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IDENTIFICATION-ROBUST FACTOR PRICING: CANADIAN EVIDENCE*

Marie-Claude BEAULIEU
Chaire RBC en innovations financières
Département de finance, assurance et immobilier
Université Laval
Centre interuniversitaire sur le risque, les politiques économiques et l'emploi (CIRPÉE)
marie-claude.beaulieu@fsa.ulaval.ca

Jean-Marie DUFOUR
William Dow Professor of Economics
Department of Economics
McGill University
Centre interuniversitaire de recherche en économie quantitative (CIREQ)
Centre interuniversitaire de recherche en analyse des organisations (CIRANO)
jean-marie.dufour@mcgill.ca

Lynda KHALAF
Economics Department
Carleton University
Groupe de recherche en économie de l'énergie, de l'environnement et des ressources naturelles (GREEN)
Université Laval
Centre interuniversitaire de recherche en économie quantitative (CIREQ)
khalaf@carleton.ca

ABSTRACT–We analyze factor models based on the Arbitrage Pricing Theory (APT). using identification-robust inference methods. Such models involve nonlinear reduced-rank restrictions whose identification may raise serious non-regularities and lead to a failure of standard asymptotic theory. We build confidence sets for structural parameters based on

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inverting Hotelling-type pivotal statistics. These confidence sets provide much more information than the corresponding tests. Our approach may be interpreted as a multivariate extension of the Fieller method for inference on mean ratios. We also introduce a formal definition for a redundant factor linking the presence of such factors to unbounded confidence sets, and we document their perverse effects on minimum-root-based model tests.

Results are applied to multifactor asset-pricing models with Canadian data, the Fama-French-Carhart benchmarks and monthly returns of 25 portfolios from 1991 to 2010. Despite evidence of weak identification, several findings deserve notice when data are analyzed over ten-year subperiods. With equally weighted portfolios, the three-factor model is rejected before 2000, but weakly supported thereafter. In contrast, the three-factor model is not rejected with value-weighted portfolios. Interestingly in this case, the market factor is priced before 2000 along with size, while both Fama-French factors are priced thereafter. The momentum factor severely compromises identification, which calls for caution in interpreting existing work documenting momentum effects on the Canadian market. This empirical analysis underscores the practical usefulness of our analytical confidence sets.

Introduction

Equilibrium-based financial econometric models commonly involve reduced-rank (RR) restrictions. Well known applications include work on asset-pricing based on the Arbitrage Pricing Theory (APT) and factor models (Black, 1972; Ross, 1976)[1]. From a methodological perspective, RR restrictions raise statistical challenges. Even with linear multivariate models, discrepancies between standard asymptotic and finite-sample distributions can be severe, due for example to high dimensionality; see Dufour and Khalaf (2002) and the references therein. Rank restrictions pose further identification non-regularities which can lead to the failure of standard asymptotics. This provides the motivation of this paper.

From an empirical perspective, we focus on RR-based inference methods relevant to factor models of asset pricing. In multivariate-regression financial models, finite-sample testing is important because tests which are only approximate and/or do not account for non-normality can lead to unreliable empirical interpretations of standard financial models; see Shanken (1996), Campbell *et al.* (1997), Dufour, Khalaf and Beaulieu (2003, 2010), and Beaulieu, Dufour and Khalaf (2007, 2009, 2010a, 2013). In parallel, an emerging literature, which builds on Kan and Zhang (1999a, 1999b) recognizes the adverse effects of large numbers of factors; see Kleibergen (2009), Kan, Robotti and Shanken (2013), Kleibergen and Zhan (2013), Gospodinov, Kan and Robotti (2014), and Harvey, Liu and Zhu (2015). In this paper, we develop inference methods immune to both dimensionality and identification difficulties.

The few finite-sample rank-based asset-pricing methods (Shanken, 1985; Shanken, 1986; Zhou, 1991 and 1995; Shanken and Zhou, 2007) focus on testing

1. See also Gibbons (1982), Barone-Adesi (1985), Shanken (1986), Zhou (1991), Zhou (1995), Bekker, Dobbelstein and Wansbeek (1996), Costa, Gardini and Paruolo (1997), Campbell, Lo and MacKinlay (1997: Chapter 6), Velu and Zhou (1999), and Shanken and Zhou (2007).

rather than set inference. Kleibergen (2009) provides numerical identification-robust inference methods backed by a simulation study which reinforces the motivation for our approach[2]. In Beaulieu *et al.* (2013), we developed confidence sets for possibly weakly identified parameter in such a context, but the proposed method is limited to a model with a single factor. In this paper, we allow for several factors, which leads to vector nonlinear (possibly unidentified) parameters. This considerably expands the class of financial models considered. Overall, we make three main contributions.

First, we build confidence set (*CS*) estimates for the parameters of interest, which are based on inverting minimum-distance pivotal statistics. These include Hotelling's T^2 criterion (Hotelling, 1947). In multivariate analysis, Hotelling's statistic is mostly used for *testing* purposes, and its popularity stems from its least-squares (LS) foundations which yield exact F-based null distributions in Gaussian setups. We apply analytical solutions to the test inversion problem. Our *CS*s provides much more information than the Hotelling-type tests on which they are based, and extend their relevance beyond reduced-form specifications; see Beaulieu, Dufour and Khalaf (2015) for the underlying finite-sample statistical theory and simulation evidence.

