1 Benefits from division of labour

Consider the following vision of a society. Scarce resources, that is, (i) natural resources (land, minerals, forests, water, etc.), (ii) human capital (knowledge, skills, innovation, etc.), and (iii) physical capital (equipment, technologies etc.) are used to produce consumption goods: housing, food, entertainment, etc. People have heterogenous preferences over consumption goods. Division of labour and specialization allow to achieve more efficient outcomes when people are organized in a society than when they remain independent.

"Well then, how will our state supply these needs? It will need a farmer, a builder, and a weaver, and also, I think, a shoemaker and one or two others to provide for our bodily needs. So that the minimum state would consist of four or five men...." (Plato, *The Republic*).

Division of labor between two sexes is commonly considered as the begging of economic specialization and exchange in human society. A classic example of benefits from division of labour by Adam Smith tells that the efficiency of production of pins increased 240 times when workers started to concentrate on single subtasks instead of each carrying out the original broad task (*The In The Wealth of Nations*, 1776). Another commonly used example is efficiency gains from invention of the assembly line by Henry Ford’s engineers in 1913.

Because benefits from specialization create mutual dependence among economic agents, there is a joint decision to be made: (1) what to produce; (2) how to produce, and (3) how to allocate the produced output.¹

**Definition 1 (efficacité au sens Pareto)** Economic allocation is Pareto efficient $\iff$ it is impossible to increase well-being by one economic agent without decreasing well-being by some other agent.

¹The law of comparative advantage tells that individuals/firms benefit from specialization in consumption/production in those the areas where they have a comparative advantage. Consider the following example. Mary is an advocate. She gains 400$ a day. Her Mom taught her sewing, and she can sew a pair of pants in just one day. Instead, she can hire a professional couturier who would sew pants in two days, and charge 100$ for this work. Despite Mary sews better than a professional, it is optimal for her to focus on law, because of 400$ opportunity costs of a day spent on sewing.
2 General structure of competitive equilibrium model

Resources: natural, human, and physical\(^2\) are used to produce consumption goods.

Consumption goods are produced by profit-maximizing firms (Figure 1). Humans (consumers) benefit from consumption goods. They buy goods from firms using their budget that is composed of labour income and revenues from ownership in firms.

*There is a market for any good or service and information is perfect.*

3 Consumer choice

3.1 Individual preferences

Fig 2 illustrates Consumption possibility sets \(X\):

Individual preferences:

\[ x \succeq y \iff x \text{ is at least as good as } y \]

**Definition 2 (les préférences rationnelles) \(\succeq\) is rational \(\iff\)**

\[
\begin{cases}
\text{it is complete: } x \succeq y \text{ or } x \succeq y \\
\text{it is transitive: } x \succeq y \text{ and } y \succeq z \Rightarrow x \succeq z
\end{cases}
\]

**Limitations:** « just perceptible differences »; « framing » (Kahneman and Tversky 1984); social preferences; time inconsistency.

\(^2\)That is, created by humans.
**Definition 3**  
$x$ is better than $y$: $x \succ y \iff \begin{cases} x \succeq y \text{ and } y \not\succeq x \end{cases}$

**Definition 4**  
$x$ is as good as $y$: $x \sim y \iff x \succeq y$ and $y \succeq x$

Verify: $\succsim$ is rational $\Rightarrow$
(i) $\succ$ is irreflexive and transitive;
(ii) $\sim$ is reflexive and transitive;
(iii) $x \succ y \succeq z \Rightarrow x \succ z$.

**Representation of preferences by utility function:**

**Definition 5 (fonction d’utilité)** $\succeq$ are representable by utility function $u(\cdot): X \rightarrow R \iff$

$$x \succeq y \iff u(x) \geq u(y), \forall x \in X, y \in X$$

Two remarks:
- representation of $\succeq$ by utility function is not unique: if $u(\cdot)$ represents $\succeq \iff f(u(\cdot))$ also represents $\succeq$ where $f(\cdot)$ is monotonically increasing function (Eaton, ex. 2.11, 6 page 66).
- The existence of utility function representing $\succeq$ is not guaranteed.

**Indifference curves:** Eaton Fig. 2.2-2.4, ex. 7
Questions:
Q1. Draw indifference curves for the following preferences: (i) homothetic (ii) quasilinear; (iii) Leontief (Eaton ex 5.b)
Q2. Is it possible that two distinct indifference curves intersect?
Q3. Find representation by utility function and draw indifference curves for the following preferences: (i) Mary likes two goods, and she cares only for their total quantity; (ii) Mary likes gloves and she has two hands; (iii) Mary likes gloves and she has only one hand.

Assumptions that are necessary for \( \exists u(x): \succ \) is representable by utility function \( u: X \rightarrow \mathbb{R} \) \( \succ \) is rational.

But: \( \succ \) rational \( \not\Rightarrow \succ \) it is representable by a utility function (ex.: lexicographic preferences). The difference between necessary and sufficient conditions: all women are humans, but not all humans are women.

Questions:
Q4. Define Pareto efficiency.
Q5. Explain the concept of «invisible hand».
Q6. Describe general structure of competitive equilibrium model.
Q7. Which conditions are necessary for the existence of utility function that represents a preference relation. Are they sufficient?

Revision

Definition 6
\[
x = \lim_{n \to \infty} x_n \iff \forall \varepsilon > 0 \exists N : |x_n - x| < \varepsilon
\]

Q8: find \( \lim_{n \to \infty} x_n \) where \( x_n \) is equal to: (i) \( \frac{1}{2} \); (ii) 1; (iii) \( \frac{n}{n+1} \); (iv) \( n \); (v) \( \frac{n^2}{n+1} \); (vi) \( \text{(const)} \frac{1}{n} \).

