Asymmetric Smiles, Leverage Effects and Structural Parameters

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1. Introduction

In the empirical option pricing literature, departures from the Black-Scholes (1973) (BS) model are often characterized by an implied volatility curve, whereby the volatility extracted from the BS option pricing formula given the observed option price is graphed against the moneyness of the option. The empirical biases of the BS model have been dubbed the smile effect in reference to a symmetric implied volatility curve, but numerous distorted smiles in the shape of smirks or frowns are inferred more frequently from market data.

Since Renault (1997) and Renault and Touzi (1996), the symmetric shape of the smile is well characterized theoretically in terms of parameter heterogeneity. When the constant volatility in the BS world is made stochastic, the price of the option is expressed as an expectation of the BS price where the expectation is taken with respect to the distribution of the heterogeneous stochastic volatility factor. Such a stochastic volatility model as in Hull and White (1987) produces a symmetric implied volatility curve across moneyness[1]. While frequently observed, asymmetric shapes of the smile are only partially characterized theoretically. Platen and Schweizer (1997) explain this asymmetry by developing a model in which the diffusion process of the stock price incorporates the technical demand induced by hedging strategies. Renault (1997) has shown that a very small discrepancy between the option markets assessment of the stock price and the actual price can capture sensible asymmetries of the smile.

In this paper, we characterize the asymmetries of the smile in a stochastic dynamic asset pricing framework. We put forward how these asymmetries are produced by leverage effects. We know since Black (1975, 1976) that a stock that drops sharply in price is likely to show a higher volatility in the future than a stock that rises sharply in price. In our framework this dependence between price movements and future volatility is introduced through a set of latent state variables. These latent variables capture the volatility risk and the interest rate risk which affect option prices. The leverage effect just described is produced by a cross-correlation effect between the state variables which enter into the stochastic volatility process of the stock price and the stock price process itself. We provide a framework with a general stochastic discount factor where a general condition for asymmetric implied volatility curves is the presence of a stochastic correlation between the return on the stock and the stochastic discount factor.

To be able to put forward the asymmetric deformations of the smile, we first state necessary and sufficient conditions for symmetry in a fully general setting where the only maintained restriction is the homogeneity of the option pricing function (see Merton (1973)).This property appears as natural when the BS implied volatility is seen as a function of moneyness and not of the underlying asset price and the strike price taken separately. We provide new characterizations of the symmetry of the volatility smile in terms of the option pricing

[1] The metric for moneyness is the logarithm of the forward price over the strike price.

function and of the pricing probability measure. Since a Hull and White (1987) stochastic volatility model produces a symmetric smile, we propose a generalized option pricing formula based on a general stochastic discount factor containing state variables. Asymmetry is then characterized as the consequence of a correlation effect between the moneyness of the option and the contemporaneous state variables.

To further characterize these asymmetries and provide an economic interpretation to the cross-correlation effects, we propose an equilibrium stochastic discount factor. The general reluctance to use equilibrium models to price options is based on the unobservable character of both the endowment processes and the preference parameters. Our latent variable framework makes it possible to parametrize parsimoniously the dynamic evolution of the consumption and dividend processes. Since the process of state variables is a latent Markov process, a natural candidate is the Markov switching model introduced by Hamilton (1989) and applied to asset pricing by Cecchetti, Lam and Mark (1990, 1993) and Bonomo and Garcia (1993, 1994, 1996)[2].

The calibration of preference parameters is unavoidable in the presence of volatility and interest rate risks since risk premia parameters enter in the option price. Such parameters are absent from so-called preference-free option pricing formulas such as the Black and Scholes formula because they are hidden in the premia associated with the stock and bond prices. When a volatility risk exists, the no-arbitrage methodology cannot hide the risk premium parameter in the option price since there is no asset which trades directly the volatility risk. While the risk premium can be calibrated using observed option prices, it will not be stable over different maturities because it entails a time-varying risk. This is why this risk premium is often characterized in the option pricing literature (see the stochastic volatility model of Heston (1993) for example) by its counterpart in an equilibrium asset pricing model even though assumptions underlying the equilibrium model are left unspecified. Irrespective of their economic interpretation, the preference parameters and the parameters of the stochastic processes which govern the latent state variables can be viewed as structural parameters which will help predict prices of future options of any maturity in a more stable manner. To enrich the set of parameters with which we calibrate the option price, we set our equilibrium model in a recursive utility framework with state non-separable preferences (Epstein and Zin [1989]). In this framework, the respective roles of discounting, risk aversion and intertemporal substitution in the option pricing formula can be disentangled [3].

[2] The regime-switching model has recently enjoyed some popularity in the option pricing literature. See in particular Campbell and Li (1998), David and Veronesi (1998), Duan, Popova and Richken (1998). All these models can be embedded in our framework. A precursor paper in regime-switching option pricing is Naik (1993).

[3] Two papers have used preferences that disentangle risk aversion from intertemporal substitution in the context of option pricing. Detemple (1990) uses the ordinal certainty equivalence hypothesis proposed by Selden (1978) in a two-period economy and shows that time preferences play a distinctive and significant role in pricing options. For example, option prices change with the expected return on the stock and may decrease

When we calibrate this model to reasonable values of the parameters, we are able to reproduce the various types of implied volatility curves inferred from option market data. In other words, not only our model reproduces asymmetric smiles but it allows for reversals in the smile skewness. All the classic extensions of the basic Black-Scholes model with stochastic volatility, stochastic interest rates or jumps reviewed in Bakshi, Cao and Chen (1997) cannot produce such reversals. Indeed, as Bates (1996) emphasized, it is such changing skewness in the smile that poses a challenge to current option pricing models. In our setting, reversals in skewness occur when the source of risk changes.

A class of generalized deterministic volatility models has also been proposed to overcome the empirical biases of the BS model. These models include the constant elasticity of variance model of Cox and Ross (1976),the implied binomial tree approach of Rubinstein (1994), the deterministic volatility models of Dupire (1994) and Derman and Kani (1994), and the Kernel approach of Aït-Sahalia and Lo (1998).In this class of models, it is assumed that the local volatility of the underlying asset is a known function of time and of the path and level of the underlying asset price. While these models are attractive because they can capture empirical regularities in a no-arbitrage setting without resorting to equilibrium models and the need to estimate risk premia, they are prone to the risk of overfitting. Indeed, Dumas, Fleming and Whaley (1998) show clearly that deterministic volatility models overfit the smile in sample and loose any predictive power out of sample. Buraschi and Jackwerth (1997) bring further evidence that deterministic volatility models are not consistent with observed option prices and that stochastic volatility models are more likely to explain the smile. Our model allows also for stochastic interest rates. While this feature does not appear crucial for characterizing the short-term biases of the BS model, it is important for long-term contracts (see Bakshi, Cao and Chen (1998)).

Rubinstein (1976) and Brennan (1979) use a consumption-based representative agent framework to price options. Amin and Ng (1993) extend this framework to a joint process for consumption growth and stock returns which captures both interest rate and volatility risks. As special cases of our general option pricing formula, we obtain the formula derived by Amin and Ng (1993) and a fortiori all the other pricing formulas that were nested in the latter[4]: of course the BS formula, but also the Hull and White (1987) and Bailey and Stulz (1989) stochastic volatility option pricing formulas and the Merton (1973), Turnbull and Milne (1991), and Amin and Jarrow (1992) stochastic interest rate option pricing formulas for equity options.

The rest of the paper is organized as follows. Section 2 provides general conditions under

when the risk of the stock return increases. Ma (1993) extends the stochastic differential utility concept in Duffie and Epstein (1992) to a mixed Poisson-Brownian information structure and derives a closed-form pricing formula for European call options written on aggregate equity under Kreps-Porteus preferences.

[4]However, we do not incorporate as in Amin and Ng (1993) the effect on the option price of a systematic jump in the underlying asset price process, following Merton (1976) and Naik and Lee (1990). This extension could be easily accomodated in our framework.

which the smile is symmetric and provides a stochastic framework with state variables to price assets. Asymmetry of the smile is associated with the presence of cross-correlation or leverage effects. Section 3 specializes the stochastic pricing framework to an equilibrium model with a Markov chain structure for the state variables. Conditions for an asymmetric smile are stated in this specialized context. Section 4 calibrates the parameters to illustrate the various shapes of the smile that can be obtained under various stochastic and parameter assumptions. Section 5 concludes.

2. The symmetric and asymmetric shapes of the volatility smile: a general characterization

The BS formula plays a central role in option pricing since implied volatility, the volatility parameter derived from the BS formula and the observed option price, serves as a useful unit of measure in option markets. As yields in the bond market capture the underlying term structure of interest rates, the implied volatilities help to build a structure of volatilities with respect to the moneyness or the time to maturity of the options. Various shapes of the volatility structure are observed in practice. The search for good option pricing models is therefore guided by their ability to reproduce these various shapes. The straight line that would be observed if the Black-Scholes model were true is rarely if ever observed. Option pricing models have therefore been built to make either the volatility of the underlying asset or the interest rate, or both, stochastic. When the volatility is stochastic as in the Hull and White (1987) model, the shape of the volatility structure with respect to the moneyness of the option is symmetric. Moneyness x_t is defined as the logarithm of the ratio of the forward price over the strike price, $x_t = Log\frac{S_t}{KB(t,T)}$ (with S_t the price of the underlying asset, K the strike price and $B(t,T)$ the price of a pure discount bond maturing at T).

The interest of practitioners in the moneyness of an option and in implied volatilities suggest that the BS implied volatility should be a function of x_t only, and not of S_t and K separately. In other words, the implied volatility function should be homogeneous of degree zero with respect to the pair (S_t, K). But since Merton (1973) it is well-known that the BS option pricing formula is homogeneous of degree one with respect to this same pair. Now suppose that a new pricing formula is proposed in place of the BS formula and that BS implied volatilities are computed from option prices generated by this formula. Then, the homogeneity of degree zero property is maintained for these implied volatilities if and only if the option pricing formula keeps the homogeneity of degree one property. For this reason, we will provide new characterizations of the symmetry of the volatility smile in terms of the option pricing function and of the pricing probability measure in the context of a homogeneous option pricing formula.

2.1. The symmetry of the volatility smile

The theory for pricing contingent claims in the absence of arbitrage introduces a pricing probability measure $\mathrm{Q}_{t,T}$ under which the price $\boldsymbol{\pi}_t$ at time t of any contingent claim maturing at time T is the discounted expectation of its terminal payoff. In the case of a European call option with strike price K, it is given by:

$$\boldsymbol{\pi}_t = B(t,T)E_t^*(S_T - K)^+, \tag{2.1}$$

where E_t^* denotes the expectation operator with respect to $\mathrm{Q}_{t,T}$. Of course, $\mathrm{Q}_{t,T}$ is generally different from the data generating process of $\{S_t\}$. Existence and unicity of $\mathrm{Q}_{t,T}$ were studied by several authors since the seminal paper of Harrison and Kreps (1979)[5].

In this subsection, we will compare a general but homogeneous option pricing formula $\boldsymbol{\pi}_t(S_t, K)$ as defined in (2.1) with the BS option pricing formula defined itself by a homogeneous function $\mathbf{BS}(.,.,\sigma)$, for a given volatility parameter σ, with:

$$\begin{cases} \mathbf{BS}(S_t, K, \sigma) = S_t\phi(d_1) - KB(t,T)\phi(d_2), \\ d_1 = \dfrac{1}{\sigma\sqrt{T-t}}\left[Log x_t + \dfrac{1}{2}\sigma^2(T-t)\right], \\ d_2 = d_1 - \sigma\sqrt{T-t}. \end{cases} \tag{2.2}$$

In other words, the BS implied volatility is a function $\sigma_t^*(x_t)$ of the moneyness x_t only, and not of S_t and K separately. Starting with the defining formula:

$$\boldsymbol{\pi}_t(S_t, K) = \mathbf{BS}(S_t, K, \sigma_t^*(x_t)), \tag{2.3}$$

a direct application of the homogeneity of degree one of $\boldsymbol{\pi}_t(.,.)$ and $\mathbf{BS}(.,.,\sigma)$ with respect to the pair (S_t, K) allows one to divide each side of (2.3) by S_t and conclude that $\sigma_t^*(x_t)$ is well-defined as a function of K/S_t or (equivalently) of x_t by :

$$\pi_t(x_t) = BS(x_t, \sigma_t^*(x_t)) \tag{2.4}$$

with the following slight change of notations :

$$\begin{cases} \pi_t(x_t) & = \boldsymbol{\pi}_t\left(1, \frac{K}{S_t}\right), \\ BS(x_t, \sigma) & = \mathbf{BS}\left(1, \frac{K}{S_t}, \sigma\right). \end{cases} \tag{2.5}$$

[5] The theory of complete markets is beyond the scope of this paper where we are only interested in the existence of a pricing probability measure $Q_{t,T}$ which is well-defined and given to us, whether it is unique or not.

The property we just emphasized is in fact the homogeneity of degree zero of the BS implied volatility with respect to the pair (S_t, K). The subscripts t in the functions $\pi_t(.,.)$ and $\sigma_t^*(.)$ indicate that they may depend upon other variables in the information set I_t, the value of which is fixed at time t.

Various consequences of this setting both in terms of option pricing and option hedging are detailed in Renault and Touzi (1996), Renault (1997) and Garcia and Renault (1998). In particular, Renault and Touzi (1996) and Renault (1997) have investigated the slope of the BS implied volatility $\sigma_t^*(x)$ as a function of its distance x to the money[6]. In particular, two strike prices K_1 and K_2 are said symmetric with respect to the money if the corresponding x_1 and x_2 are symmetric with respect to zero, since in this case K_1 and K_2 are on each side of the forward price but their geometric average coincides with the forward price. Therefore, the relevant symmetry property of the volatility smile is the following:

$$\sigma_t^*(x) = \sigma_t^*(-x) \quad \text{for any} \quad x. \tag{2.6}$$

We will characterize the variations of $\pi(x)$, $BS(x, \sigma), \sigma_t^*(x)$ as functions of x for a given value of S_t. In other words, the genuine variable of interest is the strike price K, while the switch to the variable x is only a matter of rescaling for convenience.

In Proposition 2.1 below, we extend a result first stated in Renault and Touzi (1996), which characterizes the symmetry of the smile in terms of the option pricing function[7].

Proposition 2.1. *If option prices are conformable to a homogeneous option pricing formula $x \to \pi(x)$, the volatility smile is symmetric ($\sigma^*(x) = \sigma^*(-x)$ for any x) if and only if, for any x:*

$$\pi(-x) = e^x \pi(x) + 1 - e^x$$

Proof: See Appendix 1.

This characterization of the symmetry of the smile admits an equivalent formulation in terms of the pricing probability measure. While the pricing probability measure is usually characterized through the cumulative distribution function of $\frac{S_T}{S_t}$, it is convenient here to characterize it through either the cumulative distribution function $F_{V_T}(.)$ or the probability density function $f_{V_T}(.)$ of $V_T = Log\frac{S_T B(t,T)}{S_t}$. We are then able to prove (see Appendix 1) the following proposition:

Proposition 2.2. *If the cumulative distribution function $F_{V_T}(.)$ of V_T under a pricing probability measure is absolutely continuous (associated with a density function $f_{V_T}(.) = F'_{V_T}(.)$)*

[6] Moneyness x is equal to 0 at the money, that is when the strike price coincides with the forward price $\frac{S_t}{B(t,T)}$.