Second, we provide a formal definition of a statistically uninformative factor, and we link the presence of such factors to the possibility of unbounded confidence sets. We also document the perverse effects of adding uninformative factors on *J*-type minimum-root-based tests. Although unbounded confidence sets are not always uninformative, our results call for caution in exclusively relying on tests to assess models. This warning has obvious implications for asset-pricing models and motivate our empirical analysis.

Third, our results are applied to multifactor asset-pricing models with Canadian data and the Fama-French-Carhart benchmark model (Fama and French, 1992; Fama and French, 1993; Carhart, 1997). We analyze monthly returns of 25 portfolios from 1991 to 2010. The empirical literature on asset-pricing models in Canada is scarce. Most studies involving Canadian assets aim at measuring North-American financial market integration (see, for example, Jorion and Schwartz, 1986; Mittoo, 1992; Foerster and Karolyi, 1993; and more recently Beaulieu, Gagnon and Khalaf, 2008). In other cases (Griffin, 2002; and Fama and French, 2012), international multifactor asset-pricing models are tested on a large set of countries, including Canada, to measure the importance of international versus domestic asset-pricing factors in different countries.

One of the few articles on a multifactor asset-pricing model for Canadian portfolios — using exclusively Canadian factors — is L'Her, Masmoudi and Suret (2004); see also the references therein. We extend this literature to verify on a long time period the importance of domestic Fama-French factors to price Canadian

2. In contrast, Kan *et al.* (2013), Kleibergen and Zhan (2013), and Gospodinov *et al.* (2014) focus on model misspecification.

assets adding momentum to the list of factors considered by L'Her *et al.* (2004). The specificities of Canadian assets are put forward and contrasted with American assets for which abundant results are available. To achieve this goal, we use improved inference procedures based on a formal definition of a statistically non-informative asset-pricing factor.

In the context of the weak-instrument literature, our methodology may be viewed as a generalization of the Dufour and Taamouti (2005) quadric-based set estimation method beyond the linear limited-information simultaneous equation setting. For other quadric-based solutions in different contexts, see Bolduc, Khalaf and Yelou (2010) for inference on multiple ratios, and Khalaf and Urga (2014) on cointegration vectors.

In relation with Kleibergen (2009). our methodology allows one to estimate the zero-beta rate (Shanken, 1992; Campbell *et al.*, 1997: Chapter 6; Lewellen, Nagel and Shanken, 2010), which has not been considered by Kleibergen (2009). Theoretically coherent empirical models often impose restrictions on the risk prices through the zero-beta rate; see Lewellen *et al.* (2010: prescription 2). Examples include: (i) whether the zero-beta rate is equal to the risk-free rate, and (ii) whether the risk price of a traded portfolio when included as a factor equals the factor's average return in excess of the zero-beta rate (Shanken, 1992). We also formally control for factors which are tradeable portfolios. Furthermore, in contrast with Kleibergen's numerical projections, our analytical solutions can easily ascertain empty or unbounded sets, whereas numerical solutions remain subject to precision and execution time constraints. Avoiding numerical searches is particularly useful for multifactor models.

In the context of the statistical literature, our test inversion approach can be viewed as an extension of the classical inference procedure proposed by Fieller (1954) for inference on mean ratios (see also Zerbe, Laska, Meisner and Kushner, 1982; Dufour, 1997; Beaulieu *et al.*, 2013; Bolduc *et al.*, 2010) to a multivariate setting. Like standard asymptotic procedures which use the *delta* method, our generalized Fieller approach relies on LS. Yet both approaches exploit LS theory in fundamentally different ways. In contrast with the former — which excludes parameter discontinuity and yields bounded confidence intervals by construction — our test inversion procedure does not require parameter identification and allows for unbounded solutions.

Empirically, despite overwhelming evidence of weak identification, several interesting results deserve notice, particularly when data are analyzed over ten-year subperiods. With equally weighted portfolios, the three-factor model is rejected before 2000. Thereafter, the model is weakly supported with HML confirmed as the only priced factor. With value-weighted data, the three-factor model is not rejected; interestingly, the market factor is priced before 2000 and unidentified thereafter, while both Fama-French factors are priced after 2000. Although momentum is priced with equally-weighted data before 2000, the resulting sets on other factors are practically uninformative. The momentum factor thus severely compromises identification, which calls for caution in interpreting existing work documenting momentum effects for the Canadian market.

The paper is organized as follows. Section 1 sets the notation and framework. In section 2, we present our test inversion method and associated projections. Our empirical analysis is discussed in section 3. In the last section, we conclude. A technical appendix follows.

1. Multifactor Pricing Model

Let r_i, $i = 1, \ldots, n$, be a vector of T returns on n assets (or portfolios) for $t = 1,\ldots,T$, and $R = \left[R_1 \ \ldots \ R_q \right]$ a T x q matrix of observations on q risk factors. Building on multivariate regressions of the form

$$r_i = a_i \iota_T + R b_i + u_i, \qquad i = 1,\ldots,n, \tag{1}$$

our APT based empirical analysis formally accounts for tradeable factors; see Shanken (1992), Campbell *et al.* (1997, Chapter 6), and Lewellen *et al.* (2010). Note that the inference method applied by Kleibergen (2009) relaxes tradeability restrictions.