Definition 7 \( f(x) \) is continuous function \( \iff \)
\[
\lim_{n \to \infty} f(x_n) = f(x) \forall \{x_n\}_{n=1}^{\infty} : x = \lim_{n \to \infty} x_n
\]

Q9: Plot the following functions: \( f(x) = 1 \) and \( f(x) = \begin{cases} x, & x < 1 \\ x + 1, & x \geq 1 \end{cases} \).
Are they continuous?
Exercise: \( f(x) \) is a continuous function; \( x < \overline{x} \). Illustrate that:
(i) \( f(\underline{x}) < 0; f(\overline{x}) > 0, \exists x \in (\underline{x}, \overline{x}) : f(x) = 0 \);
(ii) \( f(x) = 0; f(\overline{x}) = 100, \exists x \in (\underline{x}, \overline{x}) : f(x) = a \forall a \in (0, 100) \).

Definition 8 (les préférences continues) Preferences \( \succ \) are continuous \( \iff \)
\[
\forall \{(x_n, y_n)\}_{n=1}^{\infty} : x_n \succ y_n \forall n, x = \lim_{n \to \infty} x_n, y = \lim_{n \to \infty} x_n, \implies x \succ y
\]
Verify: Lexicographic preferences are not continuous!

**Conditions that are sufficient for** \( \exists \ u(x) \): If \( \succsim \) is rational and continuous \( \implies \) \( \succsim \), it is representable by a continuous utility function.

The model assumes that individual \( \succsim \) are rational and continuous. Hence consumer optimization problem can be written as:

\[
\begin{align*}
\max_{x \in X} u(x) \\
\text{s.t. } px \leq w
\end{align*}
\]

The model makes two more assumptions about \( \succsim \). These assumptions are neither necessary, nor sufficient for the existence of utility representation, but they allow to use the first-order approach to solve consumer optimization problem

(i) “desirability”:

**Definition 9 (monotonicité)** \( \succsim \) is: (i) strictly monotone \( \iff y \succeq x \) and \( y \neq x \implies y \succ x \); (ii) monotone \( \iff y \succeq x \implies y \succ x \); (iii) LNS \( \iff \forall x \in X \text{ and } \forall \varepsilon > 0 \ \exists y \in X : \|x - y\| \leq \varepsilon \text{ and } y \succ x \).

Note: sometimes “enough is enough”

\( \succsim \) is strictly monotone \( \implies \) monotone \( \implies \) LNS;

but: LNS \( \nRightarrow \) monotone \( \nRightarrow \) strictly monotone!

**Revision:**

**Definition 10** set \( U \) is convex \( \iff \forall y \in X \text{ and } z \in X \implies \alpha y + (1 - \alpha) z \in X \).

**Definition 11 (Upper counter set)** \( UCS(x) = \{y \in X \mid y \succeq x\} \).

**Definition 12** \( \succsim \) is: (i) convex \( \iff UCS(x) \) is convex \( \forall x \in X \); (ii) strictly convex \( \iff y \succeq x, z \succeq x \implies \alpha y + (1 - \alpha) z \succ x \).

An individual with convex preferences has a taste for “diversity” (if you offer me two free weekends and many ways (including skiing) to have fun either weekend, I go skiing both weekends).

### 3.2 Consumer demand

**Revision:** Derivative of a function, the first- and the second-order conditions, the method of Lagrange.

**Consumer demand**

\[
\begin{align*}
\max_{x \in X} u(x) \\
\text{s.t. } px \leq w
\end{align*}
\]

if \( p \gg 0 \) and \( u(\cdot) \) is continuous, problem (2) has a unique solution \( x^*(p, w) \).
Figure 3: la demande de Walras.

**Definition 13** solution $x^*(p, w)$ to problem (2) is called Walrasian demand.

Characteristics of $x^*(p, w)$:
- If $\succ$ is LNS and $u(\cdot)$ is a continuous utility function representing preference relation $\succ$ ⇒
  - (i) “no money illusion”: $x^*(\alpha p, \alpha w) = x^*(p, w)$ $\forall \alpha > 0$
  - (ii) “Walras law”: $px^* = w$
  - (iii) uniqueness: if $u(\cdot)$ is strictly convex $\Rightarrow x^*(p, w)$ is unique.

Q10. Which assumption on $\succ$ are important for (i) and (ii)?

**First order conditions** for the “interior” solution of problem (2):

$$MRS(x^*) = \frac{\partial u/\partial x_1}{\partial u/\partial x_2} = \frac{p_1}{p_2};$$

$$p_1x_1^* + p_2x_2^* = w.$$

Eaton Fig 2.7, ex. 2.8-2.10

Limitations to the first-order approach:
- (i) differentiability is necessary: consider $u(x_1, x_2) = \min \{x_1, x_2\}$;
- (ii) a “corner” solution is possible (essential goods): find $x^*(p, w)$ for:
  $$u(x_1, x_2) = \ln(x_1) + \ln(x_2 - 2), p_1 = p_2 = 1, w = 1.$$
Q11. Find consumer demand for preferences that are described in question 3 when: (i) \( p_1 = p_2 = 1, w = 2 \); (ii) \( p_1 = 1, p_2 = 2, w = 2 \); (iii) \( p_1 = 2, p_2 = 2, w = 2 \); (iv) \( p_1 = 2, p_2 = 2, w = 4 \).

Q12. Find consumer demand for preferences that are described by utility function: (i) Cobb-Douglas \( u(x_1, x_2) = \beta \ln(x_1) + (1 - \beta) \ln(x_2) \); (ii) quasilinear: \( u(x, m) = m + \ln(x) \); (iii) \( u(x_1, x_2) = \sqrt{x_1} + \sqrt{x_2} \).