[7] For sake of notational simplicity, the subscripts t have been dropped.

and such that $\exp(V_T)$ *is integrable, the volatility smile is symmetric if and only if one of the following three equivalent properties is fulfilled:*

(i) For any x:

$$\pi(x) = F_{V_T}(x) - e^{-x}[1 - F_{V_T}(-x)]$$

(ii) For any x:

$$F_{V_T}(x) = E_t^*[e^{V_T}\mathbf{1}_{[V_T \geq -x]}]$$

(iii) There exists an even function g(.) such that for any x:

$$f_{V_T}(x) = e^{-x/2}g(x)$$

These characterizations offer to practitioners various ways to extend the BS formula, while keeping both a homogeneous option pricing function and a symmetric smile. Characterization (i) provides a theoretical support to descriptive approaches which replace the standard normal cumulative distribution function of the BS formula by alternative distribution functions, possibly asymmetric (see Garcia and Gencay (1998)). It shows that in order to keep a symmetric smile, the term $1 - F_{V_T}(-x)$ should replace $F_{V_T}(x)$ in the second part of the option pricing formula. Characterization (ii) should be interpreted in terms of hedging. Indeed, Garcia and Renault (1998) have shown that $E_t^*[e^{V_T}\mathbf{1}_{[V_T \geq -x]}]$ is precisely the hedging ratio, in other words the derivative of the option pricing function with respect to the stock price (the so-called delta of the option)[8]. Finally, for characterization (iii), it should be noticed that if the pricing probability measure is characterized by a conditional log-normal distribution of future returns given available information at time t :

$$V_T = Log\frac{S_T B(t,T)}{S_t} \mid I_t \rightsquigarrow_{(Q_{t,T})} \mathcal{N}(\mu_t, \sigma_t^2),$$

the condition of Proposition 2.2 means that :

$$\mu_t = -\frac{\sigma_t^2}{2},$$

which is automatically fulfilled in the absence of arbitrage since, by application of (2.1) with $K = 0$, we have:

$$S_t = B(t,T)E_t^* S_T.$$

In the next subsection, we provide sufficient conditions on the pricing probability measure to ensure the homogeneity of the option pricing function in a convenient stochastic framework with state variables.

[8] Their proposition 2.1 shows that this characterization of the hedging ratio is a necessary and sufficient condition for homogeneous option pricing. Since hedging is not the primary focus of this paper, we leave to the reader the interpretation of this fairly natural relationship between $F_{V_T}(x)$ and the delta coefficient.

2.2. A general state variable framework for homogeneous option pricing

Merton (1973) stressed that the desirable homogeneity of option prices will be maintained as soon as asset returns are serially independent. This condition expressed in terms of the data generating process (DGP) of the underlying asset can be slightly generalized by expressing it in terms of the pricing probability measure. Given the general option pricing formula (2.1), the required homogeneity property amounts to the following conditional independence property (with respect to $Q_{t,T}$)[9] :

$$\frac{S_T}{S_t} \perp (S_\tau)_{\tau \leq t} | I_t. \tag{2.7}$$

As mentioned in the introduction, we aim at finding a general option pricing formula with a functional shape that is close to the BS formula. A suitable and not too restrictive way to achieve this is to recover log-normality conditionally to the full path of a vectorial process $(U_t)_{t \in N}$ of possibly unobserved state variables. Indeed, while the BS geometric Brownian motion world is obviously unrealistic due to heterogeneity factors, it is much more general to assume that log-normality is recovered after conditioning on a number of state variables. In particular, it is a standard way to capture the leptokurtic feature of financial time series[10].

State variables have two basic distinctive features: they are exogenous and they summarize the dynamics of the variables of interest (see Renault (1998) and Garcia and Renault (1999) for a general discussion). We provide below a definition in the context of a pricing probability measure $Q_{t,T}$, $t = 1, ..., T$ without specifying its dynamics at this stage .

Definition 2.3. *: A vectorial process $(U_t)_{1 \leq t \leq T}$ (also written U_1^T) is called a state variable process*[11] *with respect to the stock price process $(S_t)_{1 \leq t \leq T}$ and the family of pricing probability measures $(Q_{t,T})_{1 \leq t \leq T}$ if for any $t = 1, ..., T-1$, the $Q_{t,T}$ joint probability distribution of $(\frac{S_T}{S_t}, U_\tau, \tau > t)$ does not depend on the past of the price process $(S_\tau)_{1 \leq \tau \leq t}$.*

[9] This condition for homogeneity is more general than the sufficient condition proposed by Merton (1973) since it is stated in terms of the pricing probability measure rather than the DGP. Indeed, we do not preclude a possible dependence of the risk premiums on the stock price S_t, which could violate assumption 2.7 for the DGP. Garcia and Renault (1998) offer a precise characterization of this property in the standard setting of continuous time arbitrage pricing.

[10] Since Clark (1973), there is a long tradition of this approach in financial econometrics. Clark (1973) stressed that non-normality is a puzzle when one has in mind the geometric temporal averaging of the returns and a corresponding central limit theorem argument. In this respect, log normality of returns can be invoked without any significant loss of generality once it is recovered after conditioning on a sufficient number of state variables. As far as option pricing is concerned, the role of heterogeneity factors has been enhanced by Renault (1997) in a general setting which encompasses in particular Merton (1976) jump-diffusion model and Hull and White (1987) stochastic volatility model.

[11] In many applications of the state variable concept, Markovianity is usually postulated. Then, the relevant conditioning inofrmation is summarized by a few recent lags of the state variable process. Since this Markovianity assumption is not needed at this stage, we maintain in full generality the whole past U_1^T of this process.

This definition implicitly means that $Q_{t,T}$ is a family of transition probabilities indexed by a conditioning set I_t which includes the joint past $(S_\tau, U_\tau)_{1\leq\tau\leq t}$ of prices and state variables. By writing the usual factorization of probability density functions:

$$l\left[\frac{S_T}{S_t}, U_{t+1}^T | I_t\right] = l\left[U_{t+1}^T | I_t\right].l\left[\frac{S_T}{S_t} | U_{t+1}^T, I_t\right], \tag{2.8}$$

the above definition means that the state variable process U_t is exogenous in the sense of being independent of the past of the price process conditionally to its own past and that the future returns are conditionally independent of past returns given the full path of state variables which, in this sense, summarize the dynamics of the price process. To make this point clear, in the particular case where the available information at time t is described by the sigma field:

$$I_t = \sigma\left[S_\tau, U_\tau, \tau \leq t\right], \tag{2.9}$$

the factorization (2.8) can be rewritten as follows, given the state variable concept just described above:

$$l\left[\frac{S_T}{S_t}, U_{t+1}^T | I_t\right] = l\left[U_{t+1}^T | U_1^t\right].l\left[\frac{S_T}{S_t} | U_1^T\right]. \tag{2.10}$$

It is then clear in this context that (2.10) is a sufficient condition for the homogeneity property (2.7).

The symmetric smile condition of Proposition 2.2 is also valid in the more general setting just described. If V_T follows under $Q_{t,T}$ a conditional Gaussian distribution $\mathcal{N}[\mu_t(U_t^T), \sigma_t^2(U_t^T)]$ given I_t and the path U_t^T (between t and T) of some state variables U, the symmetry condition will be fulfilled (by integration over U_t^T) as soon as:

$$\mu_t(U_t^T) = -\frac{\sigma_t^2(U_t^T)}{2}.$$

This is the case for instance for an Hull and White world, which explains the main result of Renault and Touzi (1996): if option prices are conformable to the Hull and White option pricing formula, the volatility smile is symmetric. Proposition 2.2 characterizes precisely which type of symmetry of the pricing probability measure is required for the symmetry of the smile. In particular, it shows that it is not the density of the log returns that should be symmetric (as it is commonly believed perhaps because of the usual log-normal setting), but the same density rescaled by a suitable exponential function.

In order to generalize the Hull and White model, we will assume from now on that the pricing probability measure $Q_{t,T}$ is defined through a stochastic discount factor (SDF) $\mathbf{m}_{t,T}$:

$$\boldsymbol{\pi}_t = E_t \mathbf{m}_{t,T}(S_T - K)^+ \tag{2.11}$$

where E_t denotes now the conditional expectation operator with respect to the DGP. Hansen and Richard (1987) provide some general conditions under which the existence and positivity of $\mathbf{m}_{t,T}$ is guaranteed. First, to ensure homogeneity of the option pricing formula, we will make two sufficient assumptions about the joint dynamics of the stochastic discount factor, the stock returns and the state variables.

Assumption A: The SDF $\mathbf{m}_{t,T}$ can be factorized as follows:

$$\mathbf{m}_{t,T} = \lambda_{t,T}(U_1^T) \prod_{\tau=t}^{T-1} m_{\tau+1}, \tag{2.12}$$

where it is assumed that:

A1: The process $(m_{\tau+1}, \frac{S_{\tau+1}}{S_\tau})_{1\leq\tau\leq T-1}$ does not cause the process $(U_\tau)_{1\leq\tau\leq T}$;

A2: The variables $(m_{\tau+1}, \frac{S_{\tau+1}}{S_\tau})_{1\leq\tau\leq T-1}$ are serially independent given U_1^T.

The conditions (A1) and (A2) are consistent with the concept of state variables introduced in Definition 2.3[12]. The non-causality property may be interpreted equivalently in Granger (1969) or in Sims (1972) terms. Granger causality means that, given the past U_1^t of state variables, the past observation of processes m_τ and S_τ does not bring any relevant information to forecast U_{t+1} (which is in this sense exogenous). Sims causality means that the probability distribution of $(m_{t+1}, \frac{S_{t+1}}{S_t})$ given I_t and U_{t+1}^T does not depend upon U_{t+1}^T. Jointly with the conditional independence assumption A2, assumption A1 permits to characterize the joint probability distribution of $(m_{\tau+1}, \frac{S_{\tau+1}}{S_\tau}, U_{\tau+1})_{\tau\geq t}$ given I_t as the following product:

$$\begin{aligned}
\ell\left[(m_{\tau+1}, \frac{S_{\tau+1}}{S_\tau}, U_{\tau+1})_{\tau\geq t}|I_t\right] &= l\left[U_{t+1}^T|U_1^t\right].l\left[(m_{\tau+1}, \frac{S_{\tau+1}}{S_\tau})_{\tau\geq t}|U_1^T\right] \\
&= \prod_{h=1}^{T-t} \ell\left[U_{t+h}|U_1^{t+h-1}\right].\prod_{h=1}^{T-t} \ell\left[m_{t+h}, \frac{S_{t+h}}{S_{t+h-1}}|U_1^{t+h}\right].
\end{aligned} \tag{2.13}$$

Proposition 2.4. : *Under (A1) and (A2) there exists a deterministic function $\Psi_{t,T}$ such that the option price (2.11) can be written as:*

$$\boldsymbol{\pi}_t = \Psi_{t,T}\left[U_1^t, \frac{K}{S_t}\right] S_t.$$

Proposition 2.4 establishes that the option pricing formula is homogeneous of degree one with respect to the pair (S_t, K).

[12] See Garcia and Renault (1999) for a detailed account of this point under both the pricing probability measure and the DGP.

2.3. A generalized Black-Scholes and Hull-White formula

To obtain a generalized BS and Hull-White option pricing formula starting from (2.11), one needs only, in addition to the previous assumptions (A1) and (A2), a joint log-normality assumption of $\mathbf{m}_{t,T}$ and $\frac{S_T}{S_t}$ given $I_t = \sigma[m_\tau, S_\tau, U_\tau, \tau \leq t]$ and a path U_{t+1}^T of state variables.

Assumption A3: The conditional probability distribution of $(\log m_{t+1}, \log \frac{St+1}{S_t})$ given U_1^{t+1} is, for t=1,...,T-1, a bivariate normal:

$$\aleph\left[\begin{pmatrix} \mu_{mt+1} \\ \mu_{st+1} \end{pmatrix}, \begin{bmatrix} \sigma_{mt+1}^2 & \sigma_{mst+1} \\ \sigma_{mst+1} & \sigma_{st+1}^2 \end{bmatrix}\right].$$

Proposition 2.5. *Under assumptions A1, A2 and A3:*

$$\frac{\boldsymbol{\pi}_t}{S_t} = \pi_t(x) = E_t\left\{Q_{ms}(t,T)\Phi(d_1(x)) - \frac{\widetilde{B}(t,T)}{B(t,T)}e^{-x}\Phi(d_2(x))\right\} \tag{2.14}$$

where:

$$d_1(x) = \frac{x}{\overline{\sigma}_{t,T}} + \frac{\overline{\sigma}_{t,T}}{2} + \frac{1}{\overline{\sigma}_{t,T}} Log\left[Q_{ms}(t,T)\frac{B(t,T)}{\widetilde{B}(t,T)}\right]$$

$$d_2(x) = d_1(x) - \overline{\sigma}_{t,T}$$

$$\overline{\sigma}_{t,T}^2 = \sum_{\tau=t}^{T-1} \sigma_{s\tau+1}^2.$$

and:

$$\widetilde{B}(t,T) = \lambda_{t,T}(U_1^T)\exp(\sum_{\tau=t}^{T-1}\mu_{m\tau+1} + \frac{1}{2}\sum_{\tau=t}^{T-1}\sigma^2{}_{m\tau+1}),$$

$$Q_{ms}(t,T) = \widetilde{B}(t,T)\exp(\sum_{\tau=t+1}^{T}\sigma_{ms\tau+1})E[\frac{S_T}{S_t}|U_1^T]. \tag{2.15}$$

To put this general formula in perspective, we will compare it to the three main approaches that have been used for pricing options: equilibrium option pricing, arbitrage-based option pricing, and GARCH option pricing. The latter pricing model can be set either in an equilibrium framework or in an arbitrage framework.

Concerning the equilibrium approach, our setting is very general since it is based on a stochastic model for the SDF which does not rely on restrictive assumptions about preferences, endowments, or agent heterogeneity. Moreover, our factorization for the SDF is more general than the usual one $m_{t,T} = \frac{\eta_T}{\eta_t}$[13], with the standard interpretation as an intertemporal marginal rate of substitution. Actually, the additional factor $\lambda_{t,T}(U_1^T)$ in our SDF allows to accommodate non-separable or state-dependent preferences. The non-separable case will be illustrated in the next section by a recursive utility setting. An example of state-dependent preferences could be habit formation based on state variables.