Without loss of generality, suppose that R_1 corresponds to a vector of returns on a tradeable portfolio, for example, a market benchmark. Consider the following conformable partitions of R and b_i:

$$R = \left[R_1 \ \ F \right], F = \left[R_2 \ \ldots \ R_q \right], \tag{2}$$

$$b_i = \begin{bmatrix} b_{i1} \\ b_{iF} \end{bmatrix}, i = 1,\ldots,n, \tag{3}$$

where F is the T x $(q-1)$ submatrix containing the observations on the $q-1$ factors $R_2,\ldots,R_q$, while b_{i1} is a scalar and b_{iF} a $(q-1)$-dimensional parameter vector. In this context, the APT implies a zero-intercept in the regression of returns in excess of the zero-beta rate, denoted γ_0, on: (i) R_1 in excess of γ_0, and (ii) on $R_2,\ldots,R_q$ in excess of a risk price $(q-1)$ dimensional vector denoted γ_F, where γ_0 and γ_F are unknown parameters:

$$r_i - \iota_T \gamma_0 = \left(R_1 - \iota_T \gamma_0\right) b_{i1} + \left(F - \iota_T \gamma_F'\right) b_{iF} + u_i, \qquad i = 1,\ldots,n. \tag{4}$$

Rewriting the latter as in (1) produces a nonlinear RR restriction on its intercept which captures the zero-beta rate and risk premia as model parameters:

$$r_i = a_i \iota_T + R_1 b_{i1} + F b_{iF} + u_i, \tag{5}$$

$$a_i + \gamma_0 (b_{i1} - 1) + \gamma_F' b_{iF} = 0. \tag{6}$$

We wish to build confidence sets for the q-dimensional vector

$$\theta = \left(\gamma_0, \gamma_F'\right)' \tag{7}$$

which we interpret using a traditional cross-sectional factor pricing approach; see Campbell *et al*. (1997: Chapter 6) or Shanken and Zhou (2007). Indeed, the time series averages of (5) lead (with obvious notation) to

$$\begin{aligned}\overline{r}_i &= \gamma_0(1-b_{i1}) - \gamma_F' b_{iF} + \overline{R}_1 b_{i1} + \overline{F} b_{iF} + \overline{u}_i \\ &= \gamma_0 + \left(\overline{R}_1 - \gamma_0\right) b_{i1} + \left(\overline{F} - \gamma_F'\right) b_{iF} + \overline{u}_i, \qquad i = 1,\ldots,n. \end{aligned} \tag{8}$$

It follows that γ_0 retains its traditional cross-sectional definition, and the coefficients on betas of the non-tradeable factors correspond to $\left(\overline{F} - \gamma_F'\right)$. This implies that R_1 is not priced if $\overline{R}_1 - \gamma_0 = 0$ and each one of the remaining factors will be not be priced in turn if the corresponding component of $\overline{F} - \gamma_F'$ is zero. Our procedure as described below yields confidence intervals on γ_0 and the components of γ_F. Given these intervals, we assess whether the tradeable factor is not priced if R_1 is covered, and whether each factor besides R_1 is not priced if the average of each factor is, in turn, not covered.

Our intervals are simultaneous in the sense of joint coverage, which implies that decisions on pricing will also be simultaneous. A formal definition of simultaneity is further discussed in the next section which outlines our confidence set estimation method.

2. Identification, Estimation and Testing

Whether viewed as a RR multivariate or cross-sectional regression, identification of θ can be assessed from (4) or (8). If $b_{i1} \simeq 1$, then γ_0 almost drops out of the model. Furthermore, if any of the factor *betas* is close to zero across *i*, its risk price effectively drops out. In fact, whenever factor *betas* bunch up across or within test assets, problems akin to cross-sectional collinearity emerge and undermine the identification of θ. By (8), to recover θ (without additional information or instruments) the *betas* per factor need to vary enough across equations. In practice, reliance on portfolios to reduce dimensionality ends up reducing the dispersion of the *betas*. Identification difficulties are thus an empirical reality.

To address this problem, we extend Beaulieu *et al*. (2013) beyond the case of a single factor and scalar nonlinear parameter. In particular, we proceed by inverting a multivariate test of a hypothesis that sets θ to a given value θ_0:

$$H_0(\theta_0): \theta = \theta_0, \quad \theta_0 \text{ known}. \tag{9}$$

Inverting the test (for given level α) consists in assembling the values θ_0 which are not rejected at this level. For example, given a right-tailed statistic $T(\theta_0)$ with α -level critical point $T_c(\alpha, \theta_0)$, our procedure involves solving, over θ_0, the inequality

$$T(\theta_0) \leq T_c(\alpha, \theta_0), \tag{10}$$

where $T_c(\alpha, \theta_0)$ controls the level for θ_0. $T_c(\alpha, \theta_0)$ may or may not depend on θ_0. For Hotelling statistics discussed below (under a Gaussian error assumption), $T_c(\alpha, \theta_0)$ does

not depend on θ_0 so we can write $T_c(\alpha, \theta_0) = T_c(\alpha)$ and the critical value is the same for tested values θ_0. In other cases, we may also be able to find a $T_c(\alpha)$ which ensure that the level of the test is controlled irrespective of the value θ_0, even if the level may not be constant for different values of θ_0. The confidence set for θ is the set $CS(\theta;\alpha)$ of all values θ_0 such that (10) holds. It is then easy to see that:

$$P[\theta \in CS(\theta;\alpha)] \geq 1-\alpha \tag{11}$$

whether θ may not be identified.

