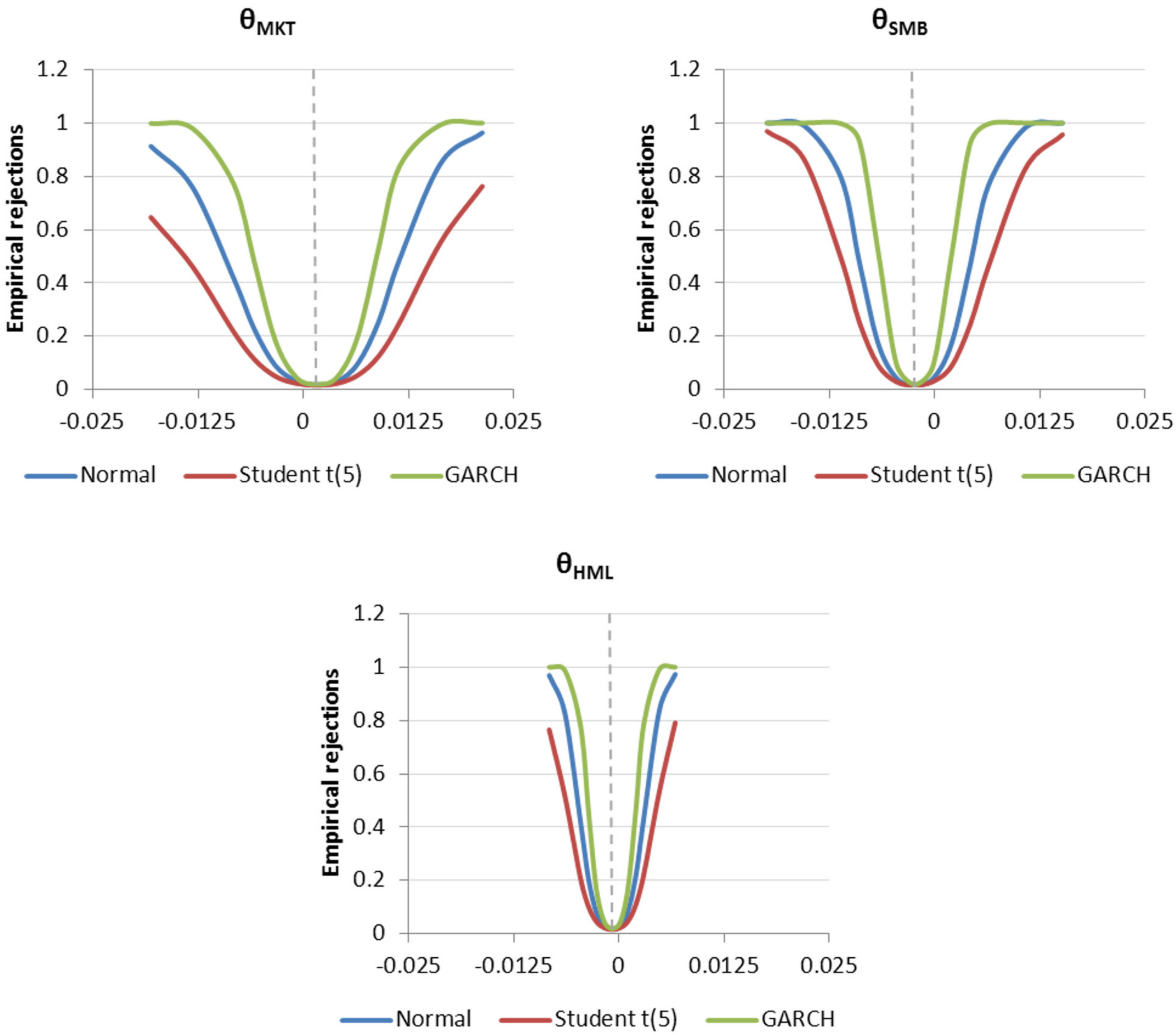
$n = 12,\ T = 624$	$r_i = a_i \iota_T + \mathscr{R}_1 b_{i1} + \mathscr{F} b_{i\mathscr{F}} + u_i$
True model	$a_i = \gamma_c - \gamma_0 b_{i1} - \gamma'_{\mathscr{F}} b_{i\mathscr{F}}, \quad \gamma_0 = \theta_{\text{MKT}}, \quad \gamma_{\mathscr{F}} = (\theta_{\text{SMB}}, \theta_{\text{HML}})'$
Empirical model	$a_i = \gamma_c - \gamma_0 b_{i1} - \gamma'_{\mathscr{F}} b_{i\mathscr{F}}$
UAPT: Inverted Statistic $\Lambda(\theta, \phi), \quad \theta = \left(\gamma_0, \gamma'_{\mathscr{F}}\right)', \quad \phi = \gamma_c$	



Note – See notes to figure 2.

Figure 5: Monte Carlo study: partialled-out tests, relaxing tradable market factor

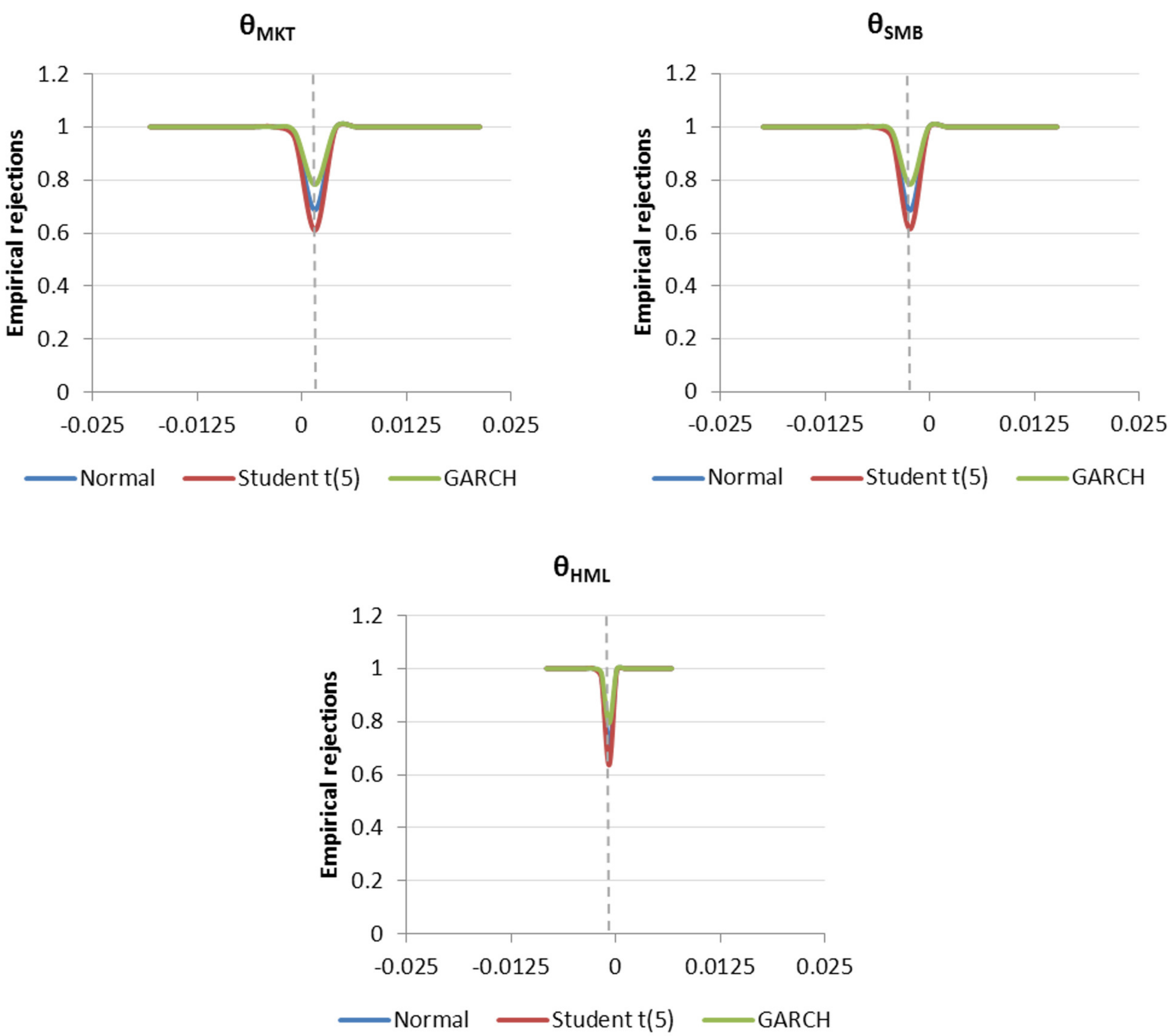
$n = 12,\ T = 624$	$r_i = a_i \iota_T + \mathscr{R}_1 b_{i1} + \mathscr{F} b_{i\mathscr{F}} + u_i$
True model	$a_i = \gamma_c - \gamma_0 b_{i1} - \gamma'_{\mathscr{F}} b_{i\mathscr{F}}, \quad \gamma_0 = \theta_{\text{MKT}}, \quad \gamma_{\mathscr{F}} = (\theta_{\text{SMB}}, \theta_{\text{HML}})'$
Empirical model	$a_i = \gamma_c - \gamma_0 b_{i1} - \gamma'_{\mathscr{F}} b_{i\mathscr{F}}$
PAPT: Inverted Statistic $\Lambda(\theta)$ on $r_i - r_n$ where $\theta = (\gamma_0, \gamma'_{\mathscr{F}})'$	



Note – See notes to figure 3 and table 1.

Figure 6: Monte Carlo study: test imposing tradable market factor when non-tradable

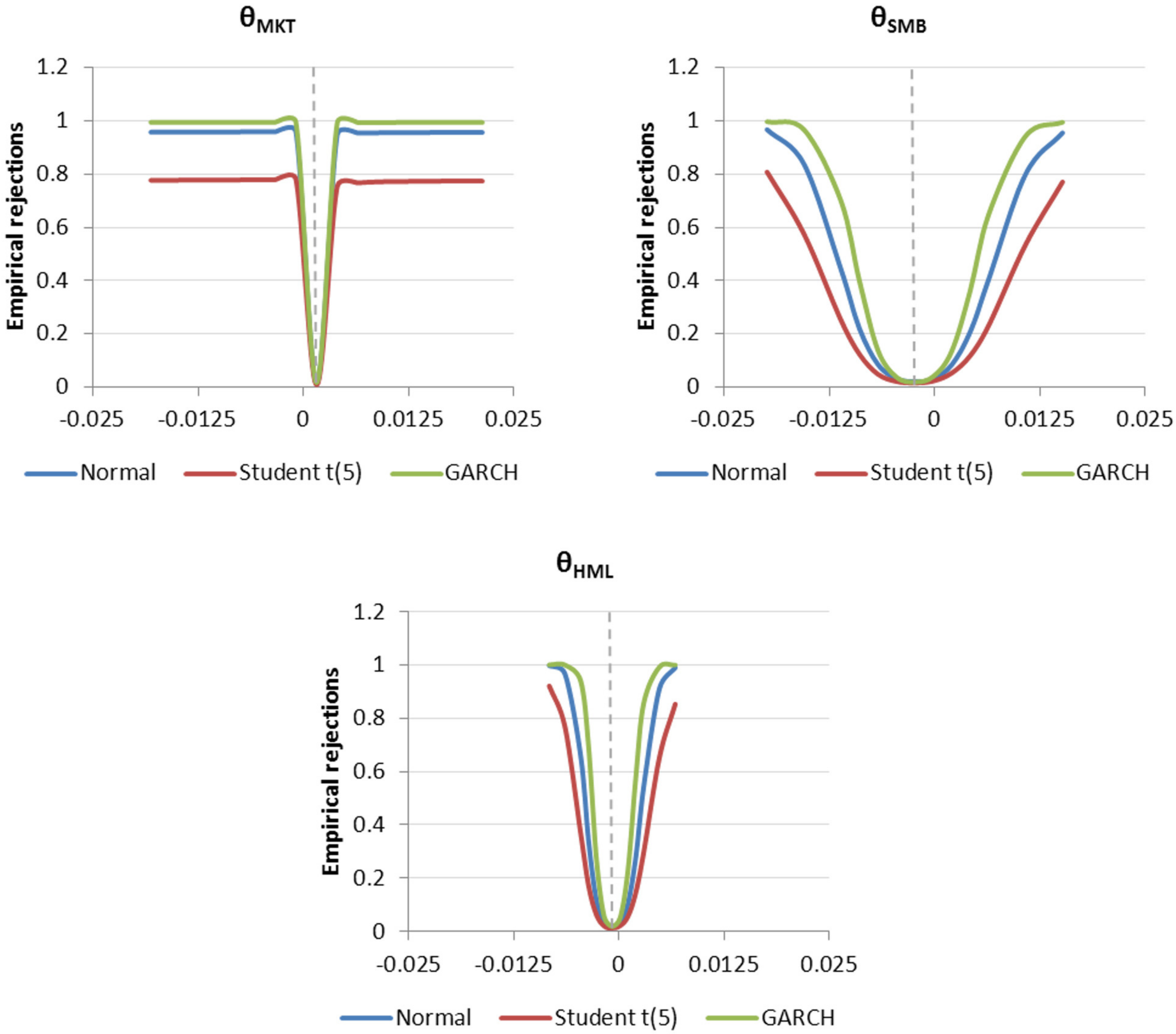
$n = 12,\ T = 624$	$r_i = a_i \iota_T + \mathscr{R}_1 b_{i1} + \mathscr{F} b_{i\mathscr{F}} + u_i$
True model	$a_i = \gamma_c - \gamma_0 b_{i1} - \gamma'_{\mathscr{F}} b_{i\mathscr{F}}, \quad \gamma_0 = \theta_{\text{MKT}}, \quad \gamma_{\mathscr{F}} = (\theta_{\text{SMB}}, \theta_{\text{HML}})'$
Empirical model	$a_i = \gamma_0(1 - b_{i1}) - \gamma'_{\mathscr{F}} b_{i\mathscr{F}}$
RAPT: Inverted Statistic $\Lambda(\theta), \quad \theta = (\gamma_0, \gamma'_{\mathscr{F}})'$	



Note – See notes to figure 1.