Of course, the most popular option pricing formulas among practitioners are based on arbitrage rather than on equilibrium in order to avoid in particular the specification of preferences. From the start, it should be stressed that our general formula (2.14) nests a large number of preference-free extensions of the Black-Scholes formula. In particular if $Q_{ms}(t,T) = 1$ and $\widetilde{B}(t,T) = \prod_{\tau=t}^{T-1} B(\tau, \tau+1)$,one can see that the option price (2.14) is nothing but the conditional expectation of the Black-Scholes price, where the expectation is computed with respect to the joint probability distribution of the rolling-over interest rate $\overline{r}_{t,T} = -\sum_{\tau=t}^{T-1} \log B(\tau, \tau+1)$ and the cumulated volatility $\overline{\sigma}_{t,T}$. This framework nests three well-known models. First, the most basic ones, the Black and Scholes (1973) and Merton (1973) formulas, when interest rates and volatility are deterministic. Second, the Hull and White (1987) stochastic volatility extension, since $\overline{\sigma}_{t,T}^2 = Var\left[\log \frac{S_T}{S_t} | U_1^T\right]$ corresponds to the cumulated volatility $\int_t^T \sigma_u^2 du$ in the Hull and White continuous-time setting[14]. Third, the formula allows for stochastic interest rates as in Turnbull and Milne (1991) and Amin and Jarrow (1992). However, the usefulness of our general formula (2.14) comes above all from the fact that it offers an explicit characterization of instances where the preference-free paradigm cannot be maintained. Usually, preference-free option pricing is underpinned by the absence of arbitrage in a complete market setting. However, our SDF-based option pricing does not preclude incompleteness and points out in which cases this incompleteness will invalidate the preference-free paradigm. The only cases of incompleteness which matter in this respect occur precisely when at least one of the two following conditions:

$$Q_{ms}(t,T) = 1 \tag{2.16}$$

$$\widetilde{B}(t,T) = \prod_{\tau=t}^{T-1} B(\tau, \tau+1) \tag{2.17}$$

is not fulfilled.

In general, preference parameters appear explicitly in the option pricing formula through $\widetilde{B}(t,T)$ and $Q_{ms}(t,T)$ since these two quantities depend on the characteristics of the SDF:

[13] See for instance Amin and Ng (1993b) and Bollerslev and Mikkelsen (1996) following Constantinides (1992) and Turnbull and Milne (1991).

[14] See Appendix 3 for a detailed comparison between standard stochastic volatility models and our state variable framework.

$\lambda_{t,T}(U_1^T), (\mu_{m\tau+1}, \sigma^2{}_{m\tau+1,}\sigma_{ms\tau+1})_{t=t}^{T-1}$. However, in so-called preference-free formulas, it happens that these parameters are eliminated from the option pricing formula through the observation of the bond price and the stock price. Actually, the bond pricing formula and the stock pricing formula provide two dynamic restrictions relating the SDF characteristics to the bond and stock price processes. To avoid cumbersome notation, we will consider for the moment a one-period option price. In this case, the bond pricing equation is given by:

$$B(t, t+1) = E_t\left[\widetilde{B}(t, t+1)\right] \tag{2.18}$$

as shown in Appendix 2. Therefore, observing the bond price will make preference parameters in $\widetilde{B}(t, t+1)$ vanish from the option price as soon as $\widetilde{B}(t, t+1) = B(t, t+1)$,that is if and only if $\widetilde{B}(t, t+1)$ belongs to the information set I_t. Since $\widetilde{B}(t, t+1) = E_t[m_{t,t+1}|U_1^{t+1}]$,it means that there is no instantaneous causality in expectation from the state variables to the SDF. We will call generically such instantaneous causality a leverage effect, in the spirit of Black (1975,1976), by reference to a standard model where returns are unpredictable but conditionally heteroskedastic[15].

A useful way of writing the stock pricing formula is:

$$E_t\left[Q_{ms}(t, t+1)\right] = 1, \tag{2.19}$$

Similarly to the bond pricing, when $Q_{ms}(t, t+1)$ is known at time t, and therefore equal to one, there is no leverage effect. As shown in Appendix 2, we can express in this case the conditional expected stock return as:

$$E\left[\frac{S_{t+1}}{S_t}|I_t\right] = \frac{1}{B(t, t+1)}\exp[-\sigma_{mst+1}],$$

which is very close to a standard conditional CAPM equation. Therefore, the absence of leverage produces both a preference-free option pricing formula and a CAPM-like stock pricing equation[16]. To conclude, it should be stressed that even in an equilibrium framework with incomplete markets, option pricing is preference-free if and only if there is no leverage effect. This result generalizes Amin and Ng (1993a), who called this effect predictability.

It is worth noting that our results of equivalence between preference-free option pricing and no instantaneous causality between state variables and asset returns are consistent with another strand of the option pricing literature, namely GARCH option pricing. Duan (1995) derived it first in an equilibrium framework, but Kallsen and Taqqu (1994) have shown that it could be obtained with an arbitrage argument. Their idea is to complete the markets by plugging the discrete-time model into a continuous time one, where conditional

[15] See Appendix 3 for a detailed comparison between leverage effect in a standard stochastic volatility framework and what we call leverage effect in our setting.

[16] A similar parallel is drawn in an unconditional two-period framework in Breeden and Litzenberger (1978).

variance is constant between two integer dates. They show that such a continuous-time embedding makes possible arbitrage pricing which is per se preference-free. It is then clear that preference-free option pricing is incompatible with the presence of an instantaneous causality effect, since it is such an effect that prevents the embedding used by Kallsen and Taqqu (1994).

This characterization of the preference-free formulas can be generalized to a multiperiod setting if the SDF can be factorized as follows:

$$\lambda_{t,T}(U_1^T) = \prod_{\tau=t}^{T-1} \lambda_{\tau,\tau+1}(U_1^{\tau+1}) \tag{2.20}$$

This factorization remains more general than the standard one $m_{t,T} = \prod_{\tau=t}^{T-1} m_{\tau,\tau+1,}$ since $\lambda_{\tau,\tau+1}$allows for time nonseparabilities, in particular, as we will see in Section 3, recursive utility. Under the factorization (2.20), the absence of leverage will imply conditions (2.16) and (2.17)[17], that is preference-free option prices.

2.4. The asymmetric distortions of the smile due to leverage effects

In this section, we focus on the option pricing formula provided by Proposition (2.5) to characterize the cases where the corresponding volatility smiles are symmetric. According to criterion i) of Proposition (2.2) for the symmetry of the smile, it means that the above option pricing formula:

$$\pi_t(x) = E_t\left\{Q_{ms}(t,T)\Phi(d_1(x))\right\} - e^{-x}E_t\left\{\frac{\widetilde{B}(t,T)}{B(t,T)}\Phi(d_2(x))\right\}$$

can be written:

$$\pi(x) = F_{V_T}(x) - e^{-x}[1 - F_{V_T}(-x)].$$

But it can be seen in the derivation of the option pricing formula (2.14) (see Appendix 2) that:

$$[1 - F_{V_T}(-x)] = E_t\left[\frac{\widetilde{B}(t,T)}{B(t,T)}\Phi(d_2(x))\right]. \tag{2.21}$$

Therefore, a necessary and sufficient condition for symmetric smiles of (2.14) is that:

$$E_t\left\{Q_{ms}(t,T)\Phi(d_1(x))\right\} = F_{V_T}(x)$$

[17] In condition (2.17), there remains interest rate risk since $\prod_{\tau=t}^{T-1} B(\tau,\tau+1)$ is not equal to $B(t,T)$.

or equivalently:

$$E_t\left\{Q_{ms}(t,T)\Phi(d_1(x))\right\} = 1 - E_t\left\{\frac{\widetilde{B}(t,T)}{B(t,T)}\Phi(d_2(-x))\right\} \tag{2.22}$$

But from (2.14):

$$d_2(-x) = -d_1(x) + \frac{2}{\overline{\sigma}_{t,T}} Log\left[Q_{ms}(t,T)\frac{B(t,T)}{\widetilde{B}(t,T)}\right].$$

Thus by taking into account that $E_t[\frac{\widetilde{B}(t,T)}{B(t,T)}] = 1$, the symmetry criterion can be rewritten:

$$E_t\left\{Q_{ms}(t,T)\Phi(d_1(x))\right\} = 1 - E_t\left\{\frac{\widetilde{B}(t,T)}{B(t,T)}\Phi(d_1(x) - \frac{2}{\overline{\sigma}_{t,T}} Log\left[Q_{ms}(t,T)\frac{B(t,T)}{\widetilde{B}(t,T)}\right])\right\}$$

We have therefore proven the following proposition:

Proposition 2.6. *A necessary and sufficient condition for a symmetric volatility smile is the following identity :*

$$E_t\left\{Q_{ms}(t,T)\Phi(d_1(x))\right\} = 1 - E_t\left\{\frac{\widetilde{B}(t,T)}{B(t,T)}\Phi(d_1(x) - \frac{2}{\overline{\sigma}_{t,T}} Log\left[Q_{ms}(t,T)\frac{B(t,T)}{\widetilde{B}(t,T)}\right])\right\} \tag{2.23}$$

A sufficient condition for a symmetric volatility smile is:

$$Q_{ms}(t,T) = \frac{\widetilde{B}(t,T)}{B(t,T)} \tag{2.24}$$

It should be stressed that the sufficient condition (2.24) is always fulfilled in expectation (see Appendix 2). Moreover, taking into account the highly nonlinear features of the two sides of the necessary and sufficient identity (2.23), Jensen effects are likely to violate it if (2.24) is not fulfilled. In other words, condition (2.24) appears at first sight not too far from being necessary. In addition, it is possible to provide a structural foundation to condition (2.24) in the particular case of a factorizable SDF (see 2.20). Indeed, in this case, we know that the global absence of leverage effects imply (2.16) and (2.17). Therefore, in the absence of leverage effects, condition (2.24) for symmetric smiles is fulfilled if and only if there is no interest risk, that is:

$$B(t,T) = \prod_{\tau=t}^{T-1} B(\tau, \tau+1).$$

More generally, it should also be stressed that, without maintaining any additional assumption, condition (2.24) for symmetry implies a stock pricing relation of the CAPM type:

$$E[\frac{S_T}{S_t}|I_t] - \frac{1}{B(t,T)} = \frac{1}{B(t,T)} \left[E_t[\exp[-\sum_{\tau=t+1}^{T} \sigma_{ms\tau}]] - 1 \right]$$

This extends the analysis of subsection 2.3 above. CAPM-like stock pricing is obtained not only when there is no leverage effects but also, more generally, when volatility smiles are symmetric.

In other words two kinds of "generalized" leverage effects may explain (besides the interest rate risk) asymmetric smiles: either a genuine leverage effect, that is an instantaneous correlation between the return on the stock and its stochastic volatility process, or a stochastic correlation between the return of the stock and the stochastic discount factor. These results provide some theoretical foundations to the observed asymmetric smiles. A link can also be established with the previous theoretical explanations of the asymmetric smile mentioned in the introduction. In Platen and Schweizer (1997), the asymmetry is introduced by the dependence of the diffusion process of the stock price on the technical demand induced by hedging strategies. This hedging strategy should introduce a conditional correlation between the current price and the future volatility similar to what is captured in our model. In Renault (1997), the asymmetry is related to a distortion in the stock price whereby option traders consider a $\widetilde{S}_t$ value for the stock price instead of the true price S_t. This can be interpreted in our model by a correction of S_t through $Q_{ms}(t,T)$.

3. A structural model for option pricing

In this section, to offer a setup that leads to a computable formula, we provide an equilibrium version of the SDF with preferences in the recursive utility class (Epstein and Zin, 1989). These preferences are richer than the usual expected utility model. In particular, the elasticity of intertemporal substitution is disentangled from the risk aversion parameter. From a practical point of view, this additional parameter might help better explain prices of long-term options such as LEAPS (Long-term Equity Anticipation Securities). We maintain a similar stochastic framework to the one in Section 2. To obtain a generalized Black-Scholes formula for option prices which is homogeneous, we outline necessary and sufficient conditions that the variables entering this particular equilibrium SDF must obey. The conditioning of variables such as the consumption growth rate or the market portfolio returns on state variables is a way to circumvent the fact that these variables are unobservable or are not available at the required frequency. The remaining task will be to calibrate the conditional moments of the variables entering the SDF and a stochastic process for the state variables. We will attend to this task in Section 4.

3.1. The equilibrium stochastic discount factor

In the recursive utility framework of Epstein and Zin (1989,1991), the stochastic discount factor is given by:

$$m_{t+1} = \beta^{\gamma}(\frac{C_{t+1}}{C_t})^{\gamma(\rho-1)} M_{t+1}^{\gamma-1} \tag{3.1}$$

where $\frac{C_{t+1}}{C_t}$ is the growth rate of consumption in the economy and M_{t+1} represents the return on the market portfolio. The parameters β, γ and ρ are preference parameters. The parameter β is the subjective rate of time preference, while $\alpha = \gamma\rho$ can be interpreted as a relative risk aversion parameter with the degree of risk aversion increasing as α falls ($\alpha \leq 1$). The parameter ρ is associated with intertemporal substitution, since the elasticity of intertemporal substitution is $1/(1-\rho)$[18]. The position of α with respect to ρ determines whether the agent has a preference towards early resolution of uncertainty ($\alpha < \rho$) or late resolution of uncertainty ($\alpha > \rho$). When $\gamma = 1$, we obtain the well-known stochastic discount factor for the expected utility case $m_{t+1} = \beta(\frac{C_{t+1}}{C_t})^{\alpha-1}$.

Given this stochastic discount factor, the price of a European option π_t maturing at t+1 is given by:

$$\pi_t = S_t E_t \left[\beta^{\gamma}(\frac{C_{t+1}}{C_t})^{\gamma(\rho-1)} M_{t+1}^{\gamma-1} Max[0, \frac{S_{t+1}}{S_t} - \frac{K}{S_t}] \right]. \tag{3.2}$$

This price depends on both the market portfolio return M_{t+1} and the stock ex-dividend return $\frac{S_{t+1}}{S_t}$. In this model, the payoff of the market portfolio at time t is the total endowment of the economy C_t. Therefore, the return on the market portfolio M_{t+1} can be written as follows:

$$M_{t+1} = \frac{P_{t+1}^M + C_{t+1}}{P_t^M}.$$

Defining $\lambda_t = \lambda(I_t) = \frac{P_t^M}{C_t}$ and $\varphi_t = \varphi(I_t) = \frac{S_t}{D_t}$ (with D_t the dividend on the stock) as the

[18] As mentioned in Epstein and Zin (1991), the association of risk aversion with α and intertemporal sustitution with ρ is not fully clear, since at a given level α of risk aversion, changing ρ affects not only the elasticity of intertemporal sustitution but also determines whether the agent will prefer early or late resolution of uncertainty.

solutions to similar Euler equations[19] than (3.2) for the price of the market portfolio and the price of the stock, the dynamic behavior of these prices, or equivalently of the associated rates of return:

$$LogM_{t+1} = Log\frac{\lambda(I_{t+1})+1}{\lambda(I_t)} + Log\frac{C_{t+1}}{C_t}, \text{and} \tag{3.3}$$

$$LogR_{t+1} = Log\frac{S_{t+1}}{S_t} = Log\frac{\varphi(I_{t+1})}{\varphi(I_t)} + \log\frac{D_{t+1}}{D_t}, \tag{3.4}$$

is determined by the joint probability distribution of the stochastic process (X_t, Y_t, I_t) where: $X_t = Log\frac{C_t}{C_{t-1}}$ and $Y_t = Log\frac{D_t}{D_{t-1}}$. We shall define this dynamics through a stationary vector-process of state variables U_t so that:

$$I_t = \vee_{\tau \leq t}[X_\tau, Y_\tau, U_\tau].$$

To obtain a homogeneous option pricing formula, we impose on X_t and Y_t similar assumptions to A1 and A2:

Assumption B1*:* (X,Y) does not cause U.
Assumption B2*:* The pairs $(\mathrm{X}_t, Y_t), t = 1, 2, ..., T$ are mutually independent knowing $U_1^T = (U_t)_{1 \leq t \leq T}$.