To derive confidence intervals for the individual components of θ, or more generally, for a given scalar function $g(\theta)$, we proceed by projecting $CS(\theta;\alpha)$, *i.e.* by minimizing and maximizing $g(\theta)$ over the θ values in $CS(\theta;\alpha)$. The resulting intervals so obtained are simultaneous, in the following sense: for any set of m continuous real valued functions of θ, $g_i(\theta) \in R$, $i = 1,\ldots, m$, let $g_i(CS(\theta;\alpha))$ denote the image of $CS(\theta;\alpha)$ by the function g_i. Then

$$P[g_i(\theta) \in g_i(CS(\theta;\alpha)),\ i = 1,\ldots,m] \geq 1-\alpha. \tag{12}$$

If T_c is defined so that (11) holds without identifying θ, then (12) would also hold whether θ is identified or not. A complete description of our methodology requires: (i) defining $T(\theta_0)$, (ii) deriving T_c to control size for any θ_0, and (iii) characterizing the solution of (10). As in Beaulieu *et al.* (2013), we find an analytical solution to this problem.

We focus on the Hotelling-type statistic

$$\Lambda(\theta) = \frac{(1,\theta')\hat{B}\hat{S}^{-1}\hat{B}'(1,\theta')'}{(1,\theta')(X'X)^{-1}(1,\theta')'}, \tag{13}$$

where we set $\theta = \theta_0$ to test $H_0(\theta_0)$ in (9).

$$\hat{B} = (X'X)^{-1}X'Y, \quad \hat{S} = \hat{U}'\hat{U}, \quad \hat{U} = Y - X\hat{B}, \tag{14}$$

X is a T x k full-column rank matrix that includes a constant regressor and the observations on all q factors so $k = q + 1$, and Y is the T x n matrix that stacks the left-hand side returns in deviation from the tradeable benchmark. Clearly, $\hat{B}$ and $\hat{U}$ are the OLS estimators for the underlying unrestricted reduced form

$$Y_t = B'X_t + U_t, \quad t=1, \ldots, T, \tag{15}$$

or equivalently

$$Y = XB + U, \tag{16}$$

where U is the disturbance matrix and the restriction on the intercept (5) is rewritten as $(1,\theta')B = 0$ in which case (9) corresponds to

$$H_0(\theta_0): (1,\theta_0')B = 0, \quad \theta_0 \text{ known.}$$

We rely on the F-approximation

$$\Lambda(\theta)\frac{\tau_n}{n} \quad F(n,\tau_n), \quad \tau_n = T-k-n+1 \tag{17}$$

which holds exactly when the regression error vectors U_t are contemporaneously correlated *i.i.d.* Gaussian assuming we can condition on X for statistical analysis. The latter distributional result does not require any identification restriction (other than usual full-rank assumptions on $X'X$ and $\hat{S}$). For proofs and further references, see Beaulieu *et al.* (2015) and references therein. Simulation studies reported in this paper confirm that fat-tailed disturbances arising from multivariate-t or GARCH do not cause size distortions for empirically relevant designs.

$\Lambda(\theta_0)$ can be interpreted relative to the financial literature and in particular the well known zero-intercept test of Gibbons, Ross and Shanken (1989), as follows. The classical Hotelling statistics provide multivariate extensions of Student-*t* statistics, and take the form:

$$\Lambda_{0j} = \frac{s_k[j]'\hat{B}\hat{S}^{-1}\hat{B}'s_k[j]}{s_k[j]'(X'X)^{-1}s_k[j]}, \tag{18}$$

where $s_k[j]$ denotes a *k*-dimensional selection vector with all elements equal to zero except for the *j*-th element which is equal to 1. The underlying hypotheses assess the joint contribution of each factor in (15). *i.e.*

$$H_{0j} : s_k[j]'B = 0, \quad j \in \{1,...,k\}. \tag{19}$$

For example $s_k[1]$ provides inference on the unrestricted intercept, and $s_k[2]$ allows one to assess the *betas* on the tradeable factor in deviation from one; see Beaulieu *et al*. (2010a) for recent applications. As in (17) the same null distribution holds for each statistic Λ_{0j} (under the same conditions). Interestingly, we show that inverting $\Lambda(\theta)$ embeds all of these tests through a sufficient condition for bounded outcomes that avoids pre-testing.