Answer to (i):
\[
x_1^*(p, w) = \frac{\beta w}{p_1}; x_2^*(p, w) = \frac{(1 - \beta)w}{p_2}.
\]

Definition 14 \( x_i \) is: (i) “normal” good \( \Leftrightarrow \frac{\partial x_i^*(p, w)}{\partial w} \geq 0 \); (ii) “inferior” good \( \Leftrightarrow \frac{\partial x_i^*(p, w)}{\partial w} < 0 \); (iii) “Giffen” good \( \Leftrightarrow \frac{\partial x_i^*(p, w)}{\partial p_i} > 0 \)

An example of an inferior good - bus to NY - substitute for airline when get more rich;

Examples of a Giffen good are rare; necessary conditions: 1. the good in question must be an inferior good, 2. there must be a lack of close substitute goods, and 3. the good must constitute a substantial percentage of the buyer’s income, but not such a substantial percentage of the buyer’s income that none of the associated normal goods are consumed.

“As Mr. Giffen has pointed out, a rise in the price of bread makes so large a drain on the resources of the poorer labouring families and raises so much the marginal utility of money to them, that they are forced to curtail their consumption of meat and the more expensive farinaceous foods: and, bread being still the cheapest food which they can get and will take, they consume more, and not less of it.” (Marshall, 1895 edition of Principles of Economics).

Demand function

Definition 15 Demand elasticity is equal to
\[
- \frac{\frac{\partial x_i^*(p, w)}{\partial p_i} \cdot p_i}{x_i^*(p, w)}.
\]

Value function

Definition 16 \( v(p, w) = u(x^*(p, w)) \) is consumer value function.

Characteristics of \( v(p, w) \):
if \( \succeq \) is LNS and \( u(\cdot) \) is a continuous utility function representing preference relation \( \succeq \) ⇒
(i) \( v(\alpha p, \alpha w) = v(p, w) \ \forall \ \alpha > 0 \);
(ii) \( \frac{\partial v}{\partial w} > 0 \); \( \frac{\partial v}{\partial p} \leq 0 \);
(iii) \( \{ (p, w) | v(p, w) \leq \tau \} \) is convex \( \forall \ \tau \);
(iv) \( v(p, w) \) is continuous in \( p \) and in \( w \).
Q13. Find $v(p, w)$ when: (i) $u(x_1, x_2) = x_1 + x_2$; (ii) $u(x_1, x_2) = \min \{x_1, x_2\}$; (iii) $u(x_1, x_2) = x_1$; (iv) $u(x_1, x_2) = \beta \ln(x_1) + (1 - \beta) \ln(x_2)$
   - answer to (iv):

   $$v(p, w) = \beta \ln \frac{w\beta}{p_1} + (1 - \beta) \ln \frac{w(1 - \beta)}{p_2}.$$  

Q14. Find $v(p, w)$ for: $u(x_1, x_2) = \sqrt{x_1} + \sqrt{x_2}$, (i) $p_1^Y = p_2^Y = 1$, $w = 6$; (ii) $p_1^N = 1$, $p_2^N = 2$, $w = 6$.

Q15. Compare value functions in Q13(i) and Q13(ii) for: (a) $p_1 = p_2 = 1$, $w = 4$; (b) $p_1 = 1$, $p_2 = 2$, $w = 4$.

Can we interpret the difference as a measure of the change of consumer well-being? The answer is “NET”, because utility representation is not unique!

How shall we measure the change in consumer well-being as a result of a price change?

3.3 "Compensated" (Hicksian) demand

$$\begin{cases} \min_{x \in X} px \\ s.t. \ u(x) \geq u \end{cases}$$ (5)

Definition 17 Solution $x^h(p, w)$ to problem (5) is Hicksian demand.
Figure 5: Les effets de la substitution et de la revenue.

Characteristics of $x^h(p,u)$:

- if $\succeq$ is LNS and $u(\cdot)$ is a continuous utility function representing preference relation $\succeq \Rightarrow \forall p > 0$

1. $x^h(\alpha p, u) = x^h(p, u) \forall \alpha > 0$;
2. $u(x^h(p, u)) = u$;
3. if $\succeq$ is strictly convex, $x^h(p, u)$ is unique.

**The first-order conditions** that describe the “interior” solution to problem (5):

$$MRS(x^h) = \frac{\partial u/\partial x_1}{\partial u/\partial x_2} = \frac{p_1}{p_2};$$

$$u(x^h_1, x^h_2) = u.$$ (6)

Example: $u(x_1, x_2) = \beta \ln(x_1) + (1 - \beta) \ln(x_2)$

$$x^h_1(p, u) = u \left( \frac{\beta p_2}{(1 - \beta)p_1} \right)^{1-\beta}; \quad x^h_2(p, u) = u \left( \frac{(1 - \beta)p_1}{\beta p_2} \right)^\beta.$$ (7)

**Expenditure function** $e(p, u) = px^h(p, u)$

Features of $e(p, u)$:

- if $\succeq$ is LNS and $u(\cdot)$ is a continuous utility function representing preference relation $\succeq \Rightarrow$

1. $e(\alpha p, u) = \alpha e(p, u) \forall \alpha > 0$;
2. $\frac{de(p, u)}{\partial a} > 0, \frac{de(p, u)}{\partial p_l} \leq 0$ where $l = 1, 2$;
(iii) $e(p, u)$ is concave in $p$; 
(iv) $e(p, u)$ is continuous in $p$ and $u$.

$$x^h_1(p, u) = \frac{\partial e(p, u)}{\partial p_1}; x^h_2(p, u) = \frac{\partial e(p, u)}{\partial p_2}.$$ 

Example: $u(x_1, x_2) = \beta \ln(x_1) + (1 - \beta) \ln(x_2)$
$$e(p, u) = up_1^\beta p_2^{1-\beta} \beta^{-\beta} (1 - \beta)^{\beta-1}.$$ 

**Duality**

If $\succsim$ is LNS and $u(\cdot)$ is a continuous utility function representing preference relation $\succsim$, $p \gg 0$
1. $x^h(p, v(p, w)) = x^*(p, w)$ and $e(p, u) = w$
2. $x^*(p, px^h(p, u)) = x^h(p, u)$ and $v(p, px^h(p, u)) = u$

For $u(x_1, x_2) = \beta \ln(x_1) + (1 - \beta) \ln(x_2)$ compare $x^h(p, v(p, w))$ and $x^*(p, w); x^*(p, px^h(p, u))$ and $x^h(p, u)$.