Given these assumptions, Garcia and Renault (1998) show that there exists a homogeneous option pricing formula:

$$\boldsymbol{\pi}_t = \Psi(U_1^t, \frac{K}{S_t})S_t. \tag{3.5}$$

3.2. An equilibrium generalized Black-Scholes and Hull and White formula

With an additional normality assumption on the probability distribution of the fundamentals X and Y given the state variables U :

Assumption B3*:*

$$\begin{pmatrix} X_t \\ Y_t \end{pmatrix} | U_1^t \sim \aleph \left[\begin{pmatrix} m_{Xt} \\ m_{Yt} \end{pmatrix}, \begin{bmatrix} \sigma_{Xt}^2 & \sigma_{XYt} \\ \sigma_{XYt} & \sigma_{Yt}^2 \end{bmatrix} \right],$$

[19] We assume that the regularity and stationarity assumptions for unique solutions to these equations are fulfilled.

where $m_{Xt}, m_{Yt}, \sigma^2_{Xt}, \sigma_{XYt}, \sigma^2_{Yt}$ are stationary and measurable functions with respect to U_1^t, so that $m_{Xt} = m_X(U_1^t), m_{Yt} = m_Y(U_1^t), \sigma^2_{Xt} = \sigma^2_X(U_1^t), \sigma^2_{Yt} = \sigma^2_Y(U_1^t), \sigma_{XYt} = \sigma_{XY}(U_1^t)$.

We arrive at the generalized Black-Scholes and Hull and White formula for pricing options[20]:

$$\frac{\pi_t}{S_t} = E_t \left\{ Q_{XY}(t,T)\Phi(d_1) - \frac{K\widetilde{B}(t,T)}{S_t}\Phi(d_2) \right\}, \tag{3.6}$$

where:

$$d_1 = \frac{Log\left[\frac{S_t Q_{XY}(t,T)}{K\widetilde{B}(t,T)}\right]}{(\sum_{\tau=t+1}^{T}\sigma^2_{Y\tau})^{1/2}} + \frac{1}{2}(\sum_{\tau=t+1}^{T}\sigma^2_{Y\tau})^{1/2}, \text{and}$$

$$d_2 = d_1 - (\sum_{\tau=t+1}^{T}\sigma^2_{Y\tau})^{1/2}.$$

and:

$$\widetilde{B}(t,T) = \beta^{\gamma(T-t)} a_t^T(\gamma) \exp((\alpha-1)\sum_{\tau=t+1}^{T} m_{X\tau} + \frac{1}{2}(\alpha-1)^2 \sum_{\tau=t+1}^{T} \sigma^2{}_{X\tau}),$$

with: $a_t^T(\gamma) = \prod_{\tau=t}^{T-1}\left[\frac{(1+\lambda(U_1^{\tau+1})}{\lambda(U_1^{\tau})}\right]^{\gamma-1}$,

$$Q_{XY}(t,T) = \widetilde{B}(t,T)\exp((\alpha-1)\sum_{\tau=t+1}^{T}\sigma_{XY\tau})E[\frac{S_T}{S_t}|U_1^T]. \tag{3.7}$$

Our setup provides a framework to capture the empirically documented relationship of the asymmetries with the business cycle and interest rate movements (see for instance the survey by Bates (1996)). More importantly, the new conclusion of our model for practitioners should be that an asymmetric smile is indicative of the relevance of preference parameters to price options. Indeed, our structural equilibrium model has shown that violations of the symmetry condition in Proposition 2.6 (due to interest rate risk or the occurrence of a leverage effect in the general sense above) correspond precisely to cases where preference parameters matter for option pricing.

Therefore, whenever an asymmetric smile is observed, the first issue to address is to specify a list of state variables as well as a set of mean, variance and covariance functions

[20] The lines of proof follow closely the proof in Appendix 2 and are provided in detail in Garcia and Renault (1998).

conformable to B3. Since the process of state variables is a latent Markov process, a natural candidate is the Markov switching model introduced by Hamilton (1989) and applied to asset pricing by Cecchetti, Lam and Mark (1990, 1993) and Bonomo and Garcia (1993, 1994, 1996). The standard procedures of estimation and identification of such a model (Hamilton (1989), Garcia (1997)) can then be used for the modeling of the bivariate process (X_t, Y_t). We provide such a setup in section 4 below.

Our approach has to be compared with a recent trend in the literature called implied binomial trees (Rubinstein (1994)). There is a formal similarity between the two approaches, because in both cases we try to calibrate a binomial tree or a discrete Markov process on the dynamics of option prices. However, while implied binomial trees inferred in Rubinstein (1994) represent the local volatility of the underlying asset, the riskless interest rate and the asset payout rate as a function of the prior path of the underlying asset price, our implied latent binomial trees are hidden Markov chains which correspond to violation 3 of the BS model in Rubinstein (1994, p. 778): "The local volatility of the underlying asset, the riskless interest rate or the asset payout rate is a function of a state variable which is not the concurrent underlying asset price or the prior path of the underlying asset price". We have explained that this type of violation is useful since it maintains the homogeneity of option prices but of course we are led to follow the route of what Rubinstein calls the unpalatable alternative of establishing an equilibrium model in which investor preferences explicitly enter. As we will see below, these parameters should help characterize the implied volatility structure in a more stable manner and their interpretation as preference parameters should not be a hindrance for this purpose.

4. A Markov-Chain Setup for the State Variables

In order to render operational the option pricing formula described in the previous section, we endow the state variable with a discrete Markov chain structure. This stochastic structure framework is useful since it provides analytical formulas and can easily be simulated as well. In this section we describe such a setup and then proceed on to simulation experiments under various parameter value configurations.

4.1. Model setup

Equilibrium requires that asset prices are such that the representative agent is satisfied to consume her expected endowment. The equilibrium market portfolio price and equilibrium stock price are determined by consumption and dividend growth which in turn depend fundamentally on the evolution of the state variable. Indeed, the cardinality of the set of values that the price-dividend ratios associated with the market portfolio and the stock can assume is the same as that of the state variable. We assume that the state variable takes one of two possible values. In this sense, the state variable determines in which of two possible equilibria the economy is at any given time.

The stochastic process describing the joint evolution of $X_t = \log \frac{C_t}{C_{t-1}}$ and $Y_t = \log \frac{D_t}{D_{t-1}}$ is then parameterized as follows:

$$\begin{aligned} X_t &= m_{X1}\mathbf{1}[U_t = 1] + m_{X2}\mathbf{1}[U_t = 2] + \sigma_X \varepsilon_{Xt} \\ Y_t &= (\sigma_{Y1}\mathbf{1}[U_t = 1] + \sigma_{Y2}\mathbf{1}[U_t = 2])\varepsilon_{Yt} \end{aligned} \tag{4.1}$$

where $\mathbf{1}[\ ]$ is the indicator function, and where the state-contingent parameter values are $m_{X1} = .0015$, $m_{X2} = -.0009$, $\sigma_X = .003$, $\sigma_{Y1} = .02$, and $\sigma_{Y2} = .12$. The noise vector $(\varepsilon_{Xt}, \varepsilon_{Yt})'$ follows a standard bivariate normal distribution with correlation coefficient $\rho_{XY} = 0.6$. The time-varying mean and variance parameters are function of the state variable process $\{U_t\}$, which we assume to be a first-order Markov chain such that U_t takes values in the set $\{1,2\}$ with $\Pr(U_t = j) = \sum_{i=1}^{2} p_{ij} \Pr(U_{t-1} = i)$ and transition probability $p_{ij} = \Pr(U_t = j|U_{t-1} = i)$.

With this specification, $U_t = 1$ can be interpreted as an expansionary state where consumption growth is positive and the variance of dividend growth is low. On the other hand when in a recessionary state, $U_t = 2$, consumption growth is negative and the variance of dividend growth is six times as large compared to the variance when $U_t = 1$. The transition probabilities governing the outcome of the state variable are set at $p_{11} = .9$, and $p_{22} = .8$. That is, given that the economy is in expansion at time t, it will remain in expansion at time $t+1$ with probability 0.9 or move into recession with complementary probability 0.1. On the other hand, with probability 0.8 a recessionary state will persist for two consecutive periods or an expansion will follow a recession with probability 0.2. Finally the subjective discount rate was set at $\beta = 0.98$.

Given a stochastic process governing the state variable and values of the structural parameters, it is then possible to explicitly compute the price of an option according to the generalized Black-Scholes formula. This can be done either analytically or numerically at very low computational cost. In the following subsection we report the results of simulation experiments where both approaches were employed. For example, under the assumption of a stochastic discount factor we consider an option with a maturity horizon of one period for which the expectation appearing in the pricing formula (3.6) was computed analytically. In the case of a deterministic discount factor, we consider longer maturity options for which the analytical computation of the expectation becomes cumbersome. For this case, we obtained approximations through Monte-Carlo integration. Specifically, the procedure by which we computed the option prices and obtained their implied volatility graphs is as follows:

1. Obtain the price-dividend ratios $\lambda(U_t)$ and $\varphi(U_t)$ by numerically solving the nonlinear system of Euler equations for the market portfolio and the stock[21].

[21] Given the assumed two-point process for the state variable, the solution set consists of two equilibrium

2. Simulate a trajectory of the state variable up to time t according to the Markov chain specification. Obtain a realization of the consumption and dividend processes up to time t according to (4.1). With the resulting value for D_t, compute the stock price as $S_t = \varphi(U_t)D_t$.

3. Generate an array of 200 strike prices in increments of 0.05$ around the value of the current stock price S_t calculated in step 2.

4. (Monte-Carlo) Conditional on the state operative at time t, draw a trajectory of the state variable over the maturity horizon $\{U_\tau\}_{\tau=t+1}^{T}$ and compute the value of the inner part of (3.6), i.e., $Q_{XY}(t,T)S_t\Phi(d_1) - K\widetilde{B}(t,T)\Phi(d_2)$ for each strike price. Repeat this step 1000 times, each time saving the quantities $\sum_{\tau=t+1}^{T}\sigma_{Y\tau}^2$ and $\widetilde{B}(t,T)$ for subsequent use. Compute the average option price across the 1000 realizations to obtain the generalized option price in (3.6) corresponding to each strike price. Similarly, the price $B(t,T)$ of the discount bond is given by the average over the 1000 realizations of the quantity $\widetilde{B}(t,T)$.

5. (Monte-Carlo) Compute the value of the inner part of (3.6) with $Q_{XY}(t,T) = 1$ and $\widetilde{B}(t,T) = B(t,T)$ for the same array of strike prices. Repeat this with each of the 1000 sums $\sum_{\tau=t+1}^{T}\sigma_{Y\tau}^2$ saved during step 4 (Monte-Carlo) and then compute the average to obtain the Hull-White type option prices for each strike price.

4′. (Exact) Given that state i is operative at time $T-1$, compute for each strike price of step 3 the price of a one-period option according to

$$GR = \sum_{j=1}^{2} p_{ij}\left\{Q_{XY}^{(j)}(T-1,T)S_t\Phi(d_1^{(j)}) - K\widetilde{B}^{(j)}(T-1,T)\Phi(d_2^{(j)})\right\}$$

where

$$d_1^{(j)} = \frac{\log\left[\frac{S_t Q_{XY}^{(j)}(T-1,T)}{K\widetilde{B}^{(j)}(T-1,T)}\right]}{\sigma_{Yj}} + \frac{1}{2}\sigma_{Yj}$$

$$d_2^{(j)} = d_1^{(j)} - \sigma_{Yj}$$

$$Q_{XY}^{(j)}(t,T) = \widetilde{B}^{(j)}(t,T)\exp((\alpha-1)\sigma_{XYj})\frac{\phi(j)}{\phi(i)}\exp(m_{Yj} + \frac{1}{2}\sigma_{Yj}^2)$$

and

$$\widetilde{B}^{(j)}(T-1,T) = \beta^{\gamma}\left[\frac{1+\lambda(j)}{\lambda(i)}\right]^{\gamma-1}\exp((\alpha-1)m_{Xj} + \frac{1}{2}(\alpha-1)^2\sigma_{Xj}^2)$$

values for each price-dividend ratio.

The price of the discount bond in this case being

$$B(T-1,T) = \sum_{j=1}^{2} p_{ij} \widetilde{B}^{(j)}(T-1,T)$$

5′. (Exact) Always for the same array of strike prices, obtain the HW-type option prices exactly as in step 4′ (Exact) with $Q_{XY}(T-1,T) = 1$ and $\widetilde{B}(T-1,T) = B(T-1,T)$.

6. Numerically invert the Black-Scholes formula to obtain the volatilities implicit in our generalized option price and Hull-White type option price across strike prices. For this we made use of a straightforward bisection search procedure.

In the application of the simulation procedure just described, the following is noteworthy. European call options on non-dividend-paying stocks must be bounded from below and above in order not to violate the no-arbitrage hypothesis. These bounds do not depend on any particular assumptions about the model by which options are priced; see Hull (1993) §7.3. In previous notation, these no-arbitrage bounds are

$$S_t - KB(t,T) \leq \pi_t \leq S_t \tag{4.2}$$

where π_t is the price of a call option. The option prices calculated from formula (3.6) satisfy the no arbitrage bounds in theory. In Monte-Carlo experiments, at best they satisfy the bounds in (4.2), however at worse they only satisfy the following:

$$S_t Q_{XY}(t,T) - K\widetilde{B}(t,T) \leq \pi_t \leq S_t Q_{XY}(t,T) \tag{4.3}$$

where π_t is the option price in (3.6) computed by Monte-Carlo integration. When the number of simulations grows without limit, the bounds in (4.3) tend to those in (4.2) since:

$$E_t[\widetilde{B}(t,T)] = B(t,T)$$

and

$$E_t[Q_{XY}(t,T)] = 1$$

One could impose restrictions directly on the simulated values of $\widetilde{B}(t,T)$ and $Q_{XY}(t,T)$ to ensure that the resulting option price lies within the no arbitrage bounds in (4.2), or alternatively proceed without restrictions and simply reject a simulated option price if it violates the no arbitrage bounds. A more serious violation can occur when simulating option prices with the formula (3.6). Proceeding in an unrestricted way can produce option prices that are below the intrinsic value bound for a call option:

$$\max(S_t - K, 0) \tag{4.4}$$

For example, in the case of a deterministic discount factor such a violation can occur when the value of $Q_{XY}(t,T)$ is below

$$\frac{K(B(t,T)-1)}{S_t}+1$$

When a cross-section of simulated option prices violates the intrinsic value bound, the implied volatilities trace aberrations departing dramatically from U-shaped smiles or smirks. For the purposes of the present study it was therefore essential to ensure that the simulated option prices respected at least the intrinsic value bound. If a simulation run produced option prices in violation of the intrinsic value bound, it was rejected and another draw for that particular cross-section of option prices was performed.