Figure 7: Monte Carlo study: tests imposing tradable market factor, performance with 10 years of data

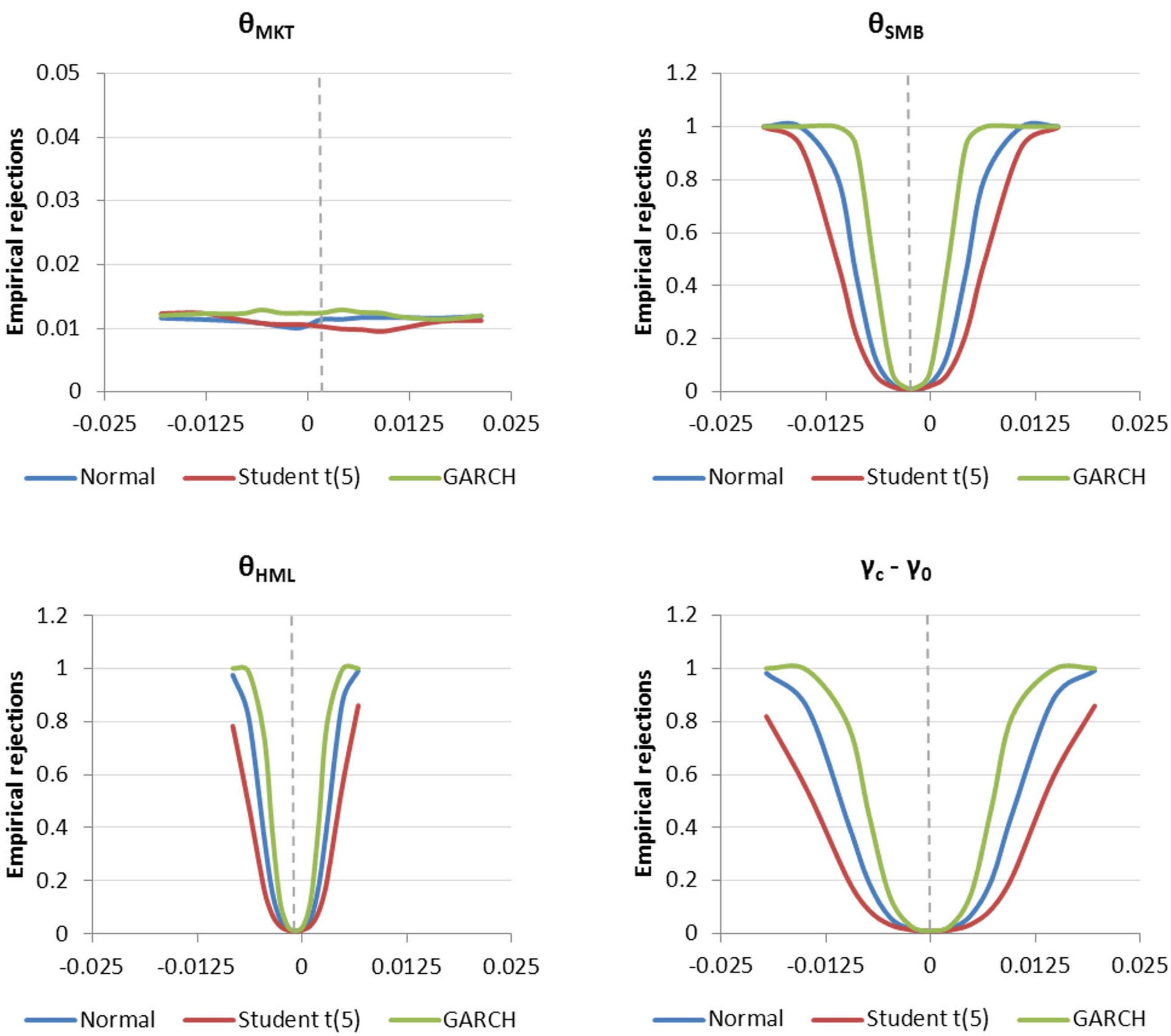
$n = 12, T = 120$	$r_i = a_i \iota_T + \mathscr{R}_1 b_{i1} + \mathscr{F} b_{i\mathscr{F}} + u_i$
True model	$a_i = \gamma_0(1 - b_{i1}) - \gamma'_{\mathscr{F}} b_{i\mathscr{F}}, \quad \gamma_0 = \theta_{\text{MKT}}, \quad \gamma_{\mathscr{F}} = (\theta_{\text{SMB}}, \theta_{\text{HML}})'$
Empirical model	$a_i = \gamma_0(1 - b_{i1}) - \gamma'_{\mathscr{F}} b_{i\mathscr{F}}$
RAPT: Inverted Statistic $\Lambda(\theta), \quad \theta = (\gamma_0, \gamma'_{\mathscr{F}})'$	



Note – See notes to figure 1 and table 1. The training sample used to generate simulation parameters is restricted to the last 10 years of data.

Figure 8: Monte Carlo study: joint tests, market factor tradable assumed non-tradable

$n = 12,\ T = 624$	$r_i = a_i \iota_T + \mathscr{R}_1 b_{i1} + \mathscr{F} b_{i\mathscr{F}} + u_i$
True model	$a_i = \gamma_c - \gamma_0 b_{i1} - \gamma_{\mathscr{F}}' b_{i\mathscr{F}}, \quad \gamma_0 = \theta_{\text{MKT}}, \quad \gamma_{\mathscr{F}} = (\theta_{\text{SMB}}, \theta_{\text{HML}})'$
Empirical model	$a_i = \gamma_c - \gamma_0 b_{i1} - \gamma_{\mathscr{F}}' b_{i\mathscr{F}}$
UAPT: Inverted Statistic $\Lambda(\theta, \phi), \quad \theta = \left(\gamma_0, \gamma_{\mathscr{F}}'\right)', \quad \phi = (\gamma_c - \gamma_0)$	



Note - See notes to figure 1 and table 1. Results are obtained by executing $\Lambda(\theta, \phi)$ on returns in deviation from $\mathscr{R}_1$. Under the null, we hold $\gamma_c = \gamma_0$, and under the alternative, γ_c moves away from the true value, whereas γ_0 does not.

following discussion thus focuses on the restricted model outcomes in table 6. Interestingly, results for the atypical 70s and 90s stand in sharp contrast with those we obtained using Carhart's model: MKT and one of the RMW and CMA factors are priced whereas SMB is not; instead, recall that SMB and MOM are jointly priced with the Carhart model, whereas the MKT factor was not.

The obvious question is, then, whether observed factors represent risks or anomalies. Yet any interpretation of our findings in this regard is hasty, given our focus thus far on the atypical subperiods. Outside these subperiods, interpretations in any direction are severely hampered by the more pernicious identification failures we observe therein. Indeed, over and above MKT and SMB, the addition of MOM and HML yields completely uninformative sets (the real line) prior to the 70s and so does the addition of RMW and CMA. In the 80s, the Carhart model is uninformative whereas MKT and RMW are priced despite unbounded sets. In contrast, post 2000, SMB is the only priced factor in the Carhart model whereas the four-factor Fama-French model is completely uninformative.

In sum, four key results are worth emphasizing. First, the three Fama-French factors are confirmed to be priced concurrently only before 1970. From there on, the factors are either: (i) jointly rejected in the sense that anomalies remain despite some evidence of pricing, or (ii) are weakly supported, in the following sense: we find no clear indication on which among the three is priced or not. Second, with regards to the historical debate on anomalies[9], we do not find convincing and uniform evidence favoring any factor relative to MKT. Third, MOM is not necessarily irrelevant despite its adverse effect on identification broadly, and may possibly proxy an outstanding anomaly relative to the three-factors model in the 70s and 90s atypical subperiods. Fourth, heterogeneity is not sufficient to distinguish a priced momentum anomaly from profitability or investment as presumably non-diversifiable risk drivers.

Size portfolios preserve some of the above findings, though globally, evidence weakens as identification is visibly weaker. The latter finding reinforces the argument in Lewellen et al. (2010, Footnote 1), namely that size and book-to-market sorted betas on MKT are close to one, a fact that seems to empirically endure since Fama and French (1993).

Notwithstanding almost inevitable resulting under-identification, table 7 broadly underscores the following, relative to industry portfolios. First, the restricted three-factor model is no longer rejected in the 70s and 90s; in fact it passes our test overall. Interestingly, the three factors are jointly priced in the 70s, whereas in the 90s, the only priced factor is MKT. Second, in all subperiods expect the 80s and 2000s in which our confidence sets on MKT risk are the real line, MKT is priced. Third, prior to the 90s, HML is always priced. Fourth, the data is not informative post-2000s, a result shared to some extent with industry sorts.

The addition of momentum provokes under-identification since almost all confidence sets for risk prices are the real line, and so does unrestricting the intercept with and without momentum (the latter results are not reported for space consideration). The same holds when adding RMW and CMA, with and without HML; a sample of these results is reported in the Appendix.

Further results including conditional models are reported in the supplementary appendix. Results confirm that the identification problems in this literature are not solved by standard conditioning, which seems instead to exacerbate complications.

5 Conclusion

One of the key goals of asset pricing is to identify factors that drive asset returns and are associated with risk premiums. This paper contributes to this literature via an identification-robust methodology to assess pricing,

[9]See Campbell et al. (1997, Chapters 5 and 6), Fama and French (2004), Perold (2004), Campbell (2003), Sentana (2009) and the recent insight in Fama and French (2015).

Table 2: Simultaneous confidence sets for factor loadings

2001-2010, Value-Weighted Industry Portfolios				
Eq.	Intercept	MKT -1	SMB	HML
1	[-.0062,.0129]	**[-.5969,-.1868]**	[-.4630,.2560]	[-.0498,.5577]
2	[-.0214,.0180]	[-.0563,.7905]	[-.2121,1.2723]	[-.0808,1.1737]
3	[-.0079,.0127]	[-.0405,.4033]	[-.1699,.6082]	[-.0701,.5874]
4	[-.0145,.0298]	[-.6664,.2849]	[-1.0358,.6319]	[-.4649,.9445]
5	[-.0063,.0147]	[-.4489,.0022]	[-.5510,.2397]	[-.0700,.5982]
6	[-.0095,.0113]	**[.1698,.6156]**	[-.0586,.7223]	**[-1.2226, -.5623]**
7	[-.0147,.0148]	[-.2125,.4201]	[-.9030,.2062]	[-.5811,.3563]
8	[-.0146,.0188]	**[-.7735,-.0573]**	[-.7852,.4705]	[-.2415,.8196]
9	[-.0102,.0113]	[-.4224,.0396]	[-.1606,.6493]	[-.2435,.4409]
10	[-.0118,.0122]	[-.5823,-.0659]	[-.8394,.0660]	[-0.3373,.4278]
11	[-.0156,.0062]	[-.1480,.3206]	[-.4751,.3463]	**[.2860,.9802]**
12	[-.0103,.0077]	[-.1198,.2679]	[-.3047,.3750]	[-.0969,.4776]
Hotelling	1.2131	16.9573	5.4253	23.2133
p-value	.284	.000	0.000	0.000

Note – See notes to table 3 for the definition of the considered sample. Intervals reported are the 95% joint (across equations) confidence sets for the coefficients (in turn) of each portfolio regression numbered 1-12. The inverted test in each case is $\Lambda(\bar{\beta}_j)$ defined in (3.5) to test H_j (2.20). The Classical Hotelling joint significance test with conforming p-value is reported at the bottom of each column to assess each of H_{0j} (2.21). $j = 1$ provides joint inference on the unrestricted regression intercepts, and as the unrestricted regression is in deviation from the tradable factor, here MKT, $j = 2$ provides joint inference on market betas in deviation from one, and $j = 3, 4$ provide inference, in turn, on SMB and HML betas. Confidence sets in bold are those that do not cover zero.

Table 3: Confidence sets for risk price: industry portfolios and three factor model

$$r_i - \iota_T \gamma_0 = (\mathscr{R}_1 - \iota_T \gamma_0) b_{i1} + (\mathscr{F} - \iota_T \gamma'_{\mathscr{F}}) b_{i\mathscr{F}} + u_i, \qquad i = 1, \ldots, n$$

PANEL A $\theta = (\gamma_0, \gamma'_{\mathscr{F}})' = (\theta_{\text{MKT}}, \theta_{\text{SMB}}, \theta_{\text{HML}})'$

	MKT		SMB		HML	
$\times 10^{-4}$	$\overline{F}$	θ_{MKT}	$\overline{F}$	θ_{SMB}	$\overline{F}$	θ_{HML}
61-70	38*	[-437,-55]	33*	[-17,14]	53*	[-46,41]
71-80	30	$\varnothing$	43	$\varnothing$	33	$\varnothing$
81-90	44*	$]-\infty, -194] \cup [1156,\infty]$	-16	$\mathbb{R}$	56*	$]-\infty$, -115] $\cup$ [611,∞[
91-00	103	$\varnothing$	4	$\varnothing$	29	$\varnothing$
00-10	14	[-79,245]	57*	[-146,3]	40	[-41,178]

$$r_i - \iota_T \gamma_c = (\mathscr{R}_1 - \iota_T \gamma_0) b_{i1} + \left(\mathscr{F} - \iota_T \gamma'_{\mathscr{F}}\right) b_{i\mathscr{F}} + u_i, , \qquad i = 1, \ldots, n$$

PANEL B $\theta = (\gamma_0, \gamma'_{\mathscr{F}}) = (\theta_{\text{MKT}}, \theta_{\text{SMB}}, \theta_{\text{HML}})$, γ_c partialled-out

	MKT		SMB		HML	
$\times 10^{-4}$	$\overline{F}$	θ_{MKT}	$\overline{F}$	θ_{SMB}	$\overline{F}$	θ_{HML}
61-70	38*	[-642, -14]	33	[-11, 83]	53	[-64, 74]
71-80	30*	[-621, -54]	43	[-26, 158]	33	[-137, 160]
81-90	44*	$]-\infty$, -198] $\cup$ [807,∞[	-16	$\mathbb{R}$	56	$\mathbb{R}$
91-00	103*	$]-\infty$, -432] $\cup$ [3087,∞[	4*	$]-\infty$, -1499] $\cup$ [97,∞[	29	$]-\infty$, -281] $\cup$ [-24,∞[
00-10	14	[-100, 1226]	57	[-1089, 66]	40	[-137, 192]

Note – Sample includes monthly observations from January 1991 to December 2010 on the US. Series include 12 equally weighted (EW) industry portfolios as well as US factors for market (MKT), size (SMB), book-to-market (HML). Confidence sets are at the 5% level. $\overline{F}$ is the factor average over the considered time period; θ captures factor pricing as defined in (2.15). * denotes evidence of pricing at the 5% significance level interpreted as follows: given the reported confidence sets, each factor is priced if its average is not covered. In Panel A, the inverted test is $\Lambda(\theta)$ defined in 3.2. This test follows our **RAPT** approach where **R** stands for "restricted" implying that tradable factor constraints are imposed, here on $\mathscr{R}_1$, in estimating and testing the model. In Panel B, the inverted test $\Lambda(\theta)$ is applied on a system on $n-1$ returns in deviation from r_n. This test follows our **PAPT** approach where **P** stands for "partialling-out" implying that all factors are assumed non-tradable but the resulting unrestricted constant is evacuated from the statistical objective function as it is based on $r_i - r_n$, $i = 1, \ldots, n-1$.