To get a confidence set from (13)-(17), we rewrite the inequation

$$\Lambda(\theta) \leq f_{n,\tau_n,\alpha}, \tag{20}$$

where $f_{n,\tau_n,\alpha}$ denotes the α -level critical value from the $F(n,\tau_n)$ distribution, as

$$(1,\theta')A(1,\theta')' \leq 0, \tag{21}$$

where A is the k x k data dependent matrix

$$A = \hat{B}\hat{S}^{-1}\hat{B}' - (X'X)^{-1} f_{n,\tau_n,\alpha}(n/\tau_n). \tag{22}$$

Simple algebraic manipulations suffice to show the above. Next, inequality (21) is re-expressed as

$$\theta' A_{22}\theta + 2A_{12}\theta + A_{11} \leq 0 \tag{23}$$

which leads to the setup of Dufour and Taamouti (2005), so projection-based *CS*s for any linear transformation of θ can be obtained as described in this paper. For

completeness sake, the solution is reproduced in the Appendix. This requires partitioning A as follows

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \tag{24}$$

where A_{11} is a scalar, A_{22} is q x q, and $A_{12} = A'_{21}$ is 1 x q. Using the partitioning

$$\hat{B} = \begin{bmatrix} \hat{a}' \\ \hat{b} \end{bmatrix}, \quad \hat{b} = \begin{bmatrix} \hat{\beta}'_2 \\ \vdots \\ \hat{\beta}'_k \end{bmatrix}, \tag{25}$$

$$(X'X)^{-1} = \begin{bmatrix} x^{11} & x^{12} \\ x^{21} & x^{22} \end{bmatrix}, \tag{26}$$

where $\hat{a}'$ is a 1 x q vector, x^{11} is a scalar, $x^{21} = x^{12}'$ is a q x 1 vector, and x^{22} is a q x q matrix, we have:

$$A_{11} = \hat{a}'\hat{S}^{-1}\hat{a} - \left(f_{n,\tau_n,\alpha}\left(n/\tau_n\right)\right)x^{11}, \tag{27}$$

$$A_{12} = A'_{21} = \hat{a}'\hat{S}^{-1}\hat{b}' - \left(f_{n,\tau_n,\alpha}\left(n/\tau_n\right)\right)x^{12}, \tag{28}$$

$$A_{22} = \hat{b}\hat{S}^{-1}\hat{b}' - \left(f_{n,\tau_n,\alpha}\left(n/\tau_n\right)\right)x^{22}. \tag{29}$$

The outcome of resulting projections can be empty, bounded, or the union of two unbounded disjoint sets. Dufour and Taamouti (2005) show that the confidence sets are unbounded if and only if A_{22} is not positive definite. Applying basic algebra to (29) reveals that the diagonal terms of A_{22} are

$$F_j = s_k[j]'\hat{B}\hat{S}^{-1}\hat{B}'s_k[j] - s_k[j]'(X'X)^{-1}s_k[j]\frac{nf_{n,\tau_n,\alpha}}{\tau_n}, \quad j = 1,...,k.$$

Comparing this expression to the definition of Λ_{0j} in (18) implies that

$$\Lambda_{0j}\left(\tau_n\right)/n < f_{n,\tau_n,\alpha} \Leftrightarrow F_j < 0, \quad j = 1,...,k.$$

So if any of the classical Hotelling tests, using the distribution in (17), is not significant at level α, then A_{22} cannot be positive definite and the confidence set will be unbounded: if the tradeable beta is not significantly different from one over all portfolios or if any of the factors is jointly redundant, information on the zero-beta rate as well as the risk price *for all factors* is compromised. The above condition is sufficient but not necessary. Thus, even though Hotelling tests on each factor

are useful, the information they provide is incomplete as for the joint usefulness of the factors in identifying risk price.

An important result from Beaulieu *et al.* (2013) regarding empty confidence set outcomes also generalizes to our multifactor case. Because the critical value underlying the inversion of $\Lambda(\theta)$, which we denoted $f_{n,\tau_n,\alpha}$ above, is the same for all θ values, then

$$\min_{\theta} \Lambda(\theta) \geq f_{n,\tau_n,\alpha} \Leftrightarrow CS(\theta;\alpha) = \varnothing.$$

Again, basic matrix algebra allows one to show (see e.g. Gouriéroux, Monfort and Renault, 1996) that minimizing $\Lambda(\theta)$ produces the Gaussian-LR statistic to test the nonlinear restriction (5) which defines θ. This suggests a bounds *J*-type test to assess the overall model fit. The outcome of this test will be revealed via the quadric solution we implement for test inversion, so it is built into our general set inference procedure. A potential unbounded confidence set guards the researcher from misreading nonsignificant tests as evidence in favour of models on which data is not informative.

In summary, our confidence sets summarize the information content of the data (from a least-squares or Gaussian quasi-likelihood perspective) on risk price without compounding type-I errors.

3. Empirical Analysis

In our empirical analysis of a multifactor asset-pricing model, we use Canadian Fama-French (1992, 1993) factors as well as momentum (Carhart, 1997). We present results for monthly returns of 25 value-weighted portfolios from 1991 to 2010. Portfolios were constructed with all Canadian stocks available on Datastream and Worldscope. The portfolios which are constructed at the end of June are the intersections of five portfolios formed on size (market equity) and five portfolios formed on the ratio of book equity to market equity. The size breakpoints for year *s* are the Toronto Stock Exchange (TSE) market equity quintiles at the end of June of year *s*. The ratio of book equity to market equity for June of year *s* is the book equity for the last fiscal year end in *s*–1 divided by market equity for December of year *s*–1. The ratios of book equity to market equity are TSE quintiles. We use the filter from Karolyi et Wu (2014) to eliminate abnormal observations in our database. This filter is minimally invasive as the filtered database contains an overall sample average of 1700 stocks.