4.2. Numerical results

4.2.1. The case of a deterministic discount function $B(t,T)$

When there is no causality from the growth rate of consumption X_t to the state variable U_t, the equilibrium price of the discount bond becomes:

$$B(t,T) = E_t \prod_{\tau=t}^{T-1} B(\tau,\tau+1)$$

In this case the term premium is preference-free in the sense that the discount factor is given by the product of predetermined short-term discount factors. When in addition the marginal probability distribution of X_t does not depend on U_t such that the consumption growth rates are iid, there is no interest rate risk yielding a deterministic discount factor. In this case, leverage effects can occur only through the stock risk. Indeed when in addition to a deterministic discount factor there is no volatility risk in the sense of no causality from Y_t to U_t, the generalized option pricing formula becomes of the HW-type.

In this section we investigate the effect of a deterministic discount factor in terms of implied volatility smiles. To this end we set

$$X_t = \overline{m}_X + \sigma_X \varepsilon_{Xt}$$

where $\overline{m}_X$ is the average of m_{X1} and m_{X2}. In this case the price-dividend ratio λ is constant and we thus have $B(t,T)$ deterministic. Furthermore the difference between the generalized BS formula and its HW preference-free counterpart resides only in the factor $Q_{XY}(t,T)$. The additional risk premium associated with the generalized option prices only involves α, the risk aversion parameter[22]. The implications on the volatility smiles in this case are of particular interest as the function used to discount the strike price is the same in both the

[22] The results of this section where qualitatively the same for admissible values of α between 1 and -10.

GBS and HW formulas. We also document the effect of increasing the time-to-maturity on the implicit volatility smiles.

Figure 1-1 depicts the option prices and implied volatility smiles for maturity horizons of 1, 3 and 5 periods[23]. The top portion shows the implications of the benchmark HW model while the bottom portion are those of the generalized option pricing model. In each frame, the uppermost solid line corresponds to an option with a maturity of 5 periods, the dashed line corresponds to an option with a maturity of 3 periods, and finally the lowermost dotted line corresponds to a one-period option.

For sake of clarity the volatility smiles associated with the two models are presented on different graphs. It is nevertheless obvious that the respective volatility smiles do not intersect. For a given maturity the volatility smile associated with the generalized option pricing formula is above its HW counterpart across moneyness.

If option prices were given by the Black-Scholes formula not only would the implicit volatilities be constant across moneyness but also constant over time-to-maturity. In reality however the term structure of implied volatilities is typically upward sloping (Ghysels et al., 1995). Another stylized fact about implied volatilities is that the smile effect is more pronounced for short maturities and decreases in amplitude as the maturity horizon lengthens. As seen in the above figure, the volatility smiles exhibit these features for both models. However the smile effect of the generalized option prices is more pronounced and more persistent than those associated with the HW option prices. Indeed the cross-section of the 3-period maturity option is closer to that of the 5-period option in the HW case indicative of a steeper term structure of implied volatilities. Moreover a comparison of the scales against which the smiles are plotted shows that at any given maturity, the curvature of the GBS smiles are greater than those of their preference-free HW counterparts. Finally we also note that, compared to longer maturities, the GBS smiles are more symmetrical around zero at shorter maturities.

4.2.2. The case of a stochastic discount function $\widetilde{B}(t,T)$

The extended option pricing formula in (3.6) depends on the preference parameters through the functions $Q_{XY}(t,T)$ and $\widetilde{B}(t,T)$. When the discount function $\widetilde{B}(t,T)$ is stochastic, preference parameters associated with intertemporal uncertainty resolution affect option prices through the term premium. We investigate in this subsection the effects of these preference parameters, γ and ρ, for the following three cases: the expected utility case where the representative agent is indifferent to the timing of uncertainty resolution, then the cases where she prefers late, and then early, resolution of uncertainty in the sense of Kreps and Porteus (1978). Within each case, we examine the effects of leverage through either the consumption risk or the stock risk, or both, in terms of the schedule of option prices and implied

[23] The same simulated trajectories of the state variable and sequences of residuals were used for the Monte-Carlo integration in each case with varying maturity horizon (T-t).

volatilities. In the figures presented in this section, the results obtained form the generalized option pricing formula in the various cases considered are represented by the solid lines. The benchmark for comparisons, across the various cases considered, are the preference-free option prices and their implied volatilities. These are computed from the generalized option pricing formula modified by setting $Q_{XY}(t,T) = 1$ and $B(t,T) = E_t[\widetilde{B}(t,T)]$, and are represented by the dashed lines in the figures.

In the absence of a leverage effect through the consumption risk, the return on the market portfolio and the mean and variance of consumption growth are predictable one period in advance. Indeed, the no-causality assumption in the general sense of $l(X_t|U_1^T) = l(X_t|U_1^{t-1})$ implies that λ, m_{Xt} and σ^2_{Xt} are function only of U_1^{t-1} such that a one-period option is priced according to (3.6) with $\widetilde{B}(T-1,T) = B(T-1,T)$. That is under the no-causality assumption at the aggregate level, the stochastic discount function entering the option pricing formula is known one period in advance.

The case of absence of a leverage effect through the stock risk while preserving one through the market risk is characterized by a no-causality assumption in the sense of $l(Y_t|U_1^T) = l(Y_t|U_1^{t-1})$. In terms of the option pricing formula (3.6), this no-causality assumption translates into one-period predictability of the following quantity:

$$E[\frac{S_T}{S_t}|U_1^T] = \frac{\varphi(U_1^T)}{\varphi(U_1^t)} \exp(\sum_{\tau=t+1}^{T} m_{Y\tau} + \frac{1}{2}\sum_{\tau=t+1}^{T} \sigma^2_{Y\tau}) \tag{4.5}$$

in addition to having m_{Yt} and σ^2_{Yt} function only of U_1^{t-1}. Notice that φ cannot be expressed as a function only of U_1^{t-1} since there remains a leverage effect through the market risk such that λ, and in turn φ, are function of the contemporaneous value of the state variable. In other words, a complete dichotomy of the stock risk leverage effect from the consumption risk leverage effect is not possible. In the context of a one-period maturity horizon, absence of a leverage effect through the stock risk only is taken to imply that

$$E[\frac{S_T}{S_{T-1}}|U_1^T] = E\left[E[\frac{S_T}{S_{T-1}}|U_1^T] \mid U_1^{T-1}\right] \tag{4.6}$$

such that, at time $T-1$, this quantity is known. That is, the expected gross return is known at the beginning of the investment horizon. Thus we see that in the absence of a leverage effect whether through the consumption or through the stock risk, the greatest uncertainty attenuation occurs for options with a one-period maturity horizon. Since we wish to emphasize the prominent role of leverage effects in option pricing, our study is conducted under various preference parameter configurations for one-period options.

The expected utility case obtains when $\gamma = 1$. We examined this case in two instances, one with $\rho = -1$ such that $\alpha = -1$ and then with $\rho = -20$ such that $\alpha = -20$. In the second instance, the representative agent is more risk averse than in the first instance while remaining within the case of expected utility. The effects in terms of the schedule of option prices and implied volatilities are shown in figures 2-1 through 2-3 for the first instance and in figures 2-4 through 2-6 for the second. In the expected utility case, consumption growth is sufficient to discount asset payoffs as in the consumption capital asset pricing model, see Epstein and Zin (1991) for further discussion. In particular, the return on the market portfolio is seen to have no impact on the determination of option prices when $\gamma = 1$ in the option pricing formula (3.6). A comparison of figure 2-1 to figure 2-2, and figure 2-4 to figure 2-5 reveals that only at a relatively high level of risk aversion $\alpha = -20$ does leverage through the consumption risk have a significant impact on the skewness of the volatility smile. Figures 2-1 and 2-2 are virtually identical indicating that when risk aversion is low, leverage through consumption risk does not matter much. On the other hand, leverage through stock risk has a significant impact. In figures 2-3 and 2-6, this leverage effect is absent and the resulting impact is a skewness bias reversal in the volatility smiles.

In the literature on option pricing, few models have been proposed that provide an economic explanation of this kind of skewness bias reversal. One notable exception is the feedback effect model of Platen and Schweizer (1997). They develop a model which incorporates the technical demand induced by hedging strategies. In their model, hedgers are Black-Scholes program traders with procyclical demand functions. The skewness effect is explained in this framework by the hedging intensity for in-the-money options relative to that for out-of-the-money options. For example, if in-the-money options are expected to be more intensively hedged than out-of-the-money options, then the implied volatilities tend to rise for increasing strike prices. When options out-of-the-money are expected to be more intensively hedged than those in-the-money, the pattern is reversed.

In order to gain an understanding of this kind of skewness bias reversal effect in our context, it is useful to interpret implied volatilities as summarizing the level of uncertainty over the life of the option; see Rubinstein (1994) and Ghysels et al. (1995). Consider then a situation with leverage effects through the consumption and stock risk and in which the implied volatilities, and hence overall uncertainty, of out-of-the-money options are higher than those of in-the-money options. If the stock risk leverage effect is not present, then the stock's expected gross return becomes predictable one period in advance. In such a case the option resembles a riskless bond, if it will be in-the-money. This induces a pattern of decreasing implied volatilities for increasing strike prices, since the more in-the-money an option is the greater its value. Whereas with a high level of uncertainty associated with out-of-the-money options, these levels are attenuated towards zero when there is no leverage effect through the stock risk. This in fact provides an upstream explanation for the kind of skewness reversals observed in the model of Platen and Schweizer (1997). Compared to a situation in which it is in-the-money options that are more intensively hedged, we might expect to observe a shift in hedging intensity towards out-of-the-money options as information about the stock's expected gross return propagates through the market.

We now examine the cases where the representative agent is no longer indifferent to the timing of uncertainty resolution. The parameters for the case of preferences for late resolution of uncertainty are set as $\gamma = 0.05$ and $\rho = -20$ such that $\alpha = -1$. The results for this case are depicted in figures 3-1 through 3-3. In figures 4-1 through 4-2 we report the results for the case of preferences for early resolution of uncertainty where $\gamma = 4$ and $\rho = -0.25$ such that $\alpha = -1$. A comparison of figure 3-1 to 3-2 and figure 4-1 to 4-2, confirms our earlier observation that leverage through consumption risk does not matter much at least at relatively low levels of risk aversion. At a given level of risk aversion, whether the representative agent has preferences for early resolution of uncertainty (figures 4-1 through 4-3) or whether she is indifferent to the timing of uncertainty resolution (figures 2-1 through 2-3), the effects in terms of the option price schedule and implied volatility graphs are observationally equivalent. As in the expected utility case examined above, absence of leverage through the stock risk induces a skewness bias reversal in the implied volatilities.

4.2.3. The smile over time

In our examination of the effects of a deterministic discount function, it was noticed that the generalized BS option prices produced volatility smiles that were more symmetrical at shorter maturities. We turn our attention now to a similar effect that occurs even in the presence of leverage effects through both consumption and stock risk. This subsection compares the evolution of the volatility smiles produced by the generalized option pricing formula to their HW-type counterparts when the maturity horizon is lengthened by one period. In figures 5-1 through 5-3 we revisit the three most interesting cases of the previous subsection. The first two of these figures are for the expected utility case and the last is for the case of preferences for late resolution of uncertainty. These figures are to be compared with their one-period counterpart, respectively figures 2-1, 2-4, and 3-1.

It is interesting to notice the differences in how the volatility smiles flatten depending on whether the representative agent has preferences for late resolution of uncertainty or a high level of risk aversion in the expected utility case. Whereas the volatility smile flattens in a more or less even fashion in figure 5-1, we see that such is not the case in figures 5-2 and 5-3. In the latter cases, the implied volatilities of out-of-the-money options are much less sensitive to the effect of a longer maturity horizon than options in-the-money. In economic terms, an agent with preferences for late resolution of uncertainty or high risk aversion perceives in-the-money options as riskier gambles similar to out-of-the-money options when the investment horizon is longer. Therefore although the volatility smile flattens overall, it is the portion associated with in-the-money options that is more sensitive to the effect of an increasing maturity horizon. Eventually, for a sufficiently long horizon, the implied volatility graph would be completely flat reflecting that the overall level of uncertainty is the same for all options across moneyness.

5. Conclusion

In this paper, we have analyzed the symmetry of the so-called implied volatility smiles, which are often used to characterize the European option pricing biases produced by the Black-Scholes formula. We have stated conditions that an option pricing formula must obey to produce a symmetric volatility smile and translated them into conditions on the pricing probability measure. Since the stochastic volatility extension of the Black-Scholes framework does not reproduce the asymmetric smiles frequently observed, we proposed an option pricing formula with a general stochastic discount factor that generalizes the stochastic volatility option pricing formula. Such a generalization is achieved through a conditioning on state variables. We have shown that two kinds of "generalized" leverage effects may explain (besides the interest rate risk) asymmetric smiles: either a genuine leverage effect, that is an instantaneous correlation between the return on the stock and its stochastic volatility process, or a stochastic correlation between the return of the stock and the stochastic discount factor. These results provide some theoretical foundations to the observed asymmetric smiles. We have also explained how these leverage effects determine if the option pricing formula is preference-free or not.

Through an equilibrium stochastic discount factor and a Markov regime-switching process for the state variables, we have shown that the model leads itself to a computable formula that can reproduce many of the shapes observed for the implied volatility curves. The remaining task is to show that the parameters estimated from the data in such an extended framework can be used to achieve smaller pricing or hedging errors out of sample. We leave such a task for future research.

Appendix 1
Appendix 1

Proof of Proposition 2.1:

We first check that, for any given value of σ,the function $\pi(.) = BS(., \sigma)$ fulfills the announced property:

$$\pi(-x) = e^x \pi(x) + 1 - e^x.$$

Indeed, from (2.2) and (2.5):

$$BS(x, \sigma) = \Phi[d_1(x, \sigma)] - e^{-x} \Phi[d_2(x, \sigma)],$$

with: $d_1(x, \sigma) = \frac{x}{\sigma} + \frac{\sigma}{2}, d_2(x, \sigma) = \frac{x}{\sigma} - \frac{\sigma}{2}$.

But: $\Phi[d_2(-x, \sigma)] = \Phi[-d_1(x, \sigma)] = 1 - \Phi[d_1(x, \sigma)]$,and: $\Phi[d_1(-x, \sigma)] = \Phi[-d_2(x, \sigma)] = 1 - \Phi[d_2(x, \sigma)]$.