Table 4: Confidence sets for risk price: industry portfolios and four factor model

$$r_i - \iota_T \gamma_0 = (\mathscr{R}_1 - \iota_T \gamma_0) b_{i1} + (\mathscr{F} - \iota_T \gamma'_{\mathscr{F}}) b_{i\mathscr{F}} + u_i, \qquad i = 1, \ldots, n$$

$$\theta = \left(\gamma_0, \gamma'_{\mathscr{F}}\right) = \left(\theta_{\text{MKT}}, \theta_{\text{SMB}}, \theta_{\text{HML}}, \theta_{\text{MOM}}\right)$$

	MKT		SMB		HML		MOM	
$\times 10^{-4}$	$\overline{F}$	θ_{MKT}	$\overline{F}$	θ_{SMB}	$\overline{F}$	θ_{HML}	$\overline{F}$	θ_{MOM}
61-70	38	$\mathbb{R}$	33	$\mathbb{R}$	53	$\mathbb{R}$	73	$\mathbb{R}$
71-80	30	[-300, 232]	43*	[-62, -17]	33	[-87, 111]	113*	[-333, 98]
81-90	44	$\mathbb{R}$	-16	$\mathbb{R}$	56	$\mathbb{R}$	66	$\mathbb{R}$
91-00	103	[-617, 123]	4*	[-387,-20]	29	[-267,36]	112*	[-995, -45]
00-10	14	[-124, 258]	57*	[-212,5]	40	[-52, 204]	-3	[-554, 129]

$$r_i - \iota_T \gamma_c = (\mathscr{R}_1 - \iota_T \gamma_0) b_{i1} + (\mathscr{F} - \iota_T \gamma'_{\mathscr{F}}) b_{i\mathscr{F}} + u_i, \qquad i = 1, \ldots, n$$

PANEL B $\theta = (\gamma_0, \gamma'_{\mathscr{F}}) = (\theta_{\text{MKT}}, \theta_{\text{SMB}}, \theta_{\text{HML}}, \theta_{\text{MOM}}), \quad \gamma_c$ partialled-out

	MKT		SMB		HML		MOM	
$\times 10^{-4}$	$\overline{F}$	θ_{MKT}	$\overline{F}$	θ_{SMB}	$\overline{F}$	θ_{HML}	$\overline{F}$	θ_{MOM}
61-70	38	$\mathbb{R}$	33	$\mathbb{R}$	53	$\mathbb{R}$	73	$\mathbb{R}$
71-80	30	[-654, 316]	43	[-53, 166]	33	[-143, 230]	113	[-499, 286]
81-90	44	$\mathbb{R}$	-16	$\mathbb{R}$	56	$\mathbb{R}$	66	$\mathbb{R}$
91-00	103	$]-\infty, 112] \cup [1959, \infty[$	4	$]-\infty, -764] \cup [-344, \infty[$	29	$\mathbb{R}$	112	$\mathbb{R}$
00-10	14	[-148, 1552]	57	[-1502, 75]	40	[-172, 227]	- 3	[-1547, 108]

Note – See notes to table 3. The considered model is the four factor case with market (MKT), size (SMB), book-to-market (HML) and momentum (MOM).

Table 5: Industry portfolios, testing the traded factor assumption

$$r_i - \mathscr{R}_1 = a_i \iota_T + \mathscr{R}_1 d_i + \mathscr{F} b_{i\mathscr{F}} + u_i, \qquad i = 1, \ldots, n,$$

$$a_i = \gamma_c^* - \gamma_0 d_i - \gamma_{\mathscr{F}}' b_{i\mathscr{F}}, \quad d_i = b_{i1} - 1, \quad \gamma_c^* = \gamma_c - \gamma_0$$

$$\theta = (\gamma_0, \gamma_{\mathscr{F}}') = (\theta_{\text{MKT}}, \theta_{\text{SMB}}, \theta_{\text{HML}})$$

	CTE	MKT		SMB		HML		MOM	
$\times 10^{-4}$	γ_c^*	$\overline{F}$	θ_{MKT}	$\overline{F}$	θ_{SMB}	$\overline{F}$	θ_{HML}	$\overline{F}$	θ_{MOM}
61-70	[-8, 101]	38*	[-777, 9]	33	[-17, 94]	53	[-81, 82]	-	-
71-80	[5, 217]	30*	[-715, -35]	43	[-34, 178]	33	[-179, 173]	-	-
81-90	$\mathbb{R}$	44*	$]-\infty$, -164] $\cup$ [707,∞[	-16	$\mathbb{R}$	56	$\mathbb{R}$	-	-
91-00	$]-\infty$, -898] $\cup$ [150,∞[	103*	$]-\infty$, -365] $\cup$ [1950,∞[	4*	$]-\infty$, -964] $\cup$ [58,∞[	29	$\mathbb{R}$	-	-
00-10	[-1876, 115]	14	[-121, 2768]	57	[-2461, 84]	40	[-254, 245]	-	-

$$r_i - \mathscr{R}_1 = a_i \iota_T + \mathscr{R}_1 d_i + \mathscr{F} b_{i\mathscr{F}} + u_i, \qquad i = 1, \ldots, n,$$

$$a_i = \gamma_c^* - \gamma_0 d_i - \gamma_{\mathscr{F}}' b_{i\mathscr{F}}, \quad d_i = b_{i1} - 1, \quad \gamma_c^* = \gamma_c - \gamma_0$$

$$\theta = \left(\gamma_0, \gamma_{\mathscr{F}}'\right) = \left(\theta_{\text{MKT}}, \theta_{\text{SMB}}, \theta_{\text{HML}}, \theta_{\text{MOM}}\right)$$

	CTE	MKT		SMB		HML		MOM	
$\times 10^{-4}$	γ_c^*	$\overline{F}$	θ_{MKT}	$\overline{F}$	θ_{SMB}	$\overline{F}$	θ_{HML}	$\overline{F}$	θ_{MOM}
61-70	$\mathbb{R}$	38	$\mathbb{R}$	33	$\mathbb{R}$	53	$\mathbb{R}$	73	$\mathbb{R}$
71-80	[-13, 224]	30	[-803, 430]	43	[-62, 182]	33	[-186, 260]	113	[-586, 375]
81-90	$\mathbb{R}$	44	$\mathbb{R}$	-16	$\mathbb{R}$	56	$\mathbb{R}$	66	$\mathbb{R}$
91-00	$]-\infty$, -714] $\cup$[-111,∞[	103	$]-\infty$, 157] $\cup$[1590,∞[	4	$]-\infty$, -600] $\cup$[-407,∞[	29	$\mathbb{R}$	112	$\mathbb{R}$
00-10	[-3548, 135]	14	[-170, 5023]	57	[-4838, 94]	40	[-398, 323]	- 3	[-4568, 129]

Note – See notes to tables 3 and 4. The inverted test is $\Lambda(\theta, \phi)$ is defined in (3.1). This test follows our **UAPT** where **U** stands for "unrestricted" implies that factors are assumed non-tradable in estimating and testing the model. The test is applied on $r_i - \mathscr{R}_1$ so inference on ϕ allows to assess whether $\gamma_c = \gamma_0$: the hypothesis that $\mathscr{R}_1$ (here, MKT) is traded is rejected at the 5% level when the confidence set on ϕ excludes zero.

Table 6: Confidence sets for risk price: industry portfolios and five-factor model, excluding HML

$$r_i - \iota_T \gamma_0 = (\mathscr{R}_1 - \iota_T \gamma_0) b_{i1} + (\mathscr{F} - \iota_T \gamma'_{\mathscr{F}}) b_{i\mathscr{F}} + u_i, \qquad i = 1, \ldots, n$$

PANEL A $\theta = (\gamma_0, \gamma'_{\mathscr{F}})' = (\theta_{\text{MKT}}, \theta_{\text{SMB}}, \theta_{\text{RMW}}, \theta_{\text{CMA}})'$

	MKT		SMB		RMW		CMA	
$\times 10^{-4}$	$\overline{F}$	θ_{MKT}	$\overline{F}$	θ_{SMB}	$\overline{F}$	θ_{RMW}	$\overline{F}$	θ_{CMA}
63-70	28	$\mathbb{R}$	57	$\mathbb{R}$	2	$\mathbb{R}$	22	$\mathbb{R}$
71-80	30*	$]-\infty, -79] \cup [34937, \infty[$	54	$]-\infty, -1511] \cup [-79, \infty[$	5*	$]-\infty, -20138] \cup [14, \infty[$	25	$]-\infty, 210] \cup [14260, \infty[$
81-90	44*	$]-\infty, -78] \cup [82, \infty[$	-20	$]-\infty, -1] \cup [42, \infty[$	39*	$]-\infty, -75] \cup [95, \infty[$	55	$]-\infty, -143] \cup [-6, \infty[$
91-00	103*	[-1568, 85]	3	[-277, 237]	32	[-118, 362]	30*	[-591, -8]
00-10	14	$\mathbb{R}$	65	$\mathbb{R}$	44	$\mathbb{R}$	35	$\mathbb{R}$

$$r_i - \iota_T \gamma_c = (\mathscr{R}_1 - \iota_T \gamma_0) b_{i1} + (\mathscr{F} - \iota_T \gamma'_{\mathscr{F}}) b_{i\mathscr{F}} + u_i, , \qquad i = 1, \ldots, n$$

PANEL B $\theta = \left(\gamma_0, \gamma'_{\mathscr{F}}\right) = (\theta_{\text{MKT}}, \theta_{\text{SMB}}, \theta_{\text{RMW}}, \theta_{\text{CMA}})$, γ_c partialled-out

	MKT		SMB		RMW		CMA	
$\times 10^{-4}$	$\overline{F}$	θ_{MKT}	$\overline{F}$	θ_{SMB}	$\overline{F}$	θ_{RMW}	$\overline{F}$	θ_{CMA}
63-70	28	$\mathbb{R}$	57	$\mathbb{R}$	2	$\mathbb{R}$	22	$\mathbb{R}$
71-80	30*	$]-\infty, -26] \cup [1108, \infty[$	54	$]-\infty, 167] \cup [282, \infty[$	5	$]-\infty, -1041] \cup [-118, \infty[$	25	$]-\infty, 202] \cup [664, \infty[$
81-90	44	$\mathbb{R}$	-20	$\mathbb{R}$	39	$\mathbb{R}$	55	$\mathbb{R}$
91-00	103	$]-\infty, 105] \cup [1790, \infty[$	3	$]-\infty, -662] \cup [-238, \infty[$	32	$\mathbb{R}$	30	$\mathbb{R}$
00-10	14	$\mathbb{R}$	65	$\mathbb{R}$	44	$\mathbb{R}$	35	$\mathbb{R}$

Note – Sample includes monthly observations from July 1963 to December 2010 on the US. Series include 12 equally weighted (EW) industry portfolios as well as US factors for market (MKT), size (SMB), profitability (RMW), and investment (CMA). Results in Panel A rely on our **RAPT** approach, and those in Panel B or its **PAPT** counterpart; see notes to notes to table 3 for further definitions.