**Slutzky equation:** if $\succsim$ is LNS and $u(\cdot)$ is a continuous utility function representing preference relation $\succsim \Rightarrow$

$$\frac{\partial h_l(p, u)}{\partial p_k} = \frac{\partial x_l(p, w)}{\partial p_k} + \frac{\partial x_l(p, w)}{\partial w} x_k(p, w)$$

where $l, k \in \{1, 2\}$, $u = v(p, w)$

**Consumer surplus**
Suppose that price for good $x_1$ changes from $p_1^V$ to $p_1^N$; $p^V = (p_1^V, p_2)$, $p^N = (p_1^N, p_2)$.

$$MMU = e(p^0, v(p^V, w)) - e(p^0, v(p^N, w))$$

(8)

$$CV = e(p^N, v(p^V, w)) - e(p^N, v(p^N, w)) = e(p^N, v(p^V, w)) - w =$$

$$= e(p^N, v(p^V, w)) - e(p^V, v(p^V, w)) = \int_{p_1^V}^{p_1^N} x_1^h(p, v(p^V, w)) dp.$$

$$EV = e(p^V, v(p^V, w)) - e(p^V, v(p^N, w)) = w - e(p^V, v(p^N, w)) =$$

$$= e(p^N, v(p^N, w)) - e(p^V, v(p^N, w)) = \int_{p_1^V}^{p_1^N} x_1^h(p, v(p^N, w)) dp.$$

**Definition 18** If preferences are quasilinear, $EV = CV$.

By definition, this value is Consumer Marshallian Surplus.

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$^3$Voir TP1.
Figure 8: Equivalent variation: "I do not mind being hurt if you pay me EV dollars".

4 Production

Let us divide goods in two categories: consumption goods whose consumption increases consumer utility; and inputs of production that are used to produce consumption goods. Production takes place in firms. A firm has a technology that allows to produce consumption good $q$ as an output from some composition of inputs $z = (z_1, z_2, ..., z_N)$. We will consider two ways to describe a technology.

Production function The first way to describe a technology is to describe how much of an output can be produced from a given composition of inputs. Figure 9 depicts technologies that use input good $z$ to produce consumption good $q$. Shaded areas are called production sets. The frontier of shaded areas is a production function - it describes maximal quantity of consumption good $q$ that can be produces out of a given quantity of input $z$.

Returns to scale

Consider constant $\alpha > 1$. Returns to scale are: (a) decreasing $\Leftrightarrow f(\alpha z) < \alpha f(z)$, (b) increasing $\Leftrightarrow f(\alpha z) > \alpha f(z)$, (c) constant $\Leftrightarrow f(\alpha z) = \alpha f(z)$.

On Figure 9(a) returns to scale are decreasing (as output increases, production becomes more and more difficult): $f(1) - f(0) > f(3) - f(1)$. On Figure 9(b) returns to scale are increasing (as output increases,

\footnote{Recall that $z$ can be a vector $(z_1, z_2, ..., z_N)$. Then, $\alpha z = (\alpha z_1, \alpha z_2, ..., \alpha z_N)$.}
production becomes more and more easy): \( f(1) - f(0) < f(3) - f(1) \). On Figure 9(c) returns to scale are constant (as output increases, production remains equally difficult): \( f(1) - f(0) = f(3) - f(1) \).

**Cost function**  The second way to describe a firm’s technology is to describe the minimal cost that is required to produce a given quantity of output. Suppose that output \( q \) is produced out of two inputs: \( z_1 \) and \( z_2 \). Let \( p_{z_1} \) be price of input \( z_1 \), \( p_{z_2} \) be price of input \( z_2 \). Then, the optimal input mix \( z^*_1(p_{z_1}, p_{z_2}, q), z^*_2(p_{z_1}, p_{z_2}, q) \) solves

\[
\begin{aligned}
\min_{z_1, z_2} & \quad p_{z_1} z_1 + p_{z_2} z_2 \\
\text{s.t.} & \quad f(z_1, z_2) \leq q
\end{aligned}
\]

The cost function is equal to \( c(p_{z_1}, p_{z_2}, q) = p_{z_1} z^*_1(p_{z_1}, p_{z_2}) + p_{z_2} z^*_2(p_{z_1}, p_{z_2}, q) \).

Figure 10 depicts cost functions for technologies with different returns to scale for some given prices of inputs. Compare Figures 9 and 10. On Figure 10(a) returns to scale are decreasing (as output increases, an additional unit of production becomes more and more costly): \( c(3) - c(1) > c(1) - c(0) \). On Figure 10(b) returns to scale are increasing (as output increases, an additional unit of production becomes less and less costly): \( c(3) - c(1) < c(1) - c(0) \). On Figure 9(c) returns to scale

---

5 Think: how much at least would it cost you to bake a cake, given that you will choose to cook it in the cheapest way.
are constant (as output increases, production remains equally difficult): $c(3) - c(1) = c(1) - c(0)$.

**The efficient scale** Figure 11 depicts the efficient scale of production with nonsunk setup cost.\(^6\) Average cost of production $AC(q) = \frac{c(q)}{q}$ is decreasing in region $q < \bar{q}$, and increasing afterwards: it is minimized at $\bar{q}$. Level $\bar{q}$ is called the *efficient scale*.

\(^6\)Nonsunk setup cost is the cost that a firm pays whenever its output is positive, regardless of output level (premises). Sunk setup cost is the cost that a firm pays regardless of whether its output is positive or null (registration).

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Figure 10: Cost function: (a) decreasing returns to scale; (b) increasing returns to scale; (c) constant returns to scale; (d) constant returns to scale with sunk setup costs.

Figure 11: efficient scale with nonsunk setup cost
**Firm’s objectives**  We assume that a firm maximizes its profits taking all prices as given. In class, we have discussed limitations of this assumption: potentially controversial objectives by different owners, and potential conflict of interests between the owners and the managers to whom the owners need to delegate decision-making.