Therefore:

$$\begin{aligned} BS(-x, \sigma) &= \Phi[d_1(-x, \sigma)] - e^x \Phi[d_2(-x, \sigma)] \\ &= e^x \Phi[d_1(x, \sigma)] - \Phi[d_2(x, \sigma)] + 1 - e^x \\ &= e^x BS(x, \sigma) + 1 - e^x. \end{aligned}$$

Let us now consider another homogeneous option pricing formula $x \to \pi(x)$. The associated BS implied volatilities are then defined by:

$$\begin{aligned} \pi(x) &= BS[x, \sigma^*(x)], \\ \pi(-x) &= BS[-x, \sigma^*(-x)]. \end{aligned}$$

Therefore, for any x:

$$\begin{aligned} \sigma^*(x) &= \sigma^*(-x) \\ &\iff \pi(-x) = BS[-x, \sigma^*(x)] \\ &\iff \pi(-x) = e^x BS[x, \sigma^*(x)] + 1 - e^x \\ &\iff \pi(-x) = e^x \pi(x) + 1 - e^x. \blacksquare \end{aligned}$$

Proof of Proposition 2.2:

a) First, we prove that the criterion of Proposition 2.1 is equivalent to the property (i) of Proposition 2.2. We can write (**??**) as:

$$\pi_t(S_t, K) = B(t,T)S_t \int_{\frac{K}{S_t}}^{+\infty} \left(\frac{S_T}{S_t} - \frac{K}{S_t}\right) dQ_{t,T}\left(\frac{S_T}{S_t}\right)$$

Therefore, by taking the derivative with respect to K, we obtain the well-known relationship between the option pricing formula and the pricing probability measure.

$$\begin{aligned} \frac{\partial \pi}{\partial K}(S_t, K) &= -B(t,T)Q_t\left[\frac{S_T}{S_t} \geq \frac{K}{S_t}\right] \\ &= -B(t,T)[1 - F_{V_T}(-x)]. \end{aligned}$$

Since, from (2.5):

$$\frac{\partial \pi}{\partial x}(x) = \frac{\partial}{\partial x}[\pi(1, \frac{K}{S_t})] = -\frac{K}{S_t}\frac{\partial}{\partial K}\pi(S_t, K)$$

we have, for any x :

$$\frac{\partial \pi}{\partial x}(x) = e^{-x}[1 - F_{V_T}(-x)].$$

Therefore, the property (i) of Proposition (2.2) may be rewritten as:

$$\pi(x) = 1 - e^{-x}\frac{\partial \pi}{\partial x}(-x) - \frac{\partial \pi}{\partial x}(x)$$

or equivalently:

$$-\frac{\partial \pi}{\partial x}(-x) = e^{x}[\pi(x) + \frac{\partial \pi}{\partial x}(x) - 1].$$

This last equality is obviously a corollary of proposition (2.2) obtained by taking the derivative with respect to x of the identity in Proposition (2.2). Conversely, this equality implies that for any x :

$$-\int_x^{+\infty} \frac{\partial \pi}{\partial u}(-u)du = \int_x^{+\infty} e^{u}[\pi(u) + \frac{\partial \pi}{\partial u}(u) - 1]du$$

This equation will provide the criterion of Proposition (2.2) if we are able to complete it by the following limit condition:

$$\lim_{x \to +\infty} \pi(-x) = \lim_{x \to +\infty} [e^{x}\pi(x) + 1 - e^{x}].$$

Therefore, the required equivalence will be proved if we show that this limit condition is always guaranteed. But, on the one hand:

$$\begin{aligned}\lim_{x\to+\infty}\pi(-x) &= \lim_{x\to-\infty}\pi(x)\\ &= \lim_{K\to+\infty}B(t,T)E_t^*Max[0,S_T-K]=0\end{aligned}$$

by virtue of the Lebesgue dominated convergence theorem since: $Max[0,S_T-K]\longrightarrow_{K\to\infty} 0$ almost surely and $0\leq Max[0,S_T-K]\leq S_T$, which is by assumption integrable with respect to the pricing probability measure. On the other hand:

$$\begin{aligned}\lim_{x\to+\infty}e^x[\pi(x)-1]+1 &= 1+\lim_{K\to0^+}\frac{1}{KB(t,T)}\{B(t,T)E_t^*Max[0,S_T-K]-B(t,T)E_t^*S_T\}\\ &= 1+\lim_{K\to0^+}\frac{1}{K}E_t^*Max[-S_T,-K]\\ &= 1-\lim_{K\to0^+}E_t^*Min[\frac{S_T}{K},1]\\ &= -\lim_{K\to0^+}E_t^*Min[\frac{S_T}{K}-1,0]=0\end{aligned}$$

by virtue of the Lebesgue dominated convergence theorem since: $Min[\frac{S_T}{K}-1,0]\longrightarrow_{K\to0^+} 0$ almost surely and $0\leq -Min[\frac{S_T}{K}-1,0]\leq 1$. This proves that: $\lim_{x\to+\infty}\pi(-x)=0=\lim_{x\to+\infty}[e^x\pi(x)+1-e^x]$ and completes the proof of the required equivalence.

b) We now check that properties (i) and (ii) of Proposition 9 are equivalent. The general definition (2.1) of the pricing probability measure implies that:

$$\pi_t(S_t,K)=B(t,T)E_t^*[S_T\mathbf{1}_{[S_T\geq K]}-B(t,T)KQ_t[S_T\geq K],$$

that is, after dividing by S_t :

$$\pi(x)=E_t^*[e^{V_T}\mathbf{1}_{[V_T\geq -x]}]-e^{-x}[1-F_{V_T}(-x)]$$

By identification of this formula with condition (i), we see that (i) is equivalent to (ii).

c) Finally, we prove that conditions (i) and (iii) are equivalent. By taking the derivative of (i), we obtain:

$$\frac{\partial\pi}{\partial x}(x)=f_{V_T}(x)-e^{-x}f_{V_T}(-x)+e^{-x}[1-F_{V_T}(-x)].$$

But, since by part a) of this proof:

$$\frac{\partial \pi}{\partial x}(x) = e^{-x}[1 - F_{V_T}(-x)]$$

we conclude that (i) implies:

$$f_{V_T}(x) = e^{-x} f_{V_T}(-x)$$

or:

$$e^{\frac{x}{2}} f_{V_T}(x) = e^{-\frac{x}{2}} f_{V_T}(-x)$$

which means that the function $x \rightarrow e^{\frac{x}{2}} f_{V_T}(x)$ is even, which is exactly condition (iii) of Proposition 9. Conversely, if this condition is fulfilled, we have, for any x:

$$\int_x^{+\infty} f_{V_T}(u)du = \int_x^{+\infty} e^{-u} f_{V_T}(-u)du.$$

This equation will provide property (i) of proposition 9 if we complete it by the following limit condition:

$$\lim_{x \rightarrow +\infty} \pi(x) = \lim_{x \rightarrow +\infty} [F_{V_T}(x) - e^{-x}[1 - F_{V_T}(-x)]].$$

Therefore, the required equivalence will be proved is we show that this limit condition always holds. But it is clear that:

$$\lim_{x \rightarrow +\infty} [F_{V_T}(x) - e^{-x}[1 - F_{V_T}(-x)]] = \lim_{x \rightarrow +\infty} F_{V_T}(x) = 1$$

and that :$\lim_{x \rightarrow +\infty} \pi(x) = 1$, since we have already shown in part a) of this proof that: $\lim_{x \rightarrow +\infty} e^x[\pi(x) - 1] = -1$. This completes the proof.■

Appendix 2

A. Proof of Proposition (2.4):

From (2.11) and the decomposition of $m_{t,T}$ conformable to (A1) and (A2):

$$\frac{\boldsymbol{\pi}_t}{S_t} = E_t\left\{\lambda_{t,T}(U_1^T)E\left[(\prod_{\tau=t}^{T-1} m_{\tau+1})\left[(\prod_{\tau=t}^{T-1}\frac{S_{\tau+1}}{S_\tau}) - \frac{K}{S_t}\right]^+ |I_t, U_1^T\right]\right\}$$

But, by (A2), the variables $(m_{\tau+1,}\frac{S_{\tau+1}}{S_\tau})_{\tau\geq t}$ are independent of $(m_{\tau+1,}\frac{S_{\tau+1}}{S_\tau})_{\tau<t}$ given U_1^T. Therefore:

$$\frac{\boldsymbol{\pi}_t}{S_t} = E_t\left\{\lambda_{t,T}(U_1^T)E\left[(\prod_{\tau=t}^{T-1} m_{\tau+1})\left[(\prod_{\tau=t}^{T-1}\frac{S_{\tau+1}}{S_\tau}) - \frac{K}{S_t}\right]^+ |U_1^T\right]\right\}$$

is a conditional expectation computed in the conditional probability distribution of U_1^T given I_t. By (A1), this probability distribution depends on I_t only through U_1^t. We are then allowed to denote this expectation by $\Psi_{t,T}(U_1^t, \frac{K}{S_t})$.

B. Proof of Proposition 2.5:

In what follows, we will derive a closed-form formula for $\Psi_{t,T}(U_1^t, \frac{K}{S_t})$ based on the log-normality assumption. We will start from the following decomposition:

$$\Psi_{t,T}(U_1^t, \frac{K}{S_t}) = E_t\left\{\lambda_{t,T}(U_1^T)\left[G_{t,T}(U_1^T) - \frac{K}{S_t}H_{t,T}(U_1^T)\right]\right\}$$

where:

$$G_{t,T}(U_1^T) = E\left[(\prod_{\tau=t}^{T-1} m_{\tau+1}\frac{S_{\tau+1}}{S_\tau})\mathbf{1}_{[S_T\geq K]}|U_1^T\right]$$

and:

$$H_{t,T}(U_1^T) = E_t\left[\prod_{\tau=t}^{T-1} m_{\tau+1}\mathbf{1}_{[S_T\geq K]}|U_1^T\right].$$

a) **Lemma 1 :** *If* $\begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix}$ *is a bivariate Gaussian vector, with:*

$E\begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} = \begin{pmatrix} m_1 \\ m_2 \end{pmatrix}, Var\begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} = \begin{pmatrix} \omega_1^2 & \rho\omega_1\omega_2 \\ \rho\omega_1\omega_2 & \omega_2^2 \end{pmatrix}$

$E[\exp(Z_1)\mathbf{1}_{[Z_2\geq 0]}] = \exp[m_1 + \frac{\omega_1^2}{2}]\Phi(\frac{m_2}{\omega_2} + \rho\omega_1)$, with Φ the cumulative normal distribution function.

Let us by Q the probability measure corresponding to the above-specified Gaussian distribution of (Z_1, Z_2) and define the probability $\widetilde{Q}$ by:

$$\frac{d\widetilde{Q}}{dQ}(Z) = \exp[(Z - m_1) - \frac{\omega_1^2}{2}].$$

Then, with obvious notation:

$E[(\exp Z_1)(\mathbf{1}_{[Z_2 \geq 0]})] \; = \exp(m_1 + \frac{\omega_1^2}{2})\widetilde{Q}[Z_2 \geq 0]$

But by Girsanov theorem, we know that under $\widetilde{Q}$, Z_2 is a Gaussian variable with mean $m_2 + \rho\omega_1\omega_2$ and variance ω_2^2. Therefore:

$$\widetilde{Q}[Z_2 \geq 0] = 1 - \Phi[\frac{-m_2 - \rho\omega_1\omega_2}{\omega_2}] = \Phi[\frac{m_2}{\omega_2} + \rho\omega_1]$$

C. A closed-form formula for $H_{t,T}(U_1^T)$ and bond pricing:

$$H_{t,T}(U_1^T) = E_t\left[\exp[\sum_{\tau=t}^{T-1}\log m_{\tau+1}]\mathbf{1}_{[\sum_{\tau=t}^{T-1}\log\frac{S_{\tau+1}}{S_\tau} \geq \log\frac{K}{S_t}]}|U_1^T\right]$$

By virtue of assumption A, this expectation is given by lemma 1 with:

$$Z_1 = \sum_{\tau=t}^{T-1}\log m_{\tau+1} \text{ and } Z_2 = \sum_{\tau=t}^{T-1}\log\frac{S_{\tau+1}}{S_\tau} - \log\frac{K}{S_t}$$

so

$$\begin{aligned}
m_1 &= \sum_{\tau=t}^{T-1}\mu_{m_{\tau+1}} \\
m_2 &= \sum_{\tau=t}^{T-1}\mu_{S_{\tau+1}} - \log\frac{K}{S_t} \\
\omega_1^2 &= \sum_{\tau=t}^{T-1}\sigma^2_{m_{\tau+1}} \\
\omega_2^2 &= \sum_{\tau=t}^{T-1}\sigma^2_{S_{\tau+1}} \\
\rho\omega_1\omega_2 &= \sum_{\tau=t}^{T-1}\sigma_{mS_{\tau+1}}
\end{aligned}$$

Therefore:

$$H_{t,T}(U_1^T) = \exp\left[\sum_{\tau=t}^{T-1}\mu_{m_{\tau+1}} + \frac{1}{2}\sum_{\tau=t}^{T-1}\sigma^2_{m_{\tau+1}}\right]\Phi\left(\frac{1}{\sqrt{\sum_{\tau=t}^{T-1}\sigma^2_{S_{\tau+1}}}}(\sum_{\tau=t}^{T-1}\mu_{S_{\tau+1}} - \log\frac{K}{S_t} + \sum_{\tau=t}^{T-1}\sigma_{mS_{\tau+1}})\right)$$

By referring to the notation introduced in proposition 2.5, we first notice that $H_{t,T}(U_1^T)$ can be written as:

$$H_{t,T}(U_1^T) = \frac{\widetilde{B}(t,T)}{\lambda_{t,T}(U_1^T)}\Phi\left(d_2(x_t)\right)$$

with $x_t = \log\frac{S_t}{KB(t,T)}$ and $d_2(x_t)$ defined in proposition 2.5 since:

$$\frac{1}{\sqrt{\sum_{\tau=t}^{T-1}\sigma^2_{S_{\tau+1}}}}(\sum_{\tau=t}^{T-1}\mu_{S_{\tau+1}} - \log\frac{K}{S_t} + \sum_{\tau=t}^{T-1}\sigma_{mS_{\tau+1}}) = \frac{1}{\overline{\sigma}_{t,T}}(x_t + \log B(t,T) + \sum_{\tau=t}^{T-1}\mu_{S_{\tau+1}} + \sum_{\tau=t}^{T-1}\sigma_{mS_{\tau+1}})$$

But:

$$E_t\left[\frac{S_T}{S_t}|U_t^T\right] = \exp(\sum_{\tau=t}^{T-1}\mu_{S_{\tau+1}} + \frac{1}{2}\overline{\sigma}^2_{t,T})$$

Therefore, the above expression can be rewritten as:

$$\begin{aligned}
& \frac{1}{\overline{\sigma}_{t,T}}\left(x_t - \frac{1}{2}\overline{\sigma}^2_{t,T} + \log\left(E_t\left[\frac{S_T}{S_t}|U_t^T\right]B(t,T)]\right) + \sum_{\tau=t}^{T-1}\sigma_{mS_{\tau+1}}\right) \\
= \; & \frac{x_t}{\overline{\sigma}_{t,T}} - \frac{\overline{\sigma}_{t,T}}{2} + \frac{1}{\overline{\sigma}_{t,T}}\log\left(Q_{mS}(t,T)\frac{B(t,T)}{\widetilde{B}(t,T)}\right) \\
= \; & d_1(x_t) - \overline{\sigma}_{t,T} = d_2(x_t)
\end{aligned}$$

where $d_1(x_t), d_2(x_t)$ and $Q_{mS}(t,T)$ correspond to the expressions given in proposition 2.5.