The benchmark factors are: 1) the excess return on the market, defined as the value-weighted return on all TSE stocks (from Datastream and Worldscope database) minus the one-month Treasury bill rate (from the Bank of Canada). 2) SMB (small minus big) defined as the average return on three small portfolios minus the average return on three big portfolios, 3) HML (high minus low) defined as the average return on two value portfolios minus the average return on two growth portfolios, and (4). MOM, the average return on the two high prior return portfolios minus

TABLE 1

CONFIDENCE SETS FOR RISK PREMIA

TEN-YEAR SUBPERIODS

$$r_i - \iota_T\gamma_0 = \left(R_1 - \iota_T\gamma_0\right)b_{i1} + \left(F - \iota_T\gamma_F'\right)b_{iF} + u_i, \qquad i = 1,\ldots,n$$

$$\theta = \left(\gamma_0, \gamma_F'\right)' = \left(\theta_{MKT}, \theta_{SMB}, \theta_{HML}\right)'$$

EW	MKT		SMB		HML		MOM	
x10^{-4}	$\bar{R}_1$	θ_{MKT}	$\bar{F}$	θ_{SMB}	$\bar{F}$	θ_{HML}	$\bar{F}$	–
91-00	51	∅	25	∅	28	∅	–56	–
00-10	48	[–1165,302]	149	[–105,786]	218*	[–1750,187]	–9	–
VW	MKT		SMB		HML		MOM	
x10^{-4}	$\bar{R}_1$	θ_{MKT}	$\bar{F}$	θ_{SMB}	$\bar{F}$	θ_{HML}	$\bar{F}$	–
91-00	51*	]–∞,–521] ∪[13490,∞[	25*	]–∞,15] ∪[4284,∞[	28	]–∞,419] ∪[7601,∞[	–56	–
00-10	48	ℝ	149*	]–∞,–711] ∪[526,∞[	218*	]–∞,–1611] ∪[3056,∞[	–9	–

$$\theta = \left(\gamma_0, \gamma_F'\right)' = \left(\theta_{MKT}, \theta_{SMB}, \theta_{HML}, \theta_{HML}\right)'$$

EW	MKT		SMB		HML		MOM	
x10^{-4}	$\bar{R}_1$	δ	$\bar{F}$	δ	$\bar{F}$	δ	$\bar{F}$	δ
91-00	51	ℝ	25	ℝ	28	ℝ	–56*	]–∞,–969] ∪[415,∞[
00-10	48	ℝ	149	ℝ	218	ℝ	–9	ℝ
VW	MKT		SMB		HML		MOM	
x10^{-4}	$\bar{R}_1$	δ	$\bar{F}$	δ	$\bar{F}$	δ	$\bar{F}$	δ
91-00	51	ℝ	25	ℝ	28	ℝ	–56	ℝ
00-10	48	ℝ	149	ℝ	218	ℝ	–9	ℝ

NOTE: Sample includes monthly observations from January 1991 to December 2010. Series are constructed with all Canadian observations from Datastream and Worldscope. They include 25 equally weighted (EW) and value-weighted (VW) portfolios as well as Canadian factors for market (MKT), size (SMB), book-to-market (HML) and momentum (MOM). Confidence sets are at the 5 % level. $\bar{F}$ is the factor average over the considered time period; θ captures factor pricing as defined in (7).

* denotes evidence of pricing at the 5 % significance level interpreted as follows: given the reported confidence sets, the tradeable factor is not priced if $\bar{R}_1$ is covered; each other factor is not priced if (its average) is not covered; see section 2.

the average return on the two low prior return portfolios. Fama-French benchmark factors, SMB and HML, are constructed from six size/book-to-market benchmark portfolios which do not include hold ranges and do not incur transaction costs. The portfolios for these factors are rebalanced annually using two independent sorts,

TABLE 2

CONFIDENCE SETS FOR RISK PREMIA

FIVE-YEAR SUBPERIODS

$$r_i - \iota_T \gamma_0 = \left(R_1 - \iota_T \gamma_0\right) b_{i1} + \left(F - \iota_T \gamma'_F\right) b_{iF} + u_i, \qquad i = 1,\ldots,n$$

$$\theta = \left(\gamma_0, \gamma'_F\right)' = \left(\theta_{\text{MKT}}, \theta_{\text{SMB}}, \theta_{\text{HML}}\right)'$$