Table 7: Confidence sets for risk price: size portfolios

$$r_i - \iota_T \gamma_0 = (\mathscr{R}_1 - \iota_T \gamma_0) b_{i1} + (\mathscr{F} - \iota_T \gamma'_{\mathscr{F}}) b_{i\mathscr{F}} + u_i, \qquad i = 1, \ldots, n$$

$$\theta = (\gamma_0, \gamma'_{\mathscr{F}}) = \left(\theta_{\text{MKT}}, \theta_{\text{SMB}}, \theta_{\text{HML}}\right)$$

	MKT		SMB		HML		MOM	
$\times 10^{-4}$	$\overline{F}$	θ_{MKT}	$\overline{F}$	θ_{SMB}	$\overline{F}$	θ_{HML}	$\overline{F}$	θ_{MOM}
61-70	38*	$]-\infty,$ -256] $\cup$[221,∞[	33	$]-\infty,$ -87] $\cup${[-71,∞[}	53*	$]-\infty,$ -128] $\cup$[1008,∞[	-	-
71-80	30*	$]-\infty,$ -229] $\cup$[25121,∞[	43*	$]-\infty,$ -253] $\cup$[9972,∞[	33*	$]-\infty,$ -30714] $\cup$[639,∞[	-	-
81-90	44	$\mathbb{R}$	-16	$\mathbb{R}$	56*	$]-\infty,$ -881] $\cup$[975,∞[	-	-
91-00	103*	[-14492,72]	4	[-732,649]	29	[-3055,886]	-	-
00-10	14	$\mathbb{R}$	57	$\mathbb{R}$	40	$\mathbb{R}$	-	-

$$r_i - \iota_T \gamma_0 = (\mathscr{R}_1 - \iota_T \gamma_0) b_{i1} + \left(\mathscr{F} - \iota_T \gamma'_{\mathscr{F}}\right) b_{i\mathscr{F}} + u_i, \qquad i = 1, \ldots, n$$

$$\theta = \left(\gamma_0, \gamma'_{\mathscr{F}}\right) = \left(\theta_{\text{MKT}}, \theta_{\text{SMB}}, \theta_{\text{HML}}, \theta_{\text{MOM}}\right)$$

	MKT		SMB		HML		MOM	
$\times 10^{-4}$	$\overline{F}$	θ_{MKT}	$\overline{F}$	θ_{SMB}	$\overline{F}$	θ_{HML}	$\overline{F}$	θ_{MOM}
61-70	38	$\mathbb{R}$	33	$\mathbb{R}$	53	$\mathbb{R}$	73	$\mathbb{R}$
71-80	30	$\mathbb{R}$	43	$\mathbb{R}$	33	$\mathbb{R}$	113	$\mathbb{R}$
81-90	44	$\mathbb{R}$	-16	$\mathbb{R}$	56	$\mathbb{R}$	66	$\mathbb{R}$
91-00	103	$\mathbb{R}$	4	$\mathbb{R}$	29	$\mathbb{R}$	112	$]-\infty,$ 3029] $\cup$[5573,∞[
00-10	14	$\mathbb{R}$	57	$\mathbb{R}$	40	$\mathbb{R}$	3	$\mathbb{R}$

Note – Sample includes monthly observations from January 1991 to December 2010 on the US. Series include 25 size sorted equally weighted (EW) and value-weighted (VW) portfolios as well as US factors for market (MKT), size (SMB), book-to-market (HML) and momentum (MOM). See notes to tables 3 and 4 for further definitions and applied inference methods.

regardless of whether betas are jointly informative or not, or heterogenous enough to identify risk price, *i.e.*, to identify factors that represent a non-diversifiable source of risk rather than an idiosyncratic association with returns.

As with Lewellen et al. (2010), our methodology is applied to models with a given and relatively small number of popular factors. The motivation [see Lewellen et al. (2010, Footnote 3)] may be traced to Fama and French (1993) whose main message is that relevant risks can be summarized by a small number of factors. Since then, the literature does not necessarily dispute this fact, in the sense that more is not necessarily viewed as better. Instead of a consensus view on a common set of explanatory factors, a plethora of different although related candidate factors has been proposed, which raises enduring empirical puzzles, statistical concerns and ultimately, spurious pricing considerations [Harvey et al. (2016)].

The main message in both strands of the literature reflected by Lewellen et al. (2010) (on analyzing models given a small number of given factors), or Harvey et al. (2016) (on factor searches globally) is that more stringent practices are needed. Our methodology serves this purpose by robustifying inference on risk price, controlling for the quality of available betas. Whether practice moves towards more parsimonious ways of summarizing information on factors, or towards reliance on test assets instead of test portfolios, our message is that statistical inference on risk price should not take identification for granted.

References

Ahmed, S., Bu, Z. and Tsvetanov, D. (2019), 'Best of the best: a comparison of factor models', *Journal of Financial and Quantitative Analysis* **54**(00), 1713–1758.

Barillas, F. and Shanken, J. (2017), 'Which alpha ?', *The Review of Financial Studies* **30**, 1316–1338.

Barillas, F. and Shanken, J. (2018), 'Comparing asset pricing models', *The Journal of Finance* **73**, 715–754.

Barone-Adesi, G., Gagliardini, P. and Urga, G. (2004), 'Testing asset pricing models with coskewness', *Journal of Business and Economic Statistics* **22**, 474–485.

Beaulieu, M.-C., Dufour, J.-M. and Khalaf, L. (2007), 'Multivariate tests of mean-variance efficiency with possibly non-Gaussian errors: An exact simulation-based approach', *Journal of Business and Economic Statistics* **25**, 398–410.

Beaulieu, M.-C., Dufour, J.-M. and Khalaf, L. (2009), 'Finite sample multivariate tests of asset pricing models with coskewness', *Computational Statistics and Data Analysis* **53**, 2008–2021.

Beaulieu, M.-C., Dufour, J.-M. and Khalaf, L. (2010), 'Asset-pricing anomalies and spanning: Multivariate multifactor tests with heavy tailed distributions', *Journal of Empirical Finance* **17**, 763–782.

Beaulieu, M.-C., Dufour, J.-M. and Khalaf, L. (2013), 'Identification-robust estimation and testing of the Zero-Beta CAPM', *The Review of Economics Studies* **80**, 892–924.

Berndt, E. R. and Savin, N. E. (1977), 'Conflict among criteria for testing hypotheses in the multivariate linear regression model', *Econometrica* **45**, 1263–1277.

Boguth, O., Carlson, M., Fisher, A. J. and Simutin, M. (2011), 'Conditional risk and performance evaluation: Volatility timing, overconditioning, and new estimates of momentum alpha', *Journal of Financial Economics* **102**, 363–389.

Bolduc, D., Khalaf, L. and Yelou, C. (2010), 'Identification robust confidence sets methods for inference on parameter ratios with application to discrete choice models', *Journal of Econometrics* **157**, 317–327.

Campbell, J. Y. (2003), 'Asset pricing at the millennium', *Journal of Finance* **55**, 1515–1567.

Campbell, J. Y., Lo, A. W. and MacKinlay, A. C. (1997), *The Econometrics of Financial Markets*, Princeton University Press, New Jersey.

Carhart, M. M. (1997), 'On persistence in mutual fund performance', *Journal of Finance* **52**, 57–82.

Chib, S. and Zeng, X. (2019), 'Which factors are risk factors in asset pricing? A model scan framework', *Journal of Business and Economic Statistics* **Forthcoming**.

Cochrane, J. H. (2005), *Asset Pricing*, Princeton University Press, New Jersey.

Davidson, R. and MacKinnon, J. G. (2004), *Econometric Theory and Methods*, Oxford University Press, New York.

Dhrymes, P. J. (1974), *Econometrics: Statistical Foundations and Applications*, Springer-Verlag, New York.

Dufour, J.-M. (2006), 'Monte Carlo tests with nuisance parameters: A general approach to finite-sample inference and nonstandard asymptotics in econometrics', *Journal of Econometrics* **133**, 443–478.

Dufour, J.-M. and Khalaf, L. (2002), 'Simulation based finite and large sample tests in multivariate regressions', *Journal of Econometrics* **111**, 303–322.

Dufour, J.-M. and Taamouti, M. (2005), 'Projection-based statistical inference in linear structural models with possibly weak instruments', *Econometrica* **73**, 1351–1365.

Dufour, J.-M. and Taamouti, M. (2007), 'Further results on projection-based inference in IV regressions with weak, collinear or missing instruments', *Journal of Econometrics* **139**, 133–153.

Engle, R. F. and Kroner, K. F. (1995), 'Multivariate generalized ARCH', *Econometric Theory* **11**, 122–150.

Fama, E. F. and French, K. R. (1992), 'The cross-section of expected stock returns', *Journal of Finance* **47**, 427–465.

Fama, E. F. and French, K. R. (1993), 'Common risk factors in the returns on stocks and bonds', *Journal of Financial Economics* **33**, 3–56.

Fama, E. F. and French, K. R. (2004), 'The capital asset pricing model: Theory and evidence', *Journal of Economic Perspectives* **18**, 25–46.

Fama, E. F. and French, K. R. (2015), 'A five-factor asset pricing model', *Journal of Financial Economics* **116**, 1–22.

Gagliardini, P., Ossola, E. and Scaillet, O. (2016), 'Time-varying risk premium in large cross-sectional equity data sets', *Econometrica* **84**, 985–1046.

Gibbons, M. R., Ross, S. A. and Shanken, J. (1989), 'A test of the efficiency of a given portfolio', *Econometrica* **57**, 1121–1152.

Gospodinov, N., Kan, R. and Robotti, C. (2014), 'Misspecification-robust inference in linear asset-pricing models with irrelevant risk factors', *Review of Financial Studies* **27**, 2139–2170.

Gospodinov, N., Kan, R. and Robotti, C. (2019), 'Too good to be true? fallacies in evaluating risk factor models', *Journal of Financial Economics* **132**, 451–471.

Gouriéroux, C., Monfort, A. and Renault, E. (1995), Inference in factor models, *in* G. S. Maddala, P. C. B. Phillips and T. N. Srinivasan, eds, 'Advances in Econometrics and Quantitative Economics', Blackwell, Oxford, U.K., chapter 13, pp. 311–353.

Gouriéroux, C., Monfort, A. and Renault, E. (1996), *Tests sur le noyau, l'image et le rang de la matrice des coefficients d'un modèle linéaire multivarié*, Oxford University press, Oxford (U.K.).

Hamilton, J. D. (2010), Macroeconomics and ARCH, *in* T. Bollerslev, J. R. Russell and M. Watson, eds, 'Festschrift in Honor of Robert F. Engle', Oxford University Press, pp. 79–96.

Harvey, C. R., Liu, Y. and Zhu, H. (2016), '... and the cross-section of expected returns', *Review of Financial Studies* **29**(1), 5–68,.

Harville, D. A. (1997), *Matrix Algebra from a Statistician's Perspective*, Springer-Verlag, New York.

Hotelling, H. (1947), *Multivariate Quality Control Illustrated by the Air Testing of Sample Bomb Sights,Techniques of Statistical Analysis, Ch. II*, McGraw-Hill, New York.

Hou, K., Mo, H., Xue, C. and Zhang, L. (2018), ‘Which factors?’, *Review of Finance* **23**, 1–35.

Jegadeesh, N., Noh, J., Pukthuanthong, K., Roll, R. and Wang, J. (2019), ‘Empirical tests of asset pricing models with individual assets: Resolving the errors-in-variables bias in risk premium estimation’, *Journal of Financial Economics* **133**(2), 273–298.

Johansen, S. (1995), *Likelihood-based Inference in Cointegrated Vector Autoregressive Model*, second edn, Oxford University Press, New York.

Kan, R., Robotti, C. and Shanken, J. (2013), ‘Pricing model performance and the two-pass cross-sectional regression methodology’, *The Journal of Finance* **68**, 2617–2649.

Kan, R. and Zhang, C. (1999), ‘Two-pass tests of asset pricing models with useless factors’, *Journal of Finance* **54**, 204–235.

Kandel, S. (1984), ‘The likelihood ratio test statistics of mean-variance efficiency without a riskless asset’, *Journal of Financial Economics* **13**, 575–592.

Kandel, S. (1986), ‘The geometry of the likelihood estimator of the zero beta return’, *Journal of Finance* **31**, 339–346.

Khalaf, L. and Schaller, H. (2016), ‘Identification and inference in two-pass asset pricing models’, *Journal of Economic Dynamics and Control* **70**, 165–177.

Khalaf, L. and Urga, G. (2014), ‘Identification robust inference in cointegrating regressions’, *Journal of Econometrics* **182**, 385–396.

Kim, S. and Skoulakis, G. (2018), ‘Ex-post risk premia estimation and asset pricing tests using large cross sections: The regression-calibration approach’, *Journal of Econometrics* **204**, 159–188.

Kleibergen, F. (2009), ‘Tests of risk premia in linear factor models’, *Journal of Econometrics* **149**, 149–173.

Kleibergen, F., Lingwei, K. and Zhan, Z. (2019), ‘Identification robust testing of risk premia in finite samples’, *The Journal of Financial Econometrics* **Forthcoming**.

Kleibergen, F. and Zhan, Z. (2015), ‘Unexplained factors and their effects on second pass R-squared’s and t-tests’, *Journal of Econometrics* **189**(1), 106–116.

Kleibergen, F. and Zhan, Z. (2020), ‘Robust inference for consumption-based asset pricing’, *The Journal of Finance* (1), 507–550.

Lewellen, J., Nagel, S. and Shanken, J. (2010), ‘A skeptical appraisal of asset-pricing tests’, *Journal of Financial Economics* **92**, 175–194.

McKeon, J. J. (1974), ‘F approximation to the distribution of hotelling’s t^2’, *Journal of Business and Economic Statistics* **15**, 74–81.

Penaranda, F. and Sentana, E. (2016), ‘Duality in mean-variance frontiers with conditioning information’, *Journal of Empirical Finance* **38(B)**, 762–785.

Perold, A. F. (2004), ‘The capital asset pricing model’, *Journal of Economic Perspectives* **18**, 3–24.

Pukthuanthong, K., Roll, R. and Subrahmanyam, A. (2019), ‘A protocol for factor identification’, *The Review of Financial Studies* **32**, 1573–1607.

Rao, C. R. (1973), *Linear Statistical Inference and its Applications*, second edn, John Wiley & Sons, New York.

Sentana, E. (2009), ‘The econometrics of mean-variance efficiency tests: A survey’, *Econometrics Journal* **12**, C65–C101.

Shanken, J. (1985), ‘Multivariate tests of the zero-beta CAPM’, *Journal of Financial Economics* **14**, 325–348.

Shanken, J. (1992), ‘On the estimation of beta-pricing models’, *Review of Financial Studies* **5**, 1–33.

Shanken, J. and Weinstein, M. I. (2006), ‘Economic forces and the stock market revisited’, *Journal of Empirical Finance* **13**, 129–144.

Shanken, J. and Zhou, G. (2007), ‘Estimating and testing beta pricing models: Alternative methods and their performance in simulations’, *Journal of Financial Economics* **84**, 40–86.