**Firm’s supply**  A firm’s whose technology is described by production function \( q = f(z) \), chooses input mix \( z^* = (z_1^*, z_2^*, ..., z_N^*) \) that solves problem:

\[
\max_{z} pf(z) - p_z z
\]

where \( p \) is the output’s price, and \( p_z = (p_{z_1}, p_{z_2}, ..., p_{z_N}) \) is a vector of input prices. The firm *supply* is equal to \( q^* = f(z^*) \), and its *profits* are equal to \( \pi(z^*) = pf(z^*) - p_z z^* \). Figure 12 illustrates profit-maximizing input mix and production for strictly concave production function (decreasing returns to scale). Note, that \(^7\)

\[
p \frac{\partial f(z^*)}{\partial z_i} \leq p_{z_i} \text{ with equality if } z_i^* > 0, \ i = 1...N.
\]

A firm’s whose technology is described by cost function \( c(q) \), chooses to produce output \( q^* \) that solves problem:

\[
\max_{q} pq - c(q)
\]

For strictly convex cost function (decreasing returns to scale)

\[
p \leq c'(q^*) \text{ with equality if } q^* > 0.
\]

\(^7\)Be careful not to use the first order approach for increasing or constant returns to scale.
That is, a firm production, if it takes place, equals marginal cost \( c'(q^*) \), that is, the cost of producing “the last” additional unit with price that is charged for this unit, as illustrated on figure 13. Suppose several firms produce the same output. For given prices, profit-maximizing firm with less efficient technology, chooses to produce less.

5 Partial Equilibrium

Let us study market for one good in isolation. For illustrative purposes, consider an economy with two consumers; two firms and two goods: numeriare good \( m \) and consumption good \( x \).

Preferences by consumer \( i = 1, 2 \) are described by quasilinear utility function

\[
u_i(m_i, x_i) = m_i + \varphi(x_i),\]

where \( x_i \) and \( m_i \) denote consumption levels. Let us assume that \( \varphi(x_i) \) is a concave function (recall our discussion of convexity of consumer preferences in section 1).

Production technology by firm \( j = 1, 2 \) is described by cost function \( c_j(q_j) \): firm \( j \) inquires cost \( c_j(q_j) \) in order to produce \( q_j \) units of output. Let us assume that \( c_j(q_j) \) is a convex function (recall our discussion of returns to scale).

Consumer \( i \) has a right to keep share \( \theta_{ij} \) of profits in firm \( j \). Initially, consumer \( i \) has \( m_i \) units of good \( m \): no good \( x \) is available before production takes place.

Equilibrium allocation Let us normalize the price of numeriare good to be 1 (recall our discussion of “no money illusion”). Allocation \( m_1^*, m_2^*, x_1^*, x_2^*, q_1^*, q_2^* \) and price \( p^* \) of good \( x \) constitute an equilibrium if and only if
1. \( q_j^* = q_j^*(p^*) \) maximizes profits by firm \( j \) when price of the output is equal to \( p^* \), that is, it solves

\[
\max_{q_j} p^* q_j - c_j(q_j)
\]  

(Hence, profits by firm \( j \) is \( \pi_j^* = p^* q_j^* - c_j(q_j^*) \)).

2. \( x_i^* = x_i^*(p^*) \) solves optimization problem by consumer \( i \) when price of consumption good is equal to \( p^* \):  

\[
\max_{x_i} x_i ; m_i \ s.t. \ p^* x_i + m_i \leq \bar{m}_i + \theta_{i1} \pi_1^* + \theta_{i2} \pi_2^*
\]  

3. Price of good \( x \) balances the market, that is, aggregate supply of good \( x \) is equal to aggregate demand for good \( x \):

\[ q_1^*(p^*) + q_2^*(p^*) = x_1^*(p^*) + x_2^*(p^*) \]

By first-order approach, equilibrium allocation is characterized by:\(^8\)

\[ \varphi_i'(x_i^*) \leq p^* \text{ with equality if } x_i^* > 0 \]  
\[ p^* \leq c_j'(q_j^*) \text{ with equality if } q_j^* > 0 \]  
\[ q_1^* + q_2^* = x_1^* + x_2^* \]  

**Pareto optimal allocation**  
Allocation \( m_1^*, m_2^*, x_1^*, x_2^*, q_1^*, q_2^* \) is *Pareto optimal* if and only if it is impossible to find some other allocation \( m_1, m_2, x_1, x_2, q_1, q_2 \) that increases utility by one consumer without decreasing utility by the other consumer.\(^9\)

Suppose that perfectly informed benevolent social planner picks output in each firm and allocates total output between the consumers so as to maximize joint “happiness” that is measured by sum of consumer utilities. She “solves”:

\[
\max_{x_i, q_1, q_2} m_1 + \varphi_1(x_1) + m_2 + \varphi_2(x_2) \\
\text{s.t. } x_1 + x_2 \leq q_1 + q_2 \\
m_1 + m_2 + c_1(q_1) + c_2(q_2) \leq \bar{m}_1 + \bar{m}_2
\]

or, equivalently

\[
\max_{x_i, q_1, q_2} \varphi_1(x_1) + \varphi_2(x_2) - c_1(q_1) - c_2(q_2) \\
\text{s.t. } x_1 + x_2 \leq q_1 + q_2
\]

\(^8\)Recall equations (6) and (12).