Finally, it is worth noticing that $\widetilde{B}(t,T)$ can be interpreted in terms of bond pricing. Actually, the general pricing formula (2.11) implies that:

$$B(t,T) = E_t[m_{t,T}] = E_t\left[\lambda_{t,T}(U_1^T)H_{t,T}(U_1^T)\right]$$

when $H_{t,T}(U_1^T)$ is computed in the limit case $K = +\infty$, that is,

$$H_{t,T}(U_1^T) = \frac{\widetilde{B}(t,T)}{\lambda_{t,T}(U_1^T)}, \text{ since } \lim_{K\to+\infty} d_2(x_t) = +\infty$$

therefore the bond pricing equation is given by:

$$B(t,T) = E_t[\widetilde{B}(t,T)]$$

D. A closed-form formula for $G_{t,T}(U_1^T)$ and stock pricing:

$$G_{t,T}(U_1^T) = E\left[\exp(\sum_{\tau=t}^{T-1} \log m_{\tau+1} + \log \frac{S_{\tau+1}}{S_\tau})\mathbf{1}_{[\sum_{\tau=t}^{T-1} \log \frac{S_{\tau+1}}{S_\tau} \geq \log \frac{K}{S_t}]} | U_1^T\right]$$

But, by virtue of assumption A, this expectation is given by lemma 1 with:

$$Z_1 = \sum_{\tau=t}^{T-1} \log m_{\tau+1} + \log \frac{S_{\tau+1}}{S_\tau} \text{ and } Z_2 = \sum_{\tau=t}^{T-1} \log \frac{S_{\tau+1}}{S_\tau} - \log \frac{K}{S_t}$$

In other words, with respect to part C above, m_2 and ω_2^2 are unchanged while now:

$$\begin{aligned} m_1 &= \sum_{\tau=t}^{T-1} (\mu_{m_{\tau+1}} + \mu_{S_{\tau+1}}) \\ \omega_1^2 &= \sum_{\tau=t}^{T-1} (\sigma^2_{m_{\tau+1}} + \sigma^2_{S_{\tau+1}} + 2\sigma_{mS_{\tau+1}}) \\ \rho\omega_1\omega_2 &= \sum_{\tau=t}^{T-1} (\sigma_{mS_{\tau+1}} + \sigma^2_{S_{\tau+1}}) \end{aligned}$$

Therefore:

$$\begin{aligned} G_{t,T}(U_1^T) &= \exp\left[\sum_{\tau=t}^{T-1} (\mu_{m_{\tau+1}} + \mu_{S_{\tau+1}}) + \frac{1}{2}\sum_{\tau=t}^{T-1} (\sigma^2_{m_{\tau+1}} + \sigma^2_{S_{\tau+1}} + 2\sigma_{mS_{\tau+1}})\right] \times \\ & \Phi\left(\frac{1}{\sqrt{\sum_{\tau=t}^{T-1} \sigma^2_{S_{\tau+1}}}}\left[\sum_{\tau=t}^{T-1} \mu_{S_{\tau+1}} - \log \frac{K}{S_t} + \sum_{\tau=t}^{T-1} (\sigma_{mS_{\tau+1}} + \sigma^2_{S_{\tau+1}})\right]\right) \end{aligned}$$

But comparison with the above expressions of $H_{t,T}(U_1^T)$ and $E_t\left[\frac{S_T}{S_t} | U_t^T\right]$ we see that:

$$\begin{aligned} G_{t,T}(U_1^T) &= \frac{\widetilde{B}(t,T)}{\lambda_{t,T}(U_1^T)} \exp\left[\sum_{\tau=t}^{T-1} (\mu_{m_{\tau+1}} + \mu_{S_{\tau+1}}) + \frac{1}{2}\sum_{\tau=t}^{T-1} (\sigma^2_{S_{\tau+1}} + 2\sigma_{mS_{\tau+1}})\right] \Phi\left(d_2(x_t) + \overline{\sigma}_{t,T}\right) \\ &= \frac{\widetilde{B}(t,T)}{\lambda_{t,T}(U_1^T)} E_t\left[\frac{S_T}{S_t} | U_t^T\right] \exp\left[\sum_{\tau=t}^{T-1} \sigma_{mS_{\tau+1}}\right] \Phi\left(d_1(x_t)\right) \end{aligned}$$

that is,

$$G_{t,T}(U_1^T) = \frac{Q_{mS}(t,T)}{\lambda_{t,T}(U_1^T)} \Phi\left(d_1(x_t)\right)$$

Finally, it is worth noticing that $Q_{mS}(t,T)$ can be interpreted in terms of stock pricing. Actually the stock pricing equation corresponds to the general pricing formula (2.11) in the limit case $K = 0$, that is:

$$S_t = E_t\left[\lambda_{t,T}(U_1^T) S_t G_{t,T}(U_1^T)\right]$$

where

$$G_{t,T}(U_1^T) = \frac{Q_{mS}(t,T)}{\lambda_{t,T}(U_1^T)}, \text{ since } \lim_{K\to 0} d_1(x_t) = +\infty$$

In other words, the stock pricing equation can be written:

$$1 = E_t[Q_{mS}(t,T)]$$

E. Option pricing formula:

We conclude from parts A, B and C above that the option pricing formula $\boldsymbol{\pi}_t$ is given by:

$$\begin{aligned}
\frac{\boldsymbol{\pi}_t}{S_t} &= E_t\left[\lambda_{t,T}(U_1^T) G_{t,T}(U_1^T) - \frac{K}{S_t}\lambda_{t,T}(U_1^T) H_{t,T}(U_1^T)\right] \\
&= E_t\left[Q_{mS}(t,T)\Phi\left(d_1(x_t)\right) - \frac{K\widetilde{B}(t,T)}{S_t}\Phi\left(d_2(x_t)\right)\right]
\end{aligned}$$

which coincides with the announced formula of proposition 2.5 since:

$$\frac{K\widetilde{B}(t,T)}{S_t} = \frac{\widetilde{B}(t,T)}{B(t,T)}\frac{KB(t,T)}{S_t} = \frac{\widetilde{B}(t,T)}{B(t,T)}\exp(-x_t)$$

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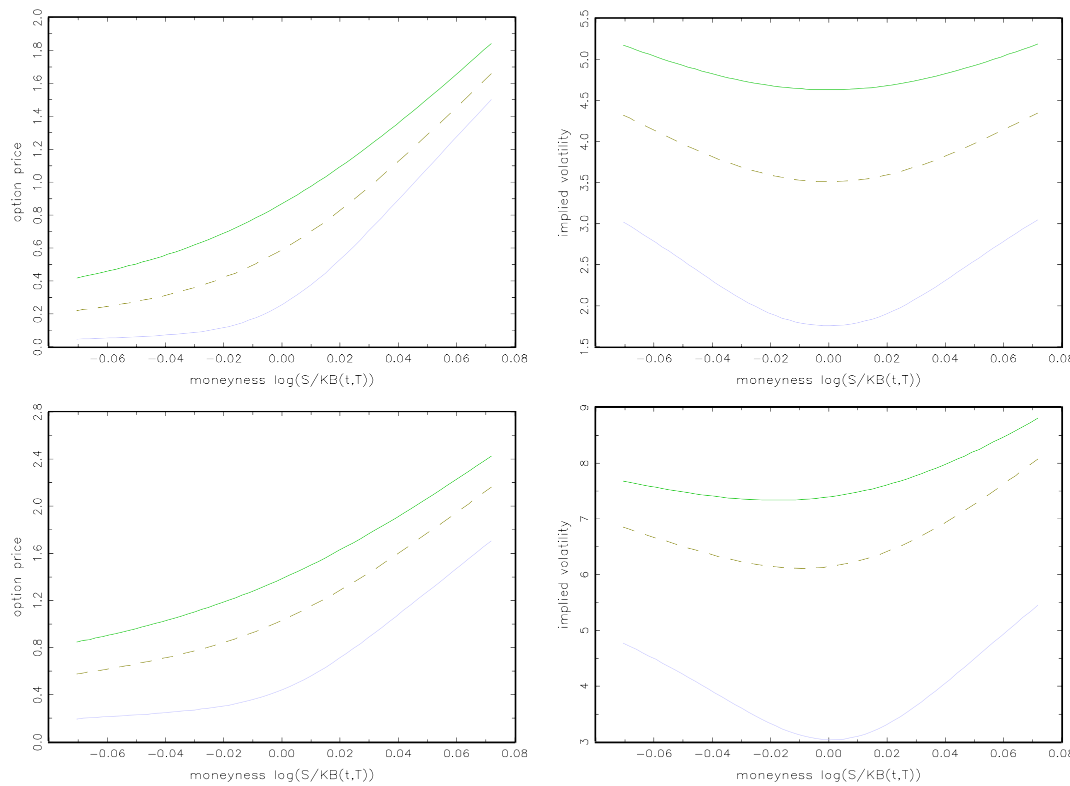

Figure 1-1: Top: HW prices and volatility smiles; Bottom: Generalized BS prices and volatility smiles.

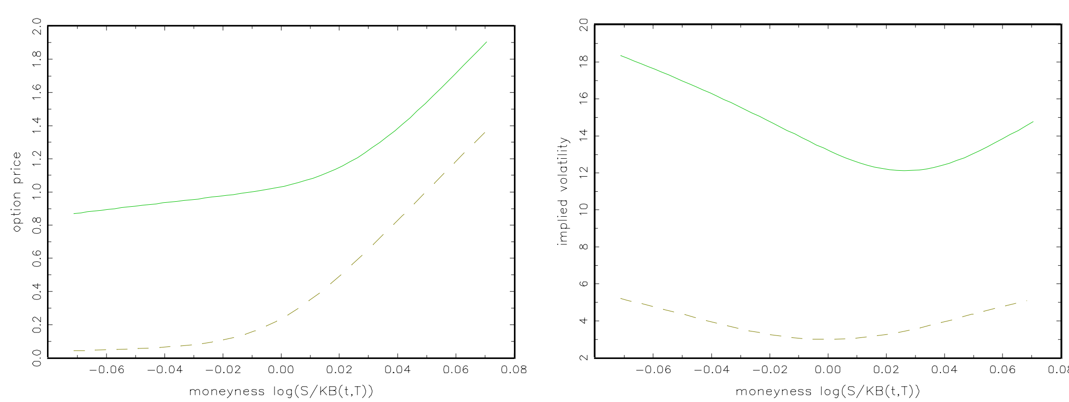

Figure 2-1: Leverage through consumption and stock risk.

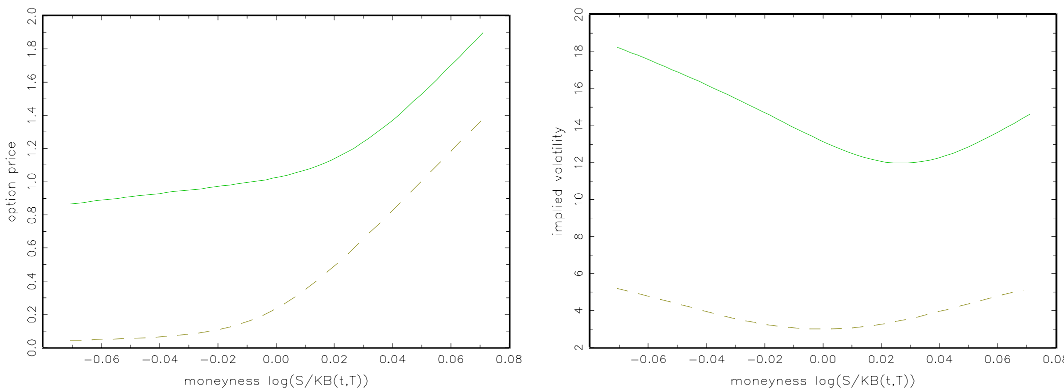

Figure 2-2: Leverage through stock risk only.

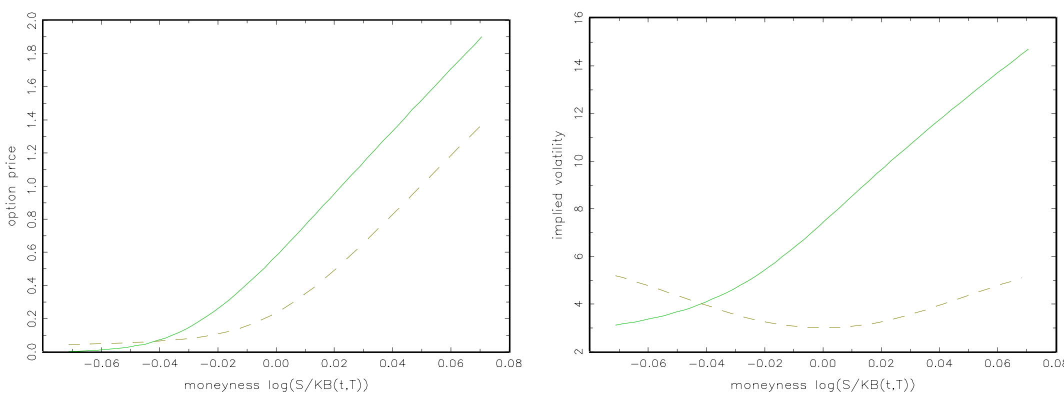

Figure 2-3: Leverage through consumption risk only.

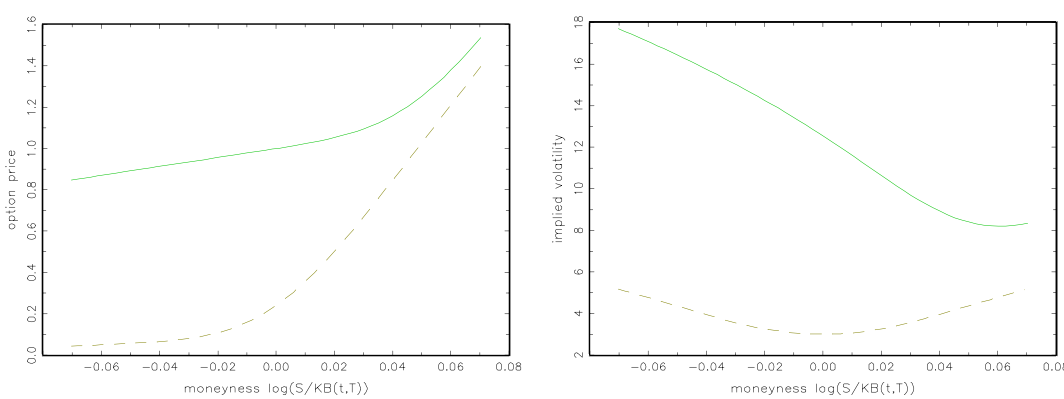

Figure 2-4: Leverage through consumption and stock risk.

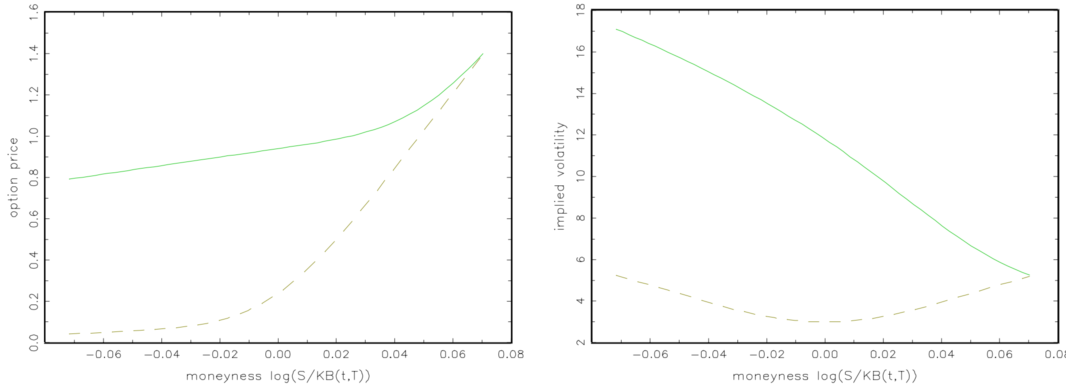

Figure 2-5: Leverage through stock risk only.