Appendix

A Eigenvalue-based confidence sets

Equation (3.21) may be re-expressed as

$$\Sigma_{11} + \Sigma_{12}\zeta = 0, \tag{A.1}$$

$$\Sigma_{21} + \Sigma_{22}\zeta = 0, \tag{A.2}$$

and solving (A.2) for ζ leads to (3.22). Substituting $\hat{\zeta}$ into (A.1) yields $\Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21} = 0$. Assuming that Σ_{11} is non-singular, on recalling that $\Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}$ is a scalar and using the formulae for the determinant of partitioned matrices

$$|\Sigma| = |\Sigma_{22}|\left|\Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}\right| = |\Sigma_{22}|\left(\Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}\right),$$

we thus see that if $\hat{\zeta}$ satisfies (A.2) then it satisfies (A.1).

We next summarize the solution of (3.10) from Dufour and Taamouti (2005). Projections based confidence sets for any linear transformation of ζ of the form $\omega'\zeta$ can be obtained as follows. Let $\tilde{A} = -A_{22}^{-1}A_{12}'$, $\tilde{D} = A_{12}A_{22}^{-1}A_{12} - A_{11}$. If all the eigenvalues of A_{22} [as defined in (3.11)] are positive so A_{22} is positive definite then:

$$\mathrm{CS}_{\alpha}(\omega'\zeta) = \left[\omega'\tilde{A} - \sqrt{\tilde{D}\left(\omega' A_{22}^{-1}\omega\right)},\, \omega'\tilde{A} + \sqrt{\tilde{D}\left(\omega' A_{22}^{-1}\omega\right)}\right], \quad if \quad \tilde{D} \geq 0, \tag{A.3}$$

$$\mathrm{CS}_{\alpha}(\omega'\zeta) = \varnothing, \quad if \quad \tilde{D} < 0. \tag{A.4}$$

If A_{22} is non-singular and has one negative eigenvalue then: (i) if $\omega' A_{22}^{-1}\omega < 0$ and $\tilde{D} < 0$:

$$\mathrm{CS}_{\alpha}(\omega'\zeta) = \left]-\infty,\, \omega'\tilde{A} - \sqrt{\tilde{D}\left(\omega' A_{22}^{-1}\omega\right)}\right] \cup \left[\omega'\tilde{A} + [\tilde{D}\left(\omega' A_{22}^{-1}\omega\right)]^{1/2}\sqrt{\tilde{D}\left(\omega' A_{22}^{-1}\omega\right)},\, +\infty\right[; \tag{A.5}$$

(ii) if $\omega' A_{22}^{-1}\omega > 0$ or if $\omega' A_{22}^{-1}\omega \leq 0$ and $\tilde{D} \geq 0$ then:

$$\mathrm{CS}_{\alpha}(\omega'\zeta) = \mathbb{R}; \tag{A.6}$$

(iii) if $\omega' A_{22}^{-1}\omega = 0$ and $\tilde{D} < 0$ then:

$$\mathrm{CS}_{\alpha}(\omega'\zeta) = \mathbb{R}\backslash\{\omega'\tilde{A}\}. \tag{A.7}$$

The projection is given by (A.6) if A_{22} is non-singular and has at least two negative eigenvalues.

B Proofs

Proof of Theorem 3.1. Equations (A.3) - - (A.7) applied with A as defined in (3.12) imply that an unbounded solution to the problem of inverting the test defined by (3.2) and (3.7) would occur if A_{22} [refer to the partitioning in (3.11) and (3.13)] is not positive definite. In this case, the diagonal term of A_{22} is given by $\mathrm{DIAG}(A_{22}) = \left[F_2 \cdots F_k\right]'$ where

$$F_i = s_k[i]'\hat{B}\hat{S}^{-1}\hat{B}'s_k[i] - s_k[i]'(X'X)^{-1}s_k[i]\frac{nf_{n,\tau_n,\alpha}}{\tau_n}. \tag{B.1}$$

Clearly, if any of the Hotelling tests based on Λ_i, $i \in \{2, \ldots, k\}$ [as in (3.5) and using the distribution in (3.7)] is not significant at level α, then by the definition of Λ_i and F_i, $\Lambda_i(\tau_n)/n < f_{n,\tau_n,\alpha} \Leftrightarrow F_i < 0$, in which case

A_{22} cannot be positive definite. On comparing (3.12) and (3.14) we see that $\Lambda_i(\tau_{n-1})/(n-1) \geq f_{n-1,\tau_{n-1},\alpha}$, $i \in \{2,\ldots,k\}$ holds for the problem of inverting the test defined by (3.1) and (3.7) as a necessary but not sufficient condition to obtain bounded CSs.■

Proof of the minimum distance computations in Theorems 3.2 and 3.3. $\min_\theta \Lambda(\theta)$ is the minimum root [denoted $\hat{\rho}$] of the determinantal equation (3.19) so the minimization problem can be cast as an equation of the (3.21) where $\zeta = \theta$ and

$$\Sigma = \hat{B}\hat{S}^{-1}\hat{B}' - \hat{\rho}(X'X)^{-1}, \tag{B.2}$$

$\Sigma_{11} = \hat{a}'\hat{S}^{-1}\hat{a} - \hat{\gamma}x^{11}$, $\Sigma_{12} = \Sigma_{21}' = \hat{a}'\hat{S}^{-1}\hat{b}' - \hat{\gamma}x^{12}$, $\Sigma_{22} = \hat{b}\hat{S}^{-1}\hat{b}' - \hat{\gamma}x^{22}$, using the partitioning (2.10). So $\hat{\theta}_{\mathbf{RAPT}}$ obtains applying (3.22) leading to (3.18).

Turning to $\Lambda(\theta,\phi)$, we have

$$\frac{\partial\Lambda(\theta,\phi)}{\partial\phi} = \frac{-2(1,\theta')\hat{B}\hat{S}^{-1}\iota_n + 2\phi\iota_n'\hat{S}^{-1}\iota_n}{(1,\theta')(X'X)^{-1}(1,\theta')'} \tag{B.3}$$

and the (non-zero) value of ϕ which sets the latter partial derivative to zero is

$$\phi(\theta) = \frac{(1,\theta')\hat{B}\hat{S}^{-1}\iota_n}{\iota_n'\hat{S}^{-1}\iota_n}. \tag{B.4}$$

Substituting $\phi(\theta)$ in (B.3) leads to

$$\Lambda(\theta,\phi(\theta)) = \frac{(1,\theta')\hat{B}\left[\hat{S}^{-1} - \hat{S}^{-1}\iota_n\left(\iota_n'\hat{S}^{-1}\iota_n\right)^{-1}\iota_n'\hat{S}^{-1}\right]\hat{B}'(1,\theta')'}{(1,\theta')(X'X)^{-1}(1,\theta')'} \tag{B.5}$$

which proves 3.23. From there on, $\min_{\theta,\phi}\Lambda(\theta,\phi)$ requires one to solve a system of the (3.21) form with $\zeta = \theta$, and

$$\Sigma = \hat{B}\left[\hat{S}^{-1} - \hat{S}^{-1}\iota_n\left(\iota_n'\hat{S}^{-1}\iota_n\right)^{-1}\iota_n'\hat{S}^{-1}\right]\hat{B}' - \hat{\nu}(X'X)^{-1} \tag{B.6}$$

$$\Sigma_{11} = \hat{a}'\left[\hat{S}^{-1} - \hat{S}^{-1}\iota_n\left(\iota_n'\hat{S}^{-1}\iota_n\right)^{-1}\iota_n'\hat{S}^{-1}\right]\hat{a} - \hat{\nu}x^{11} \tag{B.7}$$

$$\Sigma_{12} = \Sigma_{21}' = \hat{a}'\left[\hat{S}^{-1} - \hat{S}^{-1}\iota_n\left(\iota_n'\hat{S}^{-1}\iota_n\right)^{-1}\iota_n'\hat{S}^{-1}\right]\hat{b}' - \hat{\nu}x^{12} \tag{B.8}$$

$$\Sigma_{22} = \hat{b}\left[\hat{S}^{-1} - \hat{S}^{-1}\iota_n\left(\iota_n'\hat{S}^{-1}\iota_n\right)^{-1}\iota_n'\hat{S}^{-1}\right]\hat{b}' - \hat{\nu}x^{22} \tag{B.9}$$

using the partitionings (2.10) and (2.9). So a point estimate for θ [denoted $\hat{\theta}_{\mathbf{UAPT}}$] obtains applying (3.22) leading to (3.25) and an point estimate for Φ thus follows using (B.4) leading to (3.26).

Proof of Theorem 3.5. Consider the following decomposition of $G'\hat{S}G$ and $\tilde{S}_0$:

$$G'\hat{S}G = G'JW'\mathscr{M}[X]WJ'G, \tag{B.10}$$

$$C\hat{B}G - D = C(X'X)^{-1}X'\left[XB + WJ'\right]G - D = CBG - D + C(X'X)^{-1}X'WJ'G, \tag{B.11}$$

so under the null hypothesis $C\hat{B}G - D = C(X'X)^{-1}X'WJ'G$ and

$$S(C,G,D) - G'\hat{S}G = G'JW'\mathscr{M}_0[X,C]WJ'G, \tag{B.12}$$

$$S(C,G,D) = G'JW'\left(\mathscr{M}[X] + \mathscr{M}_0[X,C]\right)WJ'G, \tag{B.13}$$

which implies that (3.38) corresponds to:

$$|G'JW'\left(\mathscr{M}_0[X,C]\right)WJ'G - \lambda G'JW'\left(\mathscr{M}[X] + \mathscr{M}_0[X,C]\right)WJ'G| = 0. \tag{B.14}$$

Since J is invertible and G has full column rank, the singular value decomposition of $J'G$ gives

$$J'G = \mathscr{G}\Delta^{1/2}\Xi \tag{B.15}$$

where Δ is a g-dimensional diagonal matrix which includes the non-zero eigenvalues of $J'GG'J$, $\mathscr{G}$ is the $n \times g$ matrix which includes the corresponding eigenvectors so $\mathscr{G}'\mathscr{G} = I_g$ and Ξ is the g-dimensional matrix $\Xi = \Delta^{-1/2}\mathscr{G}'J'G$ so that $\Xi\Xi' = I_g$. Replacing the latter expressions in (B.14) leads to (3.43). In particular, under assumption (3.27), (3.43) reduces to (3.45) where $\mathscr{Z} = Z\mathscr{G}$ and in view of (3.29), the rows of $Z\mathscr{G}$ are $\overset{i.i.d.}{\sim} N(0, I_g)$. It follows that the null distribution of all test statistics which depend on the data via the roots of (3.38) are invariant to B and J. When $G = I_n$, (B.14) takes the form

$$|JW'(\mathscr{M}_0[X,C])WJ' - \lambda JW'(\mathscr{M}[X] + \mathscr{M}_0[X,C])WJ'| = 0 \tag{B.16}$$

which leads to (3.43) so B and J are evacuated.■

Proof of Theorem 3.4. Given $\tilde{H}[C, I_n, D]$ the sum of squared error ratio simplifies to

$$\begin{aligned} |\tilde{S}(C,G,D)|/|\hat{S}| &= |I_n + \hat{S}^{-1}(C\hat{B} - D)'[C(X'X)^{-1}C']^{-1}(C\hat{B} - D)| \\ &= |I_c + [C(X'X)^{-1}C']^{-1}(C\hat{B} - D)\hat{S}^{-1}(C\hat{B} - D)'| \end{aligned} \tag{B.17}$$

using a well known result on determinants, which leads to (3.39).[10]

[10]For any $n \times m$ matrix S and any $m \times n$ matrix U, $|I_n + SU| = |I_m + US|$; see e.g. Harville (1997, section 18.1, p. 416).

Arbitrage pricing, weak beta, strong beta: identification-robust and simultaneous inference

Marie-Claude Beaulieu, Jean-Marie Dufour and Lynda Khalaf

Supplementary Appendix

This appendix reports further details on data, simulations and empirical results, for completion.

S.1 Further details on data

Data on industry portfolios for the US, as in Beaulieu et al. (2013), consists of monthly returns from 1961 to 2010, obtained from the University of Chicago's Center for Research in Security Prices (CRSP), on standard 12 portfolios of New York Stock Exchange (NYSE) firms grouped by standard two-digit industrial classification (SIC).[11] For each month the industry portfolios include the firms for which the return, price per common share and number of shares outstanding are recorded by CRSP. Equally and value-weighted portfolios are analyzed.

The size portfolios from Fama and French's data base are constructed as follows. The portfolios which are constructed at the end of June are the intersections of five portfolios formed on size (market equity) and five portfolios formed on the ratio of book equity to market equity. The size breakpoints for year s are the NYSE market equity quintiles at the end of June of year s. The ratio of book equity to market equity for June of year s is the book equity for the last fiscal year end in $s-1$ divided by market equity for December of year $s-1$. The ratio of book equity to market equity is NYSE quintiles. The portfolios for July of year s to June of year $s+1$ include all NYSE, AMEX, NASDAQ stocks for which market equity data is available for December of year $s-1$ and June of year s, and (positive) book equity data for $s-1$.

Fama and French benchmark factors, SMB, HML, RMW and CMA are constructed from benchmark portfolios that do not include hold ranges and do not incur transaction costs. The portfolios for these factors are rebalanced quarterly using two independent sorts, on size (market equity, ME), book-to-market (the ratio of book equity to market equity, BE/ME), profitability (annual revenues minus cost of goods sold, interest expense, and selling, general, and administrative expenses, all divided by book equity at the end of fiscal year $s-1$) and investment (the growth of total assets for the fiscal year ending in s-1 divided by total assets at the end of $s-1$). The profitability and investment factors, RMW and CMA, are constructed in the same way as HML except the second sort is either on operating profitability (robust minus weak) or investment (conservative minus aggressive). As HML, RMW and CMA can be interpreted as averages of profitability and investment factors for small and big stocks.