\(^9\)There is only one other consumer in our economy. If there are many consumers, an allocation is *Pareto optimal* if and only if it is impossible to find some other allocation that increases utility by one consumer without decreasing utility by some other consumer.
Lagrangian for this optimization problem is equal to
\[ L = \varphi_1(x_1) + \varphi_2(x_2) - c_1(q_1) - c_2(q_2) + \mu (q_1 + q_2 - x_1 - x_2), \]
where \( \mu \) is Lagrangian multiplier associated with technological constraint. Hence, the planner picks \( x^o_1, x^o_2, q^o_1, q^o_2 \) such that
\[ \mu \leq c'_j(q^o_j) \text{ with equality if } q^o_j > 0 \tag{19} \]
\[ \varphi'_i(x^0_i) \leq \mu \text{ with equality if } x^o_i > 0 \tag{20} \]
\[ \mu (q^o_1 + q^o_2 - x^o_1 - x^o_2) = 0 \tag{21} \]
and she allocates the remaining numeraire good in any way between the consumers.

Notice, that if we take \( \mu = p \), the systems of equations (29)-(31) and (15)-(17) are equivalent. Therefore,

**The First Fundamental Welfare Theorem**: any competitive equilibrium is Pareto optimal.\(^{10}\)

A producer would increase profit by expanding production of the good if its price exceeded his marginal cost. Conversely, if he produced the good at all, he would contract production if the marginal cost were to exceed the price. This trivial result has important implications. When deciding whether to consume one more unit of the good, a consumer faces a price that is socially “the right one” and internalizes the cost of producing this extra unit (Tirole (1998), “The Theory of Industrial Organization,” *The MIT Press*).

**Marshallian surplus** Marshallian surplus is a concept of quasi-linear model. It measures social welfare. It is equal to the surface that lies below the inverse demand curve less the surface that lies below the supply curve (see figure 14).\(^{11}\) Its share above line \( p = p^\ast \) is consumer surplus, the rest is producer surplus.

Figure 15 illustrates that Marshallian surplus is maximized in equilibrium.

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\(^{10}\)Recall, that we consider an “ideal” economy.

\(^{11}\)Recall our discussion in class: an addition unit of production increases of consumer surplus (see part 1) and imposes cost on producers.
Figure 14: marginal change in Marshallian surplus.

Figure 15: maximim of Marshallian surplus.
Deadweight loss from commodity taxation  Suppose that a firm is taxed at rate \( t \) for each unit of good that it sells. Then, supply of good \( x \) is characterized by equations

\[
p^*(t) + t \leq c'_j(q^*_j) \quad \text{with equality if } q^*_j > 0,
\]

where \( p^*(t) \) denotes new equilibrium price. Aggregate supply curve “shifts up”, as depicted on figure 16. As a result, equilibrium output and consumption decrease. Taxation creates a deadweight loss: tax revenues + consumer surplus + producer surplus in equilibrium with taxes lies below Marshallian surplus without taxes.

Note, that new equilibrium price increases as compared to that without taxation: both producers and consumers share the burden of deadweight loss. How the share that is beard by consumers depend on elasticity of demand? What happens if consumers, and not producers pay a tax?

Number of firms in the market  Consider market for good \( x \). Demand is given by \( x(p) \). There is an infinite number of firms. Each firm has an access to production technology that is characterized by cost function \( c(q) \): \( c(0) = 0 \), and it can enter or exit the market. A triple \((p^*, q^*, N^*)\) is a long-run competitive equilibrium if

1. a firm’s optimization: \( q^* \) solves

\[
\max_q p^* q - c(q)
\]

2. \( x^*(p^*) = N^* q^* \) (balance on the market);
3. \( p^* q^* - c(q^*) = 0 \) (no profits).

As we have discussed in class: when returns to scale are constant, \( q^* \) and \( N^* \) cannot be determined (quantity of good \( x \) that is demanded can be generated by any number of firms with any load); when returns to scale are decreasing, there is no long-run competitive equilibrium (profit-maximization implies that profits are positive, then, however, more firms want to enter the market). Indeed, in any equilibrium with determinantal number of firms cost function must exhibit a strictly positive efficient scale (see part 2).

6 A primer in general equilibrium model: Robinson Crusoe economy

Partial equilibrium approach considers one market in isolation. Potentially, shocks on this market generate effects on other markets. Partial equilibrium approach ignores these effects. A more complicated, general equilibrium model takes these effects into the account. In class, we have considered a simple illustration of this model.

**Robinson Crusoe economy** Consider an economy with one consumer (Robinson Crusoe) who owns a single firm. Robinson benefits from two goods: consumption good \( x \) and leisure \( l \). His preferences are described by utility function \( u(l, x) \). Initially, Robinson has 24 hours available for leisure. However, he needs to work \( z \) hours in the firm in order to produce \( f(z) \) units of consumption good. That is, the firm’s technology is described by production function \( f(z) \), where \( z = 24 - l \).

**Equilibrium allocation** Let \( p \) be price of good \( x \), and \( w \) be the wage rate (price of time). \( x^*, z^* \) and \( p^*, w^* \) constitute an equilibrium, if and only if

1. \( z^* = z^*(p^*, w^*) \) maximizes profits by firm \( j \), when prices are equal to \( p^*, w^* \). That is, it solves\(^{12}\)

\[
\max_z p^* f(z) - w^* z
\]

(Hence, supply of good \( x \) is equal to \( q^* = f(z^*(p^*, w^*)) \); and profits by the firm is equal to \( \pi^* = p^* f(z^*(p^*, w^*)) - w^* z^* \).

2. \( x^*(p^*, w^*), l^*(p^*, w^*) \) solve consumer optimization problem:

\[
\max_{x, l \leq 24} u(l, x) \quad s.t. \quad p^* x \leq w^*(24 - l) + \pi^*
\]

\(^{12}\)Recall problem (10).
Figure 17: deficit of consumption good and excess supply of labour.

3. Prices balance the markets,\textsuperscript{13} that is, supply of good $x$ is equal to demand for good $x$:

$$f \left( z^* \left( p^*, w^* \right) \right) = x^* \left( p^*, w^* \right),$$

and labour supply is equal to labour demand

$$24 - l^* \left( p^*, w^* \right) = z^* \left( p^*, w^* \right).$$

Suppose that both utility function and production function are strictly concave, so that the first-order approach is valid. Then, (interior) equilibrium is characterized by:

$$f' \left( z^* \right) = \frac{p^*}{w^*}; \quad \frac{u_x'(l, x)}{u_l'(l, x)} = \frac{p^*}{w^*}; \quad 24 - l^* = z^*; \quad f \left( z^* \right) = x^*.$$

(26)

On figure 17 there is excess supply of labour and excess demand for good $x$.