EW	MKT		SMB		HML		MOM	
x10^{-4}	$\bar{R}_1$	θ_{MKT}	$\bar{F}$	θ_{SMB}	$\bar{F}$	θ_{HML}	$\bar{F}$	–
91-95	21	$\mathbb{R}$	68	$\mathbb{R}$	30	$\mathbb{R}$	–152	–
96-00	80	$\mathbb{R}$	–18	$\mathbb{R}$	25	$\mathbb{R}$	39	–
00-05	52	$\mathbb{R}$	170	$\mathbb{R}$	262	$\mathbb{R}$	72	–
06-10	43	$\mathbb{R}$	128	$\mathbb{R}$	175*	$\{-\infty,-359\} \cup \{1026,\infty\}$	–89	–
VW	MKT		SMB		HML		MOM	
x10^{-4}	$\bar{R}_1$	θ_{MKT}	$\bar{F}$	θ_{SMB}	$\bar{F}$	θ_{HML}	$\bar{F}$	–
91-95	21	$\mathbb{R}$	68	$\mathbb{R}$	30	$\mathbb{R}$	–152	–
96-00	80*	$\{-\infty,-384\} \cup \{291,\infty\}$	–18	$\mathbb{R}$	25	$\mathbb{R}$	39	–
00-05	52	$\{-\infty,-307\} \cup \{1687,\infty\}$	170	$\mathbb{R}$	262	$\mathbb{R}$	72	–
06-10	43	$\mathbb{R}$	128	$\mathbb{R}$	175*	$\{-\infty,-556\} \cup \{874,\infty\}$	–89	–

$$\theta = \left(\gamma_0, \gamma'_F\right)' = \left(\theta_{\text{MKT}}, \theta_{\text{SMB}}, \theta_{\text{HML}}, \theta_{\text{HML}}\right)'$$

EW	MKT		SMB		HML		MOM	
x10^{-4}	$\bar{R}_1$	δ	$\bar{F}$	δ	$\bar{F}$	δ	$\bar{F}$	δ
91-95	21	$\mathbb{R}$	68	$\mathbb{R}$	30	$\mathbb{R}$	–152	$\mathbb{R}$
96-00	80	$\mathbb{R}$	–18	$\mathbb{R}$	25	$\mathbb{R}$	39	$\mathbb{R}$
00-05	52	$\mathbb{R}$	170	$\mathbb{R}$	262	$\mathbb{R}$	72	$\mathbb{R}$
06-10	43	$\mathbb{R}$	128	$\mathbb{R}$	175*	$\{-\infty,-172\} \cup \{894,\infty\}$	–89	$\mathbb{R}$
VW	MRKT		SMB		HML		MOM	
x10^{-4}	$\bar{R}_1$	δ	$\bar{F}$	δ	$\bar{F}$	δ	$\bar{F}$	δ
91-95	21	$\mathbb{R}$	68	$\mathbb{R}$	30	$\mathbb{R}$	–152	$\mathbb{R}$
96-00	80	$\mathbb{R}$	–18	$\mathbb{R}$	25	$\mathbb{R}$	39	$\mathbb{R}$
00-05	52	$\mathbb{R}$	170	$\mathbb{R}$	262	$\mathbb{R}$	72	$\mathbb{R}$
06-10	43	$\mathbb{R}$	128	$\mathbb{R}$	175*	$\{-\infty,317\} \cup \{807,\infty\}$	–89	$\mathbb{R}$

NOTE: Sample includes monthly observations from January 1991 to December 2010. Series are constructed with all Canadian observations from Datastream and Worldscope. They include 25 equally weighted (EW) and value-weighted (VW) portfolios as well as Canadian factors for market (MKT). size (SMB). book-to-market (HML) and momentum (MOM). Confidence sets are at the 5 % level. $\bar{F}$ is the factor average over the considered time period; θ captures factor pricing as defined in (7).

* denotes evidence of pricing at the 5 % significance level interpreted as follows: given the reported confidence sets, the tradeable factor is not priced if $\bar{R}_1$ is covered; each other factor is not priced if (its average) is not covered; see section 2.

on size (market equity, ME) and book-to-market (the ratio of book equity to market equity, BE/ME). The size breakpoint (which determines the buy range for the small and big portfolios) is the median TSE market equity. The BE/ME breakpoints (which determine the buy range for the growth, neutral, and value portfolios) are the 30th and 70th TSE percentiles. For the construction of the MOM factor, six value-weighted portfolios formed on size and prior (2–12) returns are used. The portfolios, which are formed monthly, are the intersections of two portfolios formed on size (market equity, ME) and three portfolios formed on prior (2–12) return. The monthly size breakpoint is the median TSE market equity. The monthly prior (2–12) return breakpoints are the 30th and 70th TSE percentiles.

Results over 10 and 5 years subperiods are summarized in Tables 1 and 2, where EW and VW denotes equally weighted and value-weighted portfolios. Unless stated otherwise, significancy in what follows refers to the 5 % level. All reported confidence sets are also at the 5 % level.

Over ten-year subperiods and with equally-weighted data, the three-factor model is rejected before 2000. Thereafter, while very wide although bounded confidence intervals are observed, HML is confirmed as the only priced factor. Value-weighted data support the three-factor model before 2000 albeit weakly as all confidence sets obtained are unbounded. Interestingly, the market factor is priced along with SMB. In sharp contrast, the market risk is unidentified after 2000 and despite evidence of identification difficulties, both Fama-French factors are priced. When the momentum factor is added over and above the Fama-French factor, we find completely uninformative results on all model parameters expect for momentum using equally-weighted data before 2000, in which case we find this factor is priced despite the overwhelming evidence of weak-identification.