For the construction of the MOM factor, six value-weighted portfolios formed on size and prior (2–12) returns are used. The portfolios, which are formed monthly, are the intersections of two portfolios formed on size (market equity, ME) and three portfolios formed on prior (2–12) return. The size breakpoint (which determines the buy range for the small and big portfolios) is the median NYSE market equity. The BE/ME breakpoints are the 30th and 70th NYSE percentiles. The monthly prior (2–12) return breakpoints are also the 30th and 70th NYSE percentiles.

[11] The sectors studied include: (1) petroleum; (2) finance and real estate; (3) consumer durables; (4) basic industries; (5) food and tobacco; (6) construction; (7) capital goods; (8) transportation; (9) utilities; (10) textile and trade; (11) services; (12) leisure.

Table S.1: Proportion of empty confidence sets in reported figures

		Step								
Figure	Errors	0	-4	-3	-2	-1	1	2	3	4
1	Normal	.0110	.0099	.0098	.0093	.0102	.0112	.0113	.0115	.0121
	Student t(5)	.0103	.0110	.0112	.0101	.0103	.0098	.0094	.0095	.0098
	GARCH	.0124	.0124	.0120	.0123	.0127	.0130	.0126	.0122	.0120
2	Normal	.0055	.0050	.0049	0042	.0044	.0060	.0065	.0068	.0070
	Student t(5)	.0051	.0053	.0052	.0055	.0052	.0052	.0052	.0047	.0049
	GARCH	.0068	.0054	.0060	.0066	.0067	.0062	.0066	.0059	.0064
3	Normal	.0100	.0086	.0089	.0091	.0097	.0106	.0105	.0108	.0119
	Student t(5)	.0084	.0090	.0093	.0086	.0084	.0085	.0084	.0082	.0084
	GARCH	.0108	.0098	.0103	.0106	.0109	.0110	.0106	.0104	.0106
4	Normal	.0055	.0050	.0049	.0042	.0044	.0060	.0065	.0068	.0070
	Student t(5)	.0053	.0051	.0052	.0055	.0052	.0052	.0052	.0047	.0049
	GARCH	.0068	.0054	.0060	.0066	.0067	.0062	.0066	.0059	.0064
5	Normal	.0100	.0086	.0089	.0091	.0097	.0106	.0105	.0108	.0119
	Student t(5)	.0084	.0090	.0093	.0086	.0084	.0085	.0084	.0082	.0084
	GARCH	.0108	.0098	.0103	.0106	.0109	.0110	.0106	.0104	.0106
6	Normal	.6683	1.0	1.0	1.0	.9700	.9954	1.0	1.0	1.0
	Student t(5)	.5854	1.0	1.0	1.0	.9553	.9930	1.0	1.0	1.0
	GARCH	.7690	1.0	1.0	1.0	.9811	.9973	1.0	1.0	1.0
7	Normal	.0095	.0096	.0091	.0091	.0094	.0076	.0073	.0069	.0069
	Student t(5)	.0051	.0056	.0055	.0054	.0054	.0039	.0037	.0032	.0039
	GARCH	.0111	.0107	.0106	.0107	.0104	.0093	.0083	.0074	.0084
8	Normal	.0055	.0049	.0053	.0050	.0055	.0054	.0060	.0062	.0056
	Student t(5)	.0051	.0055	.0058	.0056	.0052	.0046	.0047	.0047	.0049
	GARCH	.0068	.0061	.0070	.0069	.0073	.0066	.0062	.0058	.0055

Note – Numbers reported are the proportion of empty confidence sets which correspond to tests that reject the specification.

S.2 Further details on simulation results

Table S.1 reports the proportion of empty joint confidence sets in the experiments underlying each figure in the main text. Table S.2 presents the unidentified experiment results. Reported results in the latter table are restricted to size, since power is expected not to exceed size in this case, a fact we verified. These results confirm that even when identification fails, the size problems documented in this literature with standard methods are solved via our proposed tests.

Perhaps equally important here is our finding in table S.1: inverting the test that imposes tradability when it does not hold produces a very large proportion of empty sets, which implies it successfully detects the false assumptions. Taken collectively, results reinforce the prescription in Lewellen et al. (2010) regarding tradable factors particularly because we provide a method to validate this assumption.

Table S.2: Monte Carlo study: tests in unidentified models

$n=12,\ T=624$	$r_i = a_i \iota_T + \mathscr{R}_1 b_{i1} + \mathscr{F} b_{i\mathscr{F}} + u_i$					
Model	$a_i = \gamma_0(1-b_{i1}) - \gamma'_{\mathscr{F}} b_{i\mathscr{F}}\ ,\ \gamma_0 = \theta_{\text{MKT}},\ \gamma_{\mathscr{F}} = (\theta_{\text{SMB}}, \theta_{\text{HML}})'$					
Inverted Statistic	$\Lambda(\theta)\ ,\quad \theta = \left(\gamma_0, \gamma'_{\mathscr{F}}\right)'$					
	MKT betas = zero			MKT betas = one		
Errors	Normal	Student $t(5)$	GARCH	Normal	Student $t(5)$	GARCH
θ_{MKT}	.0199	.0185	.0207	.0199	.0185	.0207
θ_{SMB}	.0190	.0198	.0209	.0001	.0004	.0003
θ_{HML}	.0203	.0191	.0218	.0003	.0005	.0003
Rejected	.0115	.0101	.0124	.0000	.0002	.0002
Model	$a_i = \gamma_c - \gamma_0 b_{i1} - \gamma'_{\mathscr{F}} b_{i\mathscr{F}}\ ,\ \gamma_0 = \theta_{\text{MKT}},\ \gamma_{\mathscr{F}} = (\theta_{\text{SMB}}, \theta_{\text{HML}})'$					
	MKT betas = zero					
Inverted Statistic	$\Lambda(\theta,\phi)\ ,\ \theta = (\gamma_0, \gamma'_{\mathscr{F}})',\ \phi = \gamma_c$			$\Lambda(\theta)$ on $r_i - r_n\ ,\ \theta = (\gamma_0, \gamma'_{\mathscr{F}})'$		
Errors	Normal	Student $t(5)$	GARCH	Normal	Student $t(5)$	GARCH
θ_{MKT}	.0114	.0103	.0124	.0189	.0163	.0201
θ_{SMB}	.0000	.0001	.0000	.0000	.0002	.0001
θ_{HML}	.0000	.0002	.0000	.0001	.0003	.0002
ϕ	.0000	.0002	.0002	-	-	-
Rejected	.0000	.0000	.0000	.0000	.0001	.0001

Note – Numbers reported are test sizes, for 5% tests, given models in which identification problems are provoked by setting MKT betas jointly to zeros or ones. All other model parameters and inverted test are kept as in the original designs. For this design, (unreported) power curves remain below 5%.

S.3 Further details on empirical results

Here we provide results with value-weighted industry sorts, size sorts, the Fama-French five factor model and a representative set of results using conditioning information.

We examine conditional models estimated over the full sub-period, using the full set of industry and size portfolios, and the (standard) conditioning variables as in Beaulieu et al. (2007). Assuming all betas are time varying returned real lines, the same holds when each set of portfolios was used on its own which is not surprising, given the number of regressors to add relative to the sample size.

We report a sample of results assuming that the MKT beta varies as a function of the difference between the one-month lagged returns of a three-month and a one month. This sample is representative is the following sense: as the conditioning information changes, confidence sets jump from empty to severely unbounded a result we observed even with a single benchmark conditional model.

Two points are worth emphasizing from tables S.10 and S.11. First, because industry and size sorted portfolios are used jointly, the advantage of equal or value weights no longer prevail. This reinforces our earlier findings in this regard. Second, the conditional model in question does not fare well, in view of its rejection with value-weighted portfolios. Given the extensive instruments search we experimented with leading to these tables, we do not aim to over-emphasize these results, aside from the following broad yet empirically important message. The identification problems in this literature are not restricted to unconditional asset pricing, and are not solved by standard conditioning which seems instead to exacerbate complications.[12] Our findings thus endorse identification-robust methods for assessing whether candidate factors are associated with risk premiums.

[12]For other perspectives on conditioning complications, see *e.g.* Boguth, Carlson, Fisher and Simutin (2011) and Penaranda and Sentana (2016).

Table S.3: Confidence sets for risk price: industry portfolios and three factor model

$$r_i - \iota_T \gamma_0 = (\mathscr{R}_1 - \iota_T \gamma_0) b_{i1} + (\mathscr{F} - \iota_T \gamma_{\mathscr{F}}') b_{i\mathscr{F}} + u_i, \qquad i = 1, \ldots, n$$

PANEL A $\theta = (\gamma_0, \gamma_{\mathscr{F}}')' = (\theta_{\text{MKT}}, \theta_{\text{SMB}}, \theta_{\text{HML}})'$

VW	MKT		SMB		HML	
$\times 10^{-4}$	$\overline{F}$	θ_{MKT}	$\overline{F}$	θ_{SMB}	$\overline{F}$	θ_{HML}
61-70	38*	[-612,-77]	33	[10,158]	53	[-27,54]
71-80	30	[-842,286]	43	[-44,302]	33	[-244,133]
81-90	44*	[75,259]	-16	[-78,0]	56*	[-25,24]
91-00	103	[-248, 206]	4*	[83, 596]	29	[-8,159]
00-10	14	[-60,260]	57	[-276,105]	40	[-66,85]

$$r_i - \iota_T \gamma_c = (\mathscr{R}_1 - \iota_T \gamma_0) b_{i1} + (\mathscr{F} - \iota_T \gamma_{\mathscr{F}}') b_{i\mathscr{F}} + u_i,, \qquad i = 1, \ldots, n$$

PANEL B $\theta = (\gamma_0, \gamma_{\mathscr{F}}') = (\theta_{\text{MKT}}, \theta_{\text{SMB}}, \theta_{\text{HML}})$, γ_c partialled-out

VW	MKT		SMB		HML	
$\times 10^{-4}$	$\overline{F}$	θ_{MKT}	$\overline{F}$	θ_{SMB}	$\overline{F}$	θ_{HML}
61-70	38	[-655, 137]	33	[-60, 172]	53	[-18, 166]
71-80	30	[-625, 413]	43	[-83, 244]	33	[-179, 219]
81-90	44	[-454, 204]	-16	[-56, 200]	56	[-11, 186]
91-00	103	[-230, 197]	4*	[38, 673]	29	[-19,156]
00-10	14	[-73, 284]	57	[-281, 135]	40	[-61, 124]

Note – Sample includes monthly observations from January 1991 to December 2010 on the US. Series include 12 value weighted (VW) industry portfolios as well as US factors for market (MKT), size (SMB), book-to-market (HML). Confidence sets are at the 5% level. $\overline{F}$ is the factor average over the considered time period; θ captures factor pricing. * denotes evidence of pricing at the 5% significance level interpreted as follows: given the reported confidence sets, each factor is priced if its average is not covered. In Panel A, the inverted test is $\Lambda(\theta)$. This test follows our **RAPT** approach where **R** stands for "restricted" implying that tradable factor constraints are imposed, here on $\mathscr{R}_1$, in estimating and testing the model. In Panel B, the inverted test $\Lambda(\theta)$ is applied on a system on $n-1$ returns in deviation from r_n. This test follows our **PAPT** approach where **P** stands for "partialling-out" implying that all factors are assumed non-tradable but the resulting unrestricted constant is evacuated from the statistical objective function as it is based on $r_i - r_n$, $i = 1, \ldots, n-1$.

Table S.4: Confidence sets for risk price: industry portfolios and four factor model

$r_i - \iota_T\gamma_0 = (\mathscr{R}_1 - \iota_T\gamma_0)b_{i1} + (\mathscr{F} - \iota_T\gamma'_{\mathscr{F}})b_{i\mathscr{F}} + u_i, \qquad i = 1,\ldots,n$								
$\theta = (\gamma_0, \gamma'_{\mathscr{F}}) = (\theta_{\text{MKT}}, \theta_{\text{SMB}}, \theta_{\text{HML}}, \theta_{\text{MOM}})$								
VW	MKT		SMB		HML		MOM	
$\times 10^{-4}$	$\overline{F}$	θ_{MKT}	$\overline{F}$	θ_{SMB}	$\overline{F}$	θ_{HML}	$\overline{F}$	θ_{MOM}
61-70	38	$\mathbb{R}$	33	$\mathbb{R}$	53	$\mathbb{R}$	73	$\mathbb{R}$
71-80	30	[-1933, 657]	43	[-138, 595]	33	[-422, 171]	113	[-175, 594]
81-90	44	[-203, 554]	-16	[-258, 81]	56	[-199, 53]	66	[-1030, 81]
91-00	103	[-253, 209]	4*	[81, 606]	29	[-37, 174]	112	[-212, 443]
00-10	14	$\mathbb{R}$	57	$\mathbb{R}$	40	$\mathbb{R}$	-3	$\mathbb{R}$

$r_i - \iota_T\gamma_c = (\mathscr{R}_1 - \iota_T\gamma_0)b_{i1} + (\mathscr{F} - \iota_T\gamma'_{\mathscr{F}})b_{i\mathscr{F}} + u_i, \qquad i = 1,\ldots,n$								
PANEL B $\theta = (\gamma_0, \gamma'_{\mathscr{F}}) = (\theta_{\text{MKT}}, \theta_{\text{SMB}}, \theta_{\text{HML}}, \theta_{\text{MOM}})$, γ_c partialled-out								
VW	MKT		SMB		HML		MOM	
$\times 10^{-4}$	$\overline{F}$	θ_{MKT}	$\overline{F}$	θ_{SMB}	$\overline{F}$	θ_{HML}	$\overline{F}$	θ_{MOM}
61-70	38	$\mathbb{R}$	33	$\mathbb{R}$	53	$\mathbb{R}$	73	$\mathbb{R}$
71-80	30	[-945, 3459]	43	[-928, 324]	33	[-227, 945]	113	[-1267, 366]
81-90	44	[-3856, 581]	-16	[-240, 1333]	56	[-178, 956]	66	[-2194, 70]
91-00	103	$\mathbb{R}$	4	$\mathbb{R}$	29	$\mathbb{R}$	112	$\mathbb{R}$
00-10	14	$]-\infty, 496] \cup [1223, \infty[$	57	$\mathbb{R}$	40	$]-\infty, 136] \cup [273, \infty[$	-3	$]-\infty, 734] \cup [2344, \infty[$

Note – See notes to table S.3. The considered model is the four factor case with market (MKT), size (SMB), book-to-market (HML) and momentum (MOM).