Consequently price ratio $\frac{w}{p}$ decreases, so as to balance the markets: figure 18.

**Pareto optimal allocation** Suppose that Robinson gives up with trading with himself and simply decides how much to work and how much of good $x$ to consume. That is, he solves:

$$\max_{x,l \in 24} u(l, x) \quad \text{s.t.} \quad x = f(24 - l)$$

\textsuperscript{13}Indeed, if one market is balanced, the other is also balanced.
Because we have assumed that utility and production function are both strictly concave, there is the unique Pareto optimal allocation $x^o, l^o$:

\[ u'_i(l^o, f(24 - l^o)) = u'_x(l^o; 24 - l^o) f'(24 - l^o), \quad x^o = f(24 - l^o). \]

It is the same as equilibrium allocation (The First Fundamental Welfare Theorem): compare figures 19 and 18.

7 A scope for public intervention: an illustrative example

"An ideal family" Marie and Pierre live together. An individual's utility depends on consumption of food $x$, and on the amount of money $m$:

\[ U_M = M + \ln(1 + x_M), \quad (27) \]
\[ U_P = M + 2 \ln(1 + x_P). \quad (28) \]

Hence, (i) Mary and Pierre like money equally; (ii) Pierre likes food twice more than Mary:

There are two ways to get food out of money. The first way is to send Mary to the market. She is able to bring home $q$ units of food in exchange for $\frac{q^2}{2}$ dollars. If Peter goes shopping, he has to spend twice more money to bring home the same amount of food. That is, there are two technologies that allow to produce consumption good out of the numeriare, with cost functions:

\[ c_M(q) = \frac{q^2}{2}, \quad c_P(q) = q^2. \]
Figure 19: optimal allocation.

Figure 20: Tastes for food
Mary and Peter have 100 dollars in the pocket each. The optimal shopping/consumption plan \((x^*_M, x^*_P, q^*_M, q^*_P)\) solves\(^{14}\)

\[
\max_{x_M, x_P, q_M, q_P} \ln(1 + x_M) + 2 \ln(1 + x_P) - \frac{q^2_M}{2} - q^2_P
\]
\[
s.t. : \quad x_M + x_P = q_M + q_P
\]

Lagrangian for this optimization problem is equal to

\[
L = \ln(1 + x_M) + 2 \ln(1 + x_P) - \frac{q^2_M}{2} - q^2_P + \lambda (q_M + q_P - x_M - x_P),
\]

where \(\lambda\) is the Lagrangian multiplier associated with technological constraint.

\[
\lambda = c'_M(q^*_M) = q^*_M = c'_P(q^*_P) = 2q^*_P = \frac{1}{1 + x^*_M} = \frac{2}{1 + x^*_P}
\]

\[
\lambda (q^*_M + q_P - x^*_M - x^*_P) = 0
\]

Solving (29)-(31), we find:\(^{15}\)

\[
x^*_M = \frac{\sqrt{10} - 2}{3} \approx 0.39,
\]

\(^{14}\)We have to verify later that the solution satisfies endowment constraint \(q_i \leq 100, i = M, P.\)

\(^{15}\)Verify.
\[ x_P^* = 1 + 2x_M^* = \frac{2\sqrt{10} - 1}{3} \approx 1.77, \]  
\[ q_P^* = \frac{1}{1 + x_M^*} = \frac{3(1 + \sqrt{10})}{22} \approx 0.72, \]  
\[ q_M^* = 2q_P^* = \frac{3(1 + \sqrt{10})}{11} \approx 1.44. \]  

Hence, perfectly informed benevolent “family planner” (i) would ask Mary to buy more food than Peter: \( q_M^* > q_P^* \); and she (ii) would let Peter eat more than Mary: \( x_M^* < x_P^* \).

Suppose now that the planner knows that Peter’s preferences are described by equation (28). She also knows that (i) with probability \( \frac{1}{2} \) Mary has the same preferences as Peter, (ii) with probability \( \frac{1}{2} \) Mary’s preferences are described by equation (27). However, the planner does not know Mary’s preferences exactly. Because Mary’s utility is monotonically increasing in her consumption of food, she claims that she likes food as much as Peter (recall the failure of planned economies!)\(^\text{16}\). A way to discipline Mary is to make her pay for each additional unit of food that she demands.

**Perfect Market Equilibrium** Suppose that Peter and Mary trade food at home at price \( p \). They (i) know everything about each other preferences and production technologies, and (ii) behave as price-takers both when they sell food to each other, and when they buy food from each other.

**Supply**

For a given price \( p \), Mary’s supply of food on the home market solves

\[
\max_{q} pq - \frac{q^2}{2}
\]

Hence, it is equal to

\[ q_M(p) = p \]  

(36)

Mary’s supply of food on the home market solves

\[
\max_{q} pq - q^2
\]

\[ q_P(p) = \frac{p}{2} \]  

(37)

Therefore, at a given price \( p \), aggregate supply of food on the home market is equal to

\[ q_M(p) + q_P(p) = \frac{3p}{2}. \]  

(38)

\(^{16}\)In this case it is optimal to allocate \( \frac{\sqrt{77} - 1}{2} \) units of food to either family member.
Demand
For a given price $p$, Mary’s demand for food on the home market solves

$$\begin{align*}
\max_{x_M, q_M} & \ln(1 + x_M) + m_M \\
\text{s.t.} & \quad px_M + m_M \leq 1 + \left( pq_M - \frac{q_M^2}{2} \right)
\end{align*}$$