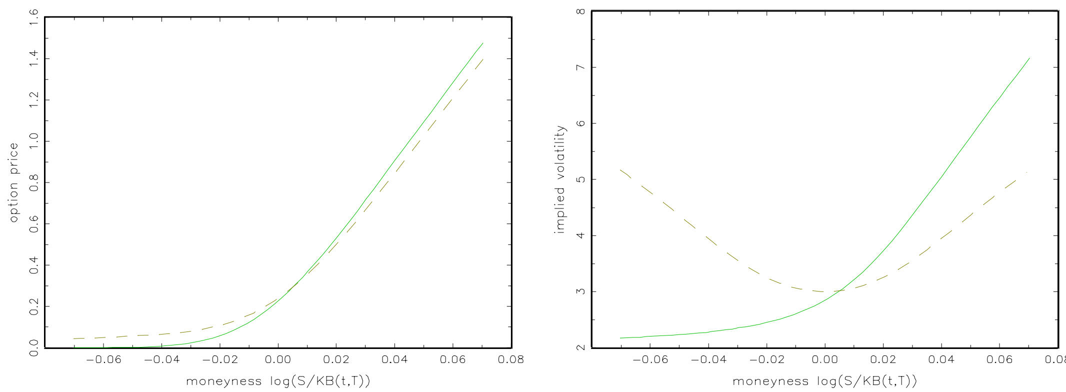

Figure 2-6: Leverage through consumption risk only.

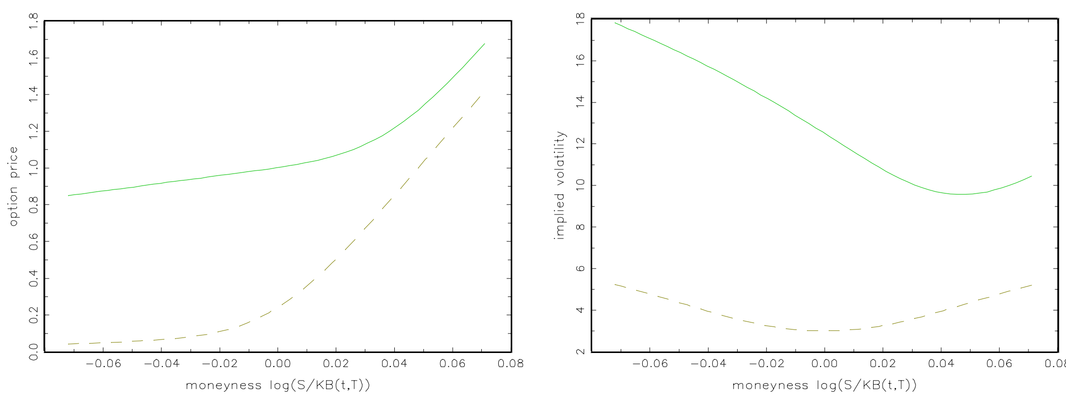

Figure 3-1: Leverage through consumption and stock risk.

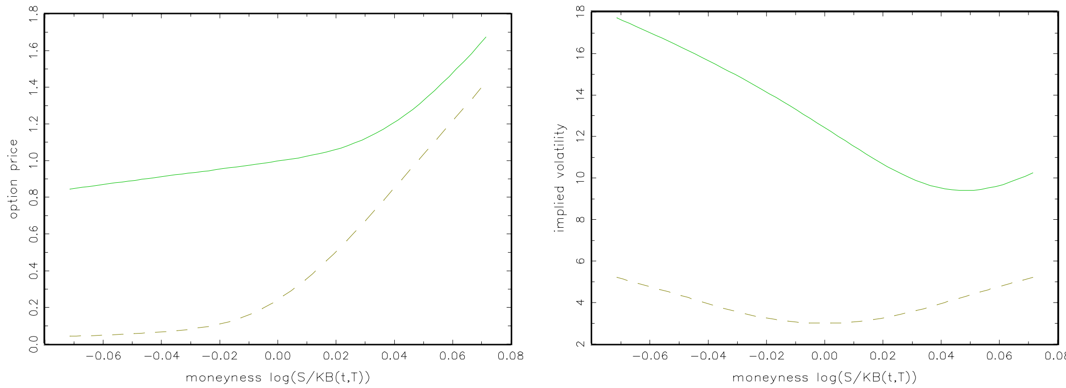

Figure 3-2: Leverage through stock risk only.

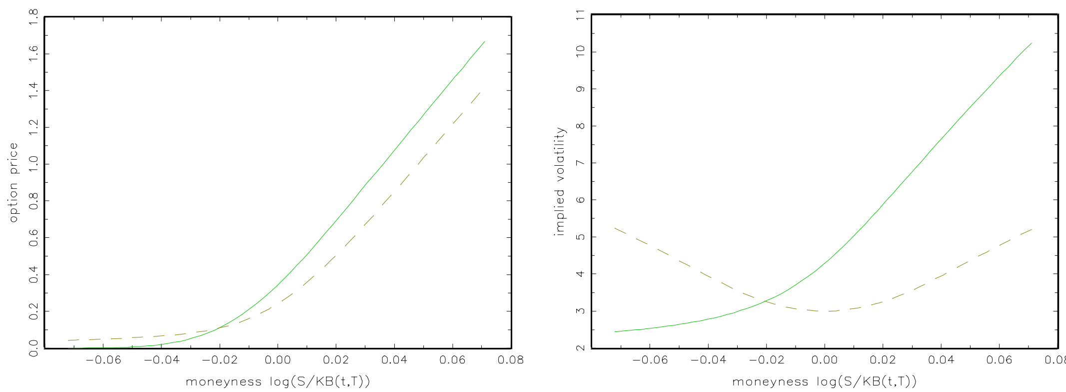

Figure 3-3: Leverage through consumption risk only.

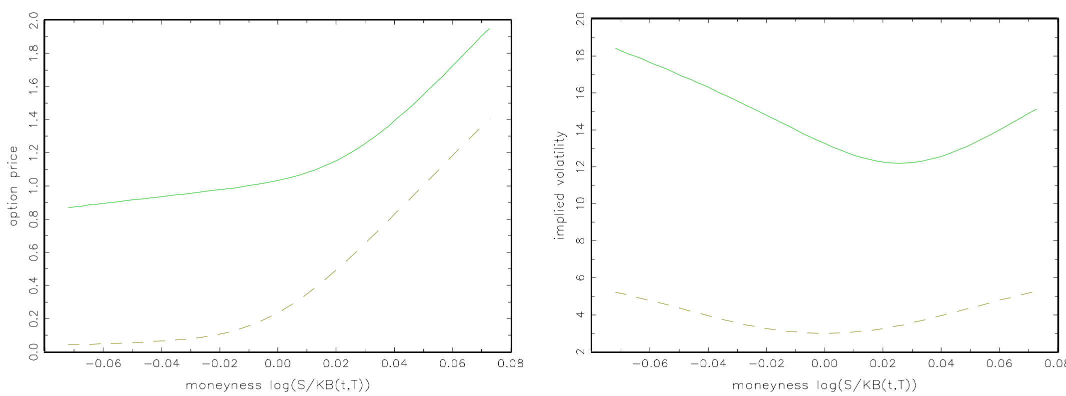

Figure 4-1: Leverage through consumption and stock risk.

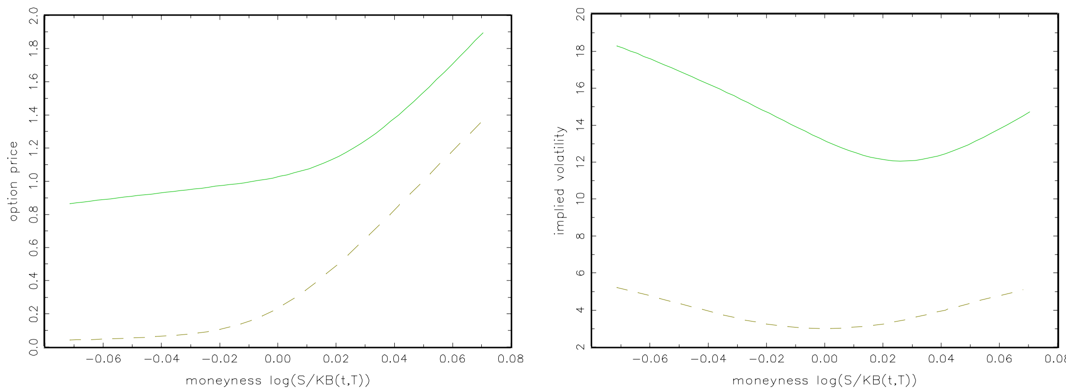

Figure 4-2: Leverage through stock risk only.

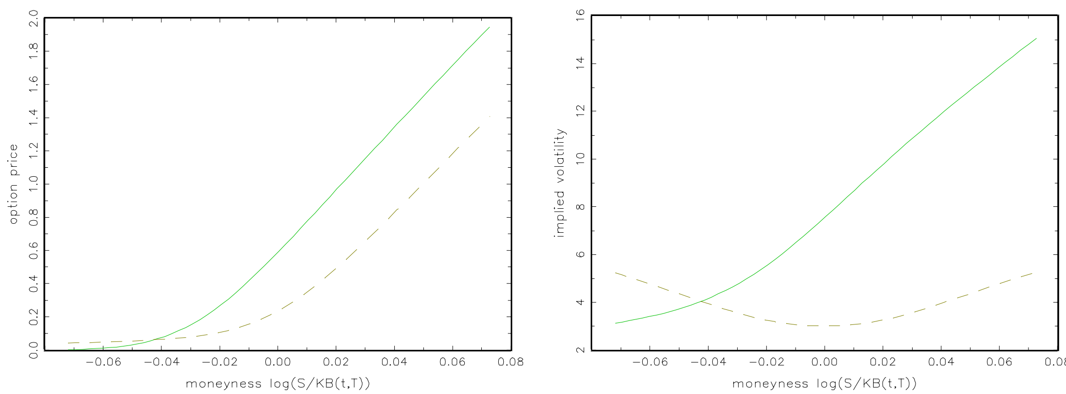

Figure 4-3: Leverage through consumption risk only.

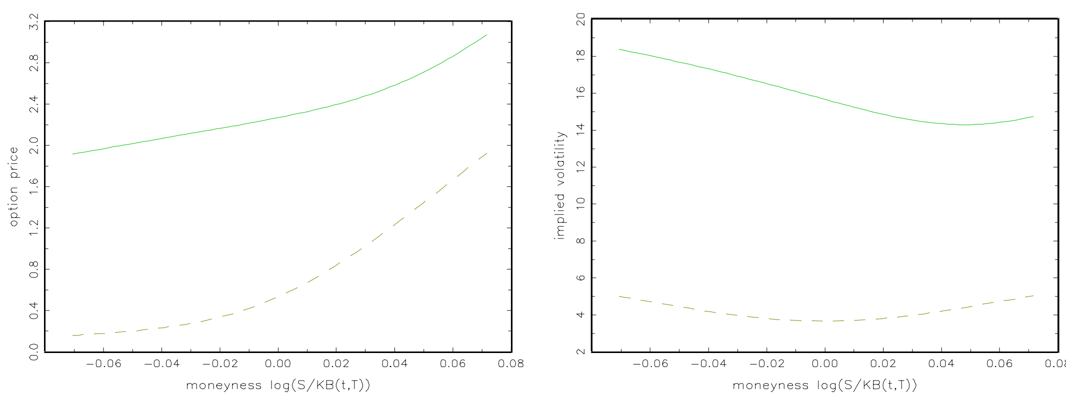

Figure 5-1: Expected utility ($\gamma = 1$) with $\rho = -1$.

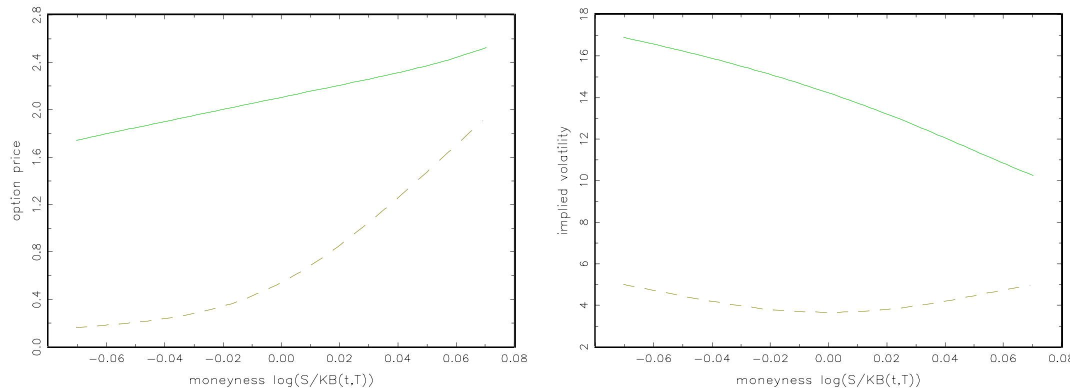

Figure 5-2: Expected utility ($\gamma = 1$) with $\rho = -20$.

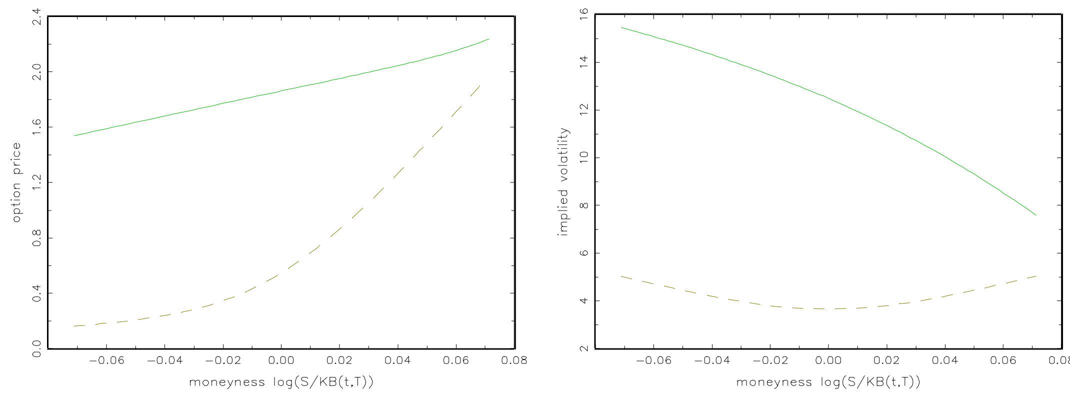

Figure 5-3: Preferences for late resolution of uncertainty with $\gamma = 0.05$ and $\rho = -20$.