Table S.5: Industry portfolios: testing traded factor assumption

$r_i - \mathscr{R}_1 = a_i \iota_T + \mathscr{R}_1 d_i + \mathscr{F} b_{i\mathscr{F}} + u_i, \qquad i = 1,\ldots,n,$										
$a_i = \gamma_c^* - \gamma_0 d_i - \gamma_{\mathscr{F}}' b_{i\mathscr{F}}, \quad d_i = b_{i1} - 1, \quad \gamma_c^* = \gamma_c - \gamma_0$										
$\theta = (\gamma_0, \gamma_{\mathscr{F}}') = (\theta_{\text{MKT}}, \theta_{\text{SMB}}, \theta_{\text{HML}})$										
VW	CTE	MKT		SMB		HML		MOM		
$\times 10^{-4}$	γ_c^*	$\overline{F}$	θ_{MKT}	$\overline{F}$	θ_{SMB}	$\overline{F}$	θ_{HML}	$\overline{F}$	θ_{MOM}	
61-70	[-2, 16]	38	[-765, 157]	33	[-61, 193]	53	[-28, 173]	-	-	
71-80	[-7, 16]	30	[-843, 510]	43	[-104, 302]	33	[-251, 250]	-	-	
81-90	[-4, 40]	44	[-707, 274]	-16	[-83, 298]	56	[-31, 255]	-	-	
91-00	[-49, 35]	103	[-249, 213]	4*	[28, 726]	29	[-24,163]	-	-	
00-10	[-6, 21]	14	[-82, 307]	57	[-315, 146]	40	[-68, 129]	-	-	
$r_i - \mathscr{R}_1 = a_i \iota_T + \mathscr{R}_1 d_i + \mathscr{F} b_{i\mathscr{F}} + u_i, \qquad i = 1,\ldots,n,$										
$a_i = \gamma_c^* - \gamma_0 d_i - \gamma_{\mathscr{F}}' b_{i\mathscr{F}}, \quad d_i = b_{i1} - 1, \quad \gamma_c^* = \gamma_c - \gamma_0$										
$\theta = (\gamma_0, \gamma_{\mathscr{F}}') = (\theta_{\text{MKT}}, \theta_{\text{SMB}}, \theta_{\text{HML}}, \theta_{\text{MOM}})$										
VW	CTE	MKT		SMB		HML		MOM		
$\times 10^{-4}$	γ_c^*	$\overline{F}$	θ_{MKT}	$\overline{F}$	θ_{SMB}	$\overline{F}$	θ_{HML}	$\overline{F}$	θ_{MOM}	
61-70	$\mathbb{R}$	38	$\mathbb{R}$	33	$\mathbb{R}$	53	$\mathbb{R}$	73	$\mathbb{R}$	
71-80	$\mathbb{R}$	30	$\mathbb{R}$]	43	$\mathbb{R}$	33	$\mathbb{R}$	113	$\mathbb{R}$	
81-90	[-20, 960]	44	[-19988, 683]	-16	[-283, 6831]	56	[-212, 4618]	66	[-12992, 82]	
91-00	$\mathbb{R}$	103	$\mathbb{R}$	4	$\mathbb{R}$	29	$\mathbb{R}$	112	$\mathbb{R}$	
00-10	$\mathbb{R}$	14	$\mathbb{R}$	57	$\mathbb{R}$	40	$\mathbb{R}$	-3	$\mathbb{R}$	

Note – See notes to tables S.3 and S.4. The inverted test is $\Lambda(\theta, \phi)$. This test follows our **UAPT** where **U** stands for "unrestricted" implies that factors are assumed non-tradable in estimating and testing the model. The test is applied on $r_i - \mathscr{R}_1$ so inference on ϕ allows to assess whether $\gamma_c = \gamma_0$: the hypothesis that $\mathscr{R}_1$ (here, MKT) is traded is rejected at the 5% level when the confidence set on ϕ excludes zero.

Table S.6: Confidence sets for risk price: size portfolios

$r_i - \iota_T\gamma_0 = (\mathscr{R}_1 - \iota_T\gamma_0)b_{i1} + (\mathscr{F} - \iota_T\gamma'_{\mathscr{F}})b_{i\mathscr{F}} + u_i, \quad i = 1,\dots,n$								
$\theta = (\gamma_0, \gamma'_{\mathscr{F}}) = (\theta_{\text{MKT}}, \theta_{\text{SMB}}, \theta_{\text{HML}})$								
VW	MKT		SMB		HML		MOM	
$\times 10^{-4}$	$\overline{F}$	θ_{MKT}	$\overline{F}$	θ_{SMB}	$\overline{F}$	θ_{HML}	$\overline{F}$	θ_{MOM}
61-70	38*	$]-\infty,\ -755] \cup [1716, \infty[$	33	$]-\infty,\ -260] \cup [-31, \infty[$	53*	$]-\infty,\ -731] \cup [2599, \infty[$	-	-
71-80	30	$\mathbb{R}$	43	$\mathbb{R}$	33	$\mathbb{R}$	-	-
81-90	44	$\mathbb{R}$	-16	$\mathbb{R}$	56	$]-\infty,\ 707] \cup [818, \infty[$	-	-
91-00	103*	[-11236, -1030]	4	[-568, 1375]	29	[-5394,730]	-	
00-10	14*	$]-\infty,\ -1788] \cup [1895, \infty[$	57	$\mathbb{R}$	40	$\mathbb{R}$	-	-
$r_i - \iota_T\gamma_0 = (\mathscr{R}_1 - \iota_T\gamma_0)b_{i1} + (\mathscr{F} - \iota_T\gamma'_{\mathscr{F}})b_{i\mathscr{F}} + u_i, \quad i = 1,\dots,n$								
$\theta = (\gamma_0, \gamma'_{\mathscr{F}}) = (\theta_{\text{MKT}}, \theta_{\text{SMB}}, \theta_{\text{HML}}, \theta_{\text{MOM}})$								
VW	MKT		SMB		HML		MOM	
$\times 10^{-4}$	$\overline{F}$	θ_{MKT}	$\overline{F}$	θ_{SMB}	$\overline{F}$	θ_{HML}	$\overline{F}$	θ_{MOM}
61-70	38	$\mathbb{R}$	33	$\mathbb{R}$	53	$\mathbb{R}$	73	$\mathbb{R}$
71-80	30	$\mathbb{R}$	43	$\mathbb{R}$	33	$\mathbb{R}$	113	$\mathbb{R}$
81-90	44	$\mathbb{R}$	-16	$\mathbb{R}$	56	$\mathbb{R}$	66	$\mathbb{R}$
91-00	103	$\mathbb{R}$	4	$\mathbb{R}$	29	$]-\infty,\ 969] \cup [3955, \infty[$	112	$]-\infty,\ 3167] \cup [5035, \infty[$
00-10	14*	$]-\infty,\ -1761] \cup [1261, \infty[$	57	$\mathbb{R}$	40	$\mathbb{R}$	-3	$\mathbb{R}$

Note – Sample includes monthly observations from January 1991 to December 2010 on the US. Series include 25 size sorted value-weighted (VW) portfolios as well as US factors for market (MKT), size (SMB), book-to-market (HML) and momentum (MOM). See notes to tables S.3 and S.4 for further definitions and applied inference methods.

Table S.7: Confidence sets for risk price: industry portfolios and five-factor model

$$r_i - \iota_T \gamma_0 = (\mathscr{R}_1 - \iota_T \gamma_0) b_{i1} + (\mathscr{F} - \iota_T \gamma'_{\mathscr{F}}) b_{i\mathscr{F}} + u_i, \qquad i = 1, \ldots, n$$

PANEL A $\theta = (\gamma_0, \gamma'_{\mathscr{F}})' = (\theta_{\text{MKT}}, \theta_{\text{SMB}}, \theta_{\text{HML}}, \theta_{\text{RMW}}, \theta_{\text{CMA}})'$

EW	MKT		SMB		HML		RMW		CMA	
$\times 10^{-4}$	$\overline{F}$	θ_{MKT}	$\overline{F}$	θ_{SMB}	$\overline{F}$	θ_{HML}	$\overline{F}$	θ_{RMW}	$\overline{F}$	θ_{CMA}
63-70	28	ℝ	57	ℝ	41	ℝ	2	ℝ	22	ℝ
71-80	30	ℝ	54	ℝ	33	ℝ	5	ℝ	25	ℝ
81-90	44	$]-\infty,\ -53] \cup [\ 9,\infty[$	-20	ℝ	57	$]-\infty,\ -101] \cup [\ 49,\infty[$	39*	$]-\infty,\ -67] \cup [\ 42,\infty[$	55	ℝ
91-00	103	[-2014, 116]	3	[-389, 255]	27	[-567, 88]	32	[-298, 368]	30*	[-927,-3]
00-10	14	ℝ	65	ℝ	41	ℝ	44	ℝ	35	ℝ
Full	45	[-289, 134]	30*	[-54, 14]	40	[-132, 79]	26	[15, 326]	34	[-532, 87]
VW	MKT		SMB		HML		RMW		CMA	
$\times 10^{-4}$	$\overline{F}$	θ_{MKT}	$\overline{F}$	θ_{SMB}	$\overline{F}$	θ_{HML}	$\overline{F}$	θ_{RMW}	$\overline{F}$	θ_{CMA}
63-70	28	ℝ	57	ℝ	41	ℝ	2	ℝ	22	ℝ
71-80	30	ℝ	54	ℝ	33	ℝ	5	ℝ	25	ℝ
81-90	44	[-149, 787]	-20	[-360, 91]	57	[-76, 123]	39	[-362, 154]	55	[-127, 217]
91-00	103	[-242, 1081]	3*	[76, 1641]	27	[-98, 167]	32	[-596, 48]	30	[4, 893]
00-10	14	ℝ	65	ℝ	41	ℝ	44	ℝ	35	ℝ
Full	45	[-92, 546]	30	[-13, 358]	40	[-25, 110]	26	[-446, 31]	34	[-31, 768]

$$r_i - \iota_T \gamma_c = (\mathscr{R}_1 - \iota_T \gamma_0) b_{i1} + (\mathscr{F} - \iota_T \gamma'_{\mathscr{F}}) b_{i\mathscr{F}} + u_i,, \qquad i = 1, \ldots, n$$

PANEL B $\theta = (\gamma_0, \gamma'_{\mathscr{F}}) = (\theta_{\text{MKT}}, \theta_{\text{SMB}}, \theta_{\text{HML}}, \theta_{\text{RMW}}, \theta_{\text{CMA}})$, γ_c partialled-out

EW	MKT		SMB		HML		RMW		CMA	
$\times 10^{-4}$	$\overline{F}$	θ_{MKT}	$\overline{F}$	θ_{SMB}	$\overline{F}$	θ_{HML}	$\overline{F}$	θ_{RMW}	$\overline{F}$	θ_{CMA}
63-70	28	ℝ	57	ℝ	41	ℝ	2	ℝ	22	ℝ
71-80	30	ℝ	54	ℝ	33	ℝ	5	ℝ	25	ℝ
81-90	44	ℝ	-20	ℝ	57	ℝ	39	ℝ	55	ℝ
91-00	103	$]-\infty,\ 134] \cup [\ 1473,\infty[$	3	$]-\infty,\ -434] \cup [\ -337,\infty[$	27	ℝ	32	ℝ	30	ℝ
00-10	14	ℝ	65	ℝ	41	ℝ	44	ℝ	35	ℝ
Full	45	[-219, 136]	30	[-71, 138]	40	[-103, 243]	26	[-179, 264]	34	[-373, 369]

Note – Sample includes monthly observations from July 1963 to December 2010 on the US. Series include 12 equally weighted (EW) and value-weighted (VW) industry portfolios as well as US factors for market (MKT), size (SMB), book-to-market (HML), profitability (RMW), and investment (CMA). Confidence sets are at the 5% level. $\overline{F}$ is the factor average over the considered time period; θ captures factor pricing. * denotes evidence of pricing at the 5% significance level interpreted as follows: given the reported confidence sets, each factor is priced if its average is not covered. The VW sets conformable with Panel B are all ℝ.