Considering five-year subperiods may help us assess whether the above is driven by instability of *betas*. Results must however be interpreted with caution since sample size considerations can be consequential. Indeed, the bulk of resulting confidence sets are uninformative and the only evidence we can confirm is that the market factor seems to be priced from 1996-2000 while the HML factor is priced after 2006.

Our empirical results reveal that for pricing Canadian assets, a standard three Fama-French factor model is the best avenue after 2000. Momentum creates important identification problems and it should not be used to price Canadian assets without raising identification questions. This corroborates the empirical results of Beaulieu, Dufour and Khalaf (2010b), and Beaulieu *et al.* (2015) for American stocks, although the case for omitting momentum is stronger in the Canadian context. An important issue further analyzed by Beaulieu *et al.* (2015) for the U.S. is the portfolio formation method. Using industry portfolios seems to improve identification relative to size-sorting, as the latter compounds factor structure dependences; see also Lewellen and Nagel (2006), Lewellen *et al.* (2010), and Kleibergen and Zhan (2013).

On balance, given the importance of Fama-French factors for pricing stocks internationally (Fama and French, 2012), and the potential for identification problems presented in this paper, future research should aim at finding ways of choosing the factors that empirically explain the cross-section of returns in a general standard context, as discussed by Harvey *et al.* (2015).

Conclusion

This paper studies the factor asset-pricing model for the Canadian market using identification-robust inference methods. We derive confidence sets for the zero-beta rate and factor price based on inverting minimum-distance Hotelling-type pivotal statistics. We use analytical solutions to the latter problem. Our confidence sets have much more informational content than usual Hotelling tests and have various useful applications in statistics, econometrics and finance. Our approach further provides multivariate extensions of the classical Fieller problem.

Empirical results illustrate, among others, severe problems with redundant factors. These findings concur with the (above cited) emerging literature on redundant factors, on tight factor structures and statistical pitfalls of asset-pricing tests, and on the importance of joint (across-portfolios) tests. In practice, our results support a standard three Fama-French factor model for the Canadian market after 2000. In contrast, we find that the momentum factor severely compromises identification which qualifies existing works in this regard.

Concerning the historical debate on the market factor (see Campbell *et al*., 1997: Chapters 5 and 6; Fama and French, 2004; Perold, 2004; Campbell, 2003; Sentana, 2009), our results suggest an alternative perspective. Perhaps the unconditional market model is neither *dead* nor *alive and well*. Instead, the traditional methods of accounting for additional factors may have confounded underlying inference. Because traditional methods severely understate true uncertainty, identification problems in the literature may have escaped concrete notice so far. More to the point here is that with reference to *e.g.* Harvey *et al*. (2015) who consider a very large number of factors, we document identification problems with only three to four factors. We thus concur with Lewellen *et al*. (2010: prescriptions 5 and 6) that it is by far more useful to report set estimates rather than model tests. However, the proliferation of factors in practice increases the likelihood of redundancies. Their associated costs support the use of our method, and motivate further refinements and improvements as important future research avenues.

APPENDIX

This appendix summarizes the solution of (23) from Dufour and Taamouti (2005). Projections based confidence sets for any linear transformation of θ of the form $\omega'\theta$ can be obtained as follows. Let $\tilde{A} = -A_{22}^{-1}A_{12}'$, $\tilde{D} = A_{12}A_{22}^{-1}A_{12} - A_{11}$. If all the eigenvalues of A_{22} (as defined in (24)) are positive so A_{22} is positive definite then:

$$CS_\alpha(\omega'\theta) = \left[\omega'\tilde{A} - \sqrt{\tilde{D}\left(\omega' A_{22}^{-1}\omega\right)}, \omega'\tilde{A} + \sqrt{\tilde{D}\left(\omega' A_{22}^{-1}\omega\right)}\right], \quad if \quad \tilde{D} \geq 0 \tag{A1}$$

$$CS_\alpha(\omega'\theta) = \varnothing, \quad if \quad \tilde{D} < 0. \tag{A2}$$

If A_{22} is non-singular and has one negative eigenvalue then:

(i) if $\omega' A_{22}^{-1}\omega < 0$ and $\tilde{D} < 0$:

$$CS_\alpha(\omega'\theta) = \left]-\infty, \omega'\tilde{A} - \sqrt{\tilde{D}\left(\omega' A_{22}^{-1}\omega\right)}\right] \cup \left[\omega'\tilde{A} + \sqrt{\tilde{D}\left(\omega' A_{22}^{-1}\omega\right)}, +\infty\right[; \tag{A3}$$

(ii) if $\omega' A_{22}^{-1}\omega > 0$ or if $\omega' A_{22}^{-1}\omega \leq 0$ and $\tilde{D} \geq 0$ then:

$$CS_\alpha(\omega'\theta) = \mathbb{R}; \tag{A4}$$

(iii) if $\omega' A_{22}^{-1}\omega = 0$ and $\tilde{D} < 0$ then:

$$CS_\alpha(\omega'\theta) = \mathbb{R} \setminus \left\{\omega'\tilde{A}\right\}. \tag{A5}$$

The projection is given by (A4) if A_{22} is non-singular and has at least two negative eigenvalues.

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