Hence, her demand for food $x_M(p)$ satisfies equation

$$\frac{1}{1 + x_M(p)} = p, \quad (39)$$

or, equivalently

$$x_M(p) = \frac{1}{p} - 1. \quad (40)$$

Peter’s demand for food on the home market solves

$$\begin{align*}
\max_{x_P, q_P} & 2 \ln(1 + x_P) + m_P \\
\text{s.t.} & \quad px_P + m_P \leq 1 + \left( pq_P - q_P^2 \right)
\end{align*}$$

Hence,

$$\frac{2}{1 + x_P(p)} = p, \quad (41)$$

which is equivalent to

$$x_P(p) = \frac{2}{p} - 1. \quad (42)$$

Consequently, at a given price $p$, aggregate demand for food on the home market is equal to

$$x_M(p) + x_P(p) = \frac{3}{p} - 2. \quad (43)$$

Balance on the Market
In equilibrium, price balances the market:\textsuperscript{17}

$$\frac{3}{p} - 2 = \frac{3p}{2}. \quad (44)$$

Notice, that if we pick $\lambda = p$, the systems of equations (29)-(31) and (36), (37), (39), and (41) are equivalent. Therefore, in equilibrium individual production and consumption are efficient: that is, they are given by equations (32)-(35): recall the First Fundamental Welfare Theorem.

\textsuperscript{17}Recall, that when all-but-one markets are balanced, all markets are balanced. Use equations (38), (43), and (44) to find equilibrium price and quantities. Draw aggregate demand and aggregate supply curves.
Missing markets  Suppose that Peter has a powerful stereo system, and he likes to listen to pop-music (denote by $h \leq 24$ his consumption of music per day):

$$U_P = m_P + 2\ln(1 + x_P) + \ln(1 + h).$$

Mary, instead, suffers when music is playing:

$$U_M = m_M + \ln(1 + x_M) - \frac{h}{2}.$$  

By listening to music Peter generates a negative externality on Mary.\(^{18}\) When Peter and Mary do not try to reach an agreement for how long shall the music play on, Peter listens to music all the time $h = 24$. This outcome is suboptimal. It would be efficient to take into the account Mary’s preferences, and let the music play so as to

$$\max_{h\leq 24} \ln(1 + h) - \frac{h}{2},$$

that is, for $h^* = 1$ hour a day.

Private bargaining over externality  A benevolent social planner could restore the efficiency (i) by requiring that the music cannot be played more than an hour (that is, by imposing a quota), or else (ii) by charging Peter $t = \frac{1}{2}$ dollar for each hour that he listens to the music (that is, by imposing a tax). However, these forms of intervention are not necessary.\(^{19}\)

Suppose that the government enforces Peter’s property rights to listen to the music as long as Peter wants. Suppose furthermore that the government allows Mary to make Peter a take-it-or-leave-it offer of a monetary transfer $T$ in exchange for playing a music for $h$ hours a day. If Peter agrees, this agreement is enforced by the government. If no agreement is reached, the outcome is as described in the previous section: Peter listens to music non-stop.

Peter agrees to Mary’s offer if and only if

$$\ln(1 + h) - T \geq \ln(25)$$  \hspace{1cm} (45)
Hence, the Mary’s best offer solves

$$\max_{T \leq 100, h \leq 24} T - \frac{h}{2} \quad \text{s.t. : (45)}$$

Indeed, Mary offers to Peter to pay him $T = \ln 25 - \ln 2$ if the music plays for one hour, and he agrees.

Assignment of property rights affects only the final distribution of wealth between Peter and Mary, and not the number of hours of music played. Indeed, suppose that the government guarantees Mary that she can lock the stereo system in a wardrobe for $h$ hours. Mary offers to Peter to pay him $T = \ln (25 - h) - \ln 2$ if the music plays for one hour, and he agrees: the higher $h$, more wealth is left to Mary.

Furthermore, when Mary and Peter can bargain, the resulting number of hours of music on does not depend on the form of bargaining. Suppose that it is not Mary, but Peter who can make a “take-it-or-leave-it” offer. Then, he proposes Mary to pay him $T = 11 - \frac{h}{2}$ if the music plays an hour, and Mary agrees.

The following three insights are general in bargaining games: a player’s payoff is higher, (i) the better his “outside option”; (ii) the higher his bargaining power; and (iii) the more patient he is (in dynamic games).

**Coase Theorem (1960):** When trade of the externality can occur, bargaining leads to an efficient outcome, no matter how property rights are allocated.

**Public goods** Peter and Mary rent an apartment together and none of them can limit the other’s access to the place. They can hire a cleaning lady who increases the level of order in the apartment by $y$ at effort costs $y^2$. Peter cares for order less than Mary:

$$U_P = m_P + 2 \ln(1 + x_P) + 3 \ln(1 + y_P + y_M),$$

$$U_M = m_M + \ln(1 + x_M) + \ln(1 + y_P + y_M).$$

Order in the apartment is a public good. Socially optimal level of order solves

$$\max_y 4 \ln(1 + y) - y^2$$

It is equal to

$$y^* = 1.$$

---

21Hence, the government should simply enforce private agreements.
In equilibrium, cleaning lady offers

\[ y = \frac{py}{2} \]

units of order, and only Mary is ready to pay for it - Peter benefits from the order that is paid by Mary: he would not like to pay so as to make their apartment even cleaner:

\[
\frac{1}{1 + y_P + y_M} \leq p_y \quad \text{with an equality when } y_P > 0
\]

\[
\frac{3}{1 + y_P + y_M} \leq p_y \quad \text{with an equality when } y_M > 0.
\]

As a result, \( y \) is suboptimal (see Fig 22): Verify that if Mary’s consumption of order is subsidized at rate \( s_M = \frac{1}{2} \), the efficiency is restored.\(^{22}\)

An alternative way to restore the efficiency is to impose a quota

\[ y \geq 1. \]

However, if the government is uncertain about Mary’s taste for order (unlike in our model), the task to find an optimal quota becomes non-trivial.

References


\(^{22}\)Note that the government plays a more efficient role than just law enforcement.