Table S.8: Confidence sets for risk price: industry portfolios and five-factor model, excluding HML

$$r_i - \iota_T \gamma_0 = (\mathscr{R}_1 - \iota_T \gamma_0) b_{i1} + (\mathscr{F} - \iota_T \gamma'_{\mathscr{F}}) b_{i\mathscr{F}} + u_i, \qquad i = 1, \ldots, n$$

PANEL A $\theta = (\gamma_0, \gamma'_{\mathscr{F}})' = (\theta_{\text{MKT}}, \theta_{\text{SMB}}, \theta_{\text{RMW}}, \theta_{\text{CMA}})'$

VW	MKT		SMB		RMW		CMA	
$\times 10$	$\overline{F}$	θ_{MKT}	$\overline{F}$	θ_{SMB}	$\overline{F}$	θ_{RMW}	$\overline{F}$	θ_{CMA}
63-70	28	[-126, 400]	57	[-113, 60]	2	[-101, 50]	22	[-50, 62]
71-80	30	$\mathbb{R}$	54	$\mathbb{R}$	5	$\mathbb{R}$	25	$\mathbb{R}$
81-90	44	[-141, 645]	-20	[-252, 75]	39	[-220, 126]	55	[-107, 122]
91-00	103	[-236, 594]	3*	[78, 1099]	32	[-370, 46]	30	[27, 477]
00-10	14	$\mathbb{R}$	65	$\mathbb{R}$	44	$\mathbb{R}$	35	$\mathbb{R}$
Full	45	[-91, 147]	30	[-13, 152]	26*	[-97, 20]	34	[12, 129]

$$r_i - \iota_T \gamma_c = (\mathscr{R}_1 - \iota_T \gamma_0) b_{i1} + (\mathscr{F} - \iota_T \gamma'_{\mathscr{F}}) b_{i\mathscr{F}} + u_i, , \qquad i = 1, \ldots, n$$

PANEL B $\theta = (\gamma_0, \gamma'_{\mathscr{F}}) = (\theta_{\text{MKT}}, \theta_{\text{SMB}}, \theta_{\text{RMW}}, \theta_{\text{CMA}}), \quad \gamma_c$ partialled-out

VW	MKT		SMB		RMW		CMA	
$\times 10^{-4}$	$\overline{F}$	θ_{MKT}	$\overline{F}$	θ_{SMB}	$\overline{F}$	θ_{RMW}	$\overline{F}$	θ_{CMA}
63-70	28	[-387, 572]	57	[-131, 285]	2	[-102, 229]	22	[-32, 407]
71-80	30	$\mathbb{R}$	54	$\mathbb{R}$	5	$\mathbb{R}$	25	$\mathbb{R}$
81-90	44	[-3897, 3713]	-20	[-1601, 1721]	39	[-1327, 1065]	55	[-1664, 1968]
91-00	103	[-221, 6116]	3*	[11, 12025]	32	[-7807, 151]	30	[13, 4639]
00-10	14	$\mathbb{R}$	65	$\mathbb{R}$	44	$\mathbb{R}$	35	$\mathbb{R}$
Full	45	[-104, 145]	30	[-18, 153]	26	[-93, 46]	34	[5, 127]

Note – Sample includes monthly observations from July 1963 to December 2010 on the US. Series include 12 value weighted (VW) industry portfolios as well as US factors for market (MKT), size (SMB), profitability (RMW), and investment (CMA). Results in Panel A rely on our **RAPT** approach, and those in Panel B or its **PAPT** counterpart; see notes to notes to table S.3 for further definitions.

Table S.9: Industry portfolios, five-factor model: testing the traded factor assumption

$$r_i - \mathscr{R}_1 = a_i \iota_T + \mathscr{R}_1 d_i + \mathscr{F} b_{i,\mathscr{F}} + u_i, \qquad i = 1, \ldots, n,$$

$$a_i = \gamma_c^* - \gamma_0 d_i - \gamma_{\mathscr{F}}' b_{i,\mathscr{F}}, \quad d_i = b_{i1} - 1, \quad \gamma_c^* = \gamma_c - \gamma_0$$

$$\theta = (\gamma_0, \gamma_{\mathscr{F}}') = (\theta_{\text{MKT}}, \theta_{\text{SMB}}, \theta_{\text{HML}}, \theta_{\text{RMW}}, \theta_{\text{CMA}})$$

EW	CTE	MKT		SMB		RMW		CMA	
$\times 10^{-4}$	γ_c^*	$\overline{F}$	θ_{MKT}	$\overline{F}$	θ_{SMB}	$\overline{F}$	θ_{RMW}	$\overline{F}$	θ_{CMA}
63-70	$\mathbb{R}$	28	$\mathbb{R}$	57	$\mathbb{R}$	2	$\mathbb{R}$	22	$\mathbb{R}$
71-80	$\mathbb{R}$	30	$\mathbb{R}$	54	$\mathbb{R}$	5	$\mathbb{R}$	25	$\mathbb{R}$
81-90	$\mathbb{R}$	44	$\mathbb{R}$	-20	$\mathbb{R}$	39	$\mathbb{R}$	55	$\mathbb{R}$
91-00	$]-\infty,\ -486] \cup [\ -168, \infty[$	103	$]-\infty,\ 165] \cup [\ 1266, \infty[$	3	$\mathbb{R}$	32	$\mathbb{R}$	30	$\mathbb{R}$
00-10	$\mathbb{R}$	14	$\mathbb{R}$	65	$\mathbb{R}$	44	$\mathbb{R}$	35	$\mathbb{R}$
VW	CTE	MKT		SMB		RMW		CMA	
$\times 10^{-4}$	γ_c^*	θ_{MKT}		θ_{SMB}		θ_{RMW}		θ_{CMA}	
63-70	$\mathbb{R}$	$\mathbb{R}$		$\mathbb{R}$		$\mathbb{R}$		$\mathbb{R}$	
71-80	$\mathbb{R}$	$\mathbb{R}$		$\mathbb{R}$		$\mathbb{R}$		$\mathbb{R}$	
81-90	$\mathbb{R}$	$\mathbb{R}$		$\mathbb{R}$		$\mathbb{R}$		$\mathbb{R}$	
91-00	$\mathbb{R}$	$\mathbb{R}$		$\mathbb{R}$		$\mathbb{R}$		$\mathbb{R}$	
00-10	$\mathbb{R}$	$\mathbb{R}$		$\mathbb{R}$		$\mathbb{R}$		$\mathbb{R}$	

$$r_i - \mathscr{R}_1 = a_i \iota_T + \mathscr{R}_1 d_i + \mathscr{F} b_{i,\mathscr{F}} + u_i, \qquad i = 1, \ldots, n,$$

$$a_i = \gamma_c^* - \gamma_0 d_i - \gamma_{\mathscr{F}}' b_{i,\mathscr{F}}, \quad d_i = b_{i1} - 1, \quad \gamma_c^* = \gamma_c - \gamma_0$$

$$\theta = (\gamma_0, \gamma_{\mathscr{F}}') = (\theta_{\text{MKT}}, \theta_{\text{SMB}}, \theta_{\text{RMW}}, \theta_{\text{CMA}})$$

EW	CTE	MKT	SMB	RMW	CMA
$\times 10^{-4}$	γ_c^*	θ_{MKT}	θ_{SMB}	θ_{RMW}	θ_{CMA}
63-70	$\mathbb{R}$	$\mathbb{R}$	$\mathbb{R}$	$\mathbb{R}$	$\mathbb{R}$
71-80	$\mathbb{R}$	$]-\infty,\ 29] \cup [359, \infty[$ *	$\mathbb{R}$	$]-\infty,\ -449] \cup [\ -171, \infty[$	$]-\infty,\ 231] \cup [\ 335, \infty[$
81-90	$\mathbb{R}$	$\mathbb{R}$	$\mathbb{R}$	$\mathbb{R}$	$\mathbb{R}$
91-00	$]-\infty,\ -568] \cup [\ -154, \infty[$	$]-\infty,\ 135] \cup [\ 1495, \infty[$	$]-\infty,\ -520] \cup [\ -277, \infty[$	$\mathbb{R}$	$\mathbb{R}$
00-10	$\mathbb{R}$	$\mathbb{R}$	$\mathbb{R}$	$\mathbb{R}$	$\mathbb{R}$
VW	CTE	MKT	SMB	RMW	CMA
$\times 10^{-4}$	γ_c^*	θ_{MKT}	θ_{SMB}	θ_{RMW}	θ_{CMA}
63-70	[-2, 62]	[-461, 1094]	[-152, 507]	[-118, 580]	[-50, 863]
71-80	$\mathbb{R}$	$\mathbb{R}$	$\mathbb{R}$	$\mathbb{R}$	$\mathbb{R}$
81-90	$\mathbb{R}$	$\mathbb{R}$	$\mathbb{R}$	$\mathbb{R}$	$\mathbb{R}$
91-00	$]-\infty,\ 47] \cup [\ 465, \infty[$	$]-\infty,\ -2123] \cup [\ -239, \infty[$	$]-\infty,\ -3915] \cup [\ -10, \infty[$	$]-\infty,\ 168] \cup [\ 2722, \infty[$	$]-\infty,\ -1493] \cup [\ 4, \infty[$
00-10	$\mathbb{R}$	$\mathbb{R}$	$\mathbb{R}$	$\mathbb{R}$	$\mathbb{R}$

Note – The inverted test is $\Lambda(\theta, \phi)$. This test follows our **UAPT** where **U** stands for "unrestricted" implies that factors are assumed non-tradable in estimating and testing the model. The test is applied on $r_i - \mathscr{R}_1$ so inference on ϕ allows to assess whether $\gamma_c = \gamma_0$: the hypothesis that $\mathscr{R}_1$ (here, MKT) is traded is rejected at the 5% level when the confidence set on ϕ excludes zero. In the upper Panel of this table, all confidence sets on the HML price are the real line; the lower Panel excludes HML.

Table S.10: Confidence sets for risk price: industry and size portfolios, five-factor model, instrumenting MKT with LagTBill31

$r_i - \iota_T\gamma_0 = (\mathscr{R}_1 - \iota_T\gamma_0)b_{i1} + (\mathscr{F} - \iota_T\gamma_{\mathscr{F}}')b_{i\mathscr{F}} + u_i, \qquad i = 1,\ldots,n$										
PANEL A $\theta = (\gamma_0, \gamma_{\mathscr{F}}')' = (\theta_{\text{MKT}}, \theta_{\text{SMB}}, \theta_{\text{HML}}, \theta_{\text{RMW}}, \theta_{\text{CMA}})'$										
EW	MKT		SMB		HML		RMW		CMA	
$\times 10^{-4}$	θ_{MKT}		θ_{SMB}		θ_{HML}		θ_{RMW}		θ_{CMA}	
Full	[-1496, 232]		[-265, 29]*		[-577, 377]		[-354, 934]		[-2391, 197]	
VW	MKT		SMB		HML		RMW		CMA	
$\times 10^{-4}$	$\overline{F}$	θ_{MKT}	$\overline{F}$	θ_{SMB}	$\overline{F}$	θ_{HML}	$\overline{F}$	θ_{RMW}	$\overline{F}$	θ_{CMA}
Full	45	$\varnothing$	30	$\varnothing$	40	$\varnothing$	26	$\varnothing$	34	$\varnothing$
$r_i - \iota_T\gamma_c = (\mathscr{R}_1 - \iota_T\gamma_0)b_{i1} + (\mathscr{F} - \iota_T\gamma_{\mathscr{F}}')b_{i\mathscr{F}} + u_i, , \qquad i = 1,\ldots,n$										
PANEL B $\theta = (\gamma_0, \gamma_{\mathscr{F}}') = (\theta_{\text{MKT}}, \theta_{\text{SMB}}, \theta_{\text{HML}}, \theta_{\text{RMW}}, \theta_{\text{CMA}})$, γ_c partialled-out										
EW	MKT		SMB		HML		RMW		CMA	
$\times 10^{-4}$	θ_{MKT}		θ_{SMB}		θ_{HML}		θ_{RMW}		θ_{CMA}	
Full	$]-\infty,\ 268]$ $\cup[\ 7594,\infty[$		$]-\infty,\ 30]$ $\cup[\ 3219,\infty[$		$]-\infty,\ 337]$ $\cup[\ 7422,\infty[$		$]-\infty,\ -10725]$ $\cup[\ -314,\infty[$		$]-\infty,\ 174]$ $\cup[\ 23510,\infty[$	
VW	MKT		SMB		HML		RMW		CMA	
$\times 10^{-4}$	$\overline{F}$	θ_{MKT}	$\overline{F}$	θ_{SMB}	$\overline{F}$	θ_{HML}	$\overline{F}$	θ_{RMW}	$\overline{F}$	θ_{CMA}
Full	45	$\varnothing$	30	$\varnothing$	40	$\varnothing$	26	$\varnothing$	34	$\varnothing$

Note – Sample includes monthly observations from July 1963 to December 2010 on the US. Series include 37 equally weighted (EW) and value-weighted (VW) industry and size portfolios as well as US factors for market (MKT), size (SMB), book-to-market (HML), profitability (RMW), and investment (CMA).

Table S.11: Industry and size portfolios, five-factor model. Instrumenting MKT with: LagTBill31, Testing Traded Factor Assumption

$r_i - R_1 = a_i\iota_T + R_1 d_i + F b_{i\mathscr{F}} + u_i, \qquad i = 1,\ldots,n,$											
$a_i = \gamma_c^* - \gamma_0 d_i - \gamma_{\mathscr{F}}' b_{i\mathscr{F}}, \quad d_i = b_{i1} - 1, \quad \gamma_c^* = \gamma_c - \gamma_0$											
$\theta = (\gamma_0, \gamma_{\mathscr{F}}') = (\theta_{\text{MKT}}, \theta_{\text{SMB}}, \theta_{\text{HML}}, \theta_{\text{RMW}}, \theta_{\text{CMA}})$											
EW	CTE	MKT		SMB		HML		RMW		CMA	
$\times 10^{-4}$	γ_c^*	θ_{MKT}		θ_{SMB}		θ_{HML}		θ_{RMW}		θ_{CMA}	
Full	$]-\infty,\ 75]$ $\cup[\ 762,\infty[$	$]-\infty,\ 411]$ $\cup[\ 1775,\infty[$		$]-\infty,\ 37]$ $\cup[\ 923,\infty[$		$]-\infty,\ 378]$ $\cup[\ 2172,\infty[$		$]-\infty,\ -3201]$ $\cup[\ -355,\infty[$		$]-\infty,\ 225]$ $\cup[\ 7036,\infty[$	
VW	CTE	MKT		SMB		HML		RMW		CMA	
$\times 10^{-4}$	γ_c^*	$\overline{F}$	θ_{MKT}	$\overline{F}$	θ_{SMB}	$\overline{F}$	θ_{HML}	$\overline{F}$	θ_{RMW}	$\overline{F}$	θ_{CMA}
Full	$\varnothing$	45	$\varnothing$	30	$\varnothing$	40	$\varnothing$	26	$\varnothing$	34	$\varnothing$

Note – The inverted test is $\Lambda(\theta,\phi)$. This test follows our **UAPT** where **U** stands for "unrestricted" implies that factors are assumed non-tradable in estimating and testing the model. The test is applied on $r_i - \mathscr{R}_1$ so inference on ϕ allows to assess whether $\gamma_c = \gamma_0$: the hypothesis that $\mathscr{R}_1$ (here, MKT) is traded is rejected at the 5% level when the confidence set on ϕ excludes zero.