EXACT PROPERTIES OF SPECTRAL ESTIMATES FOR A GAUSSIAN PROCESS ON THE CIRCLE

Roch Roy and Jean-Marie Dufour

ABSTRACT. A homogeneous random process on the circle $\{X(P): P \in S\}$ is a process whose mean is constant and whose covariance function depends only on the angular distance θ between the two points; i.e., $E[X(P)] \equiv \mu$ and $\text{cov}(X(P), X(Q)) = R(\theta)$. Given T independent realizations of a Gaussian homogeneous process $X(P)$, we first propose estimates of the mean and of the spectral parameters. The exact distribution of these estimates is derived. Further, an estimate $R^{(T)}(\theta)$ of the covariance function $R(\theta)$ is proposed. Exact expressions for its first and second-order moments are derived and it is shown that the sequence of processes $\{T^{\frac{1}{2}}[R^{(T)}(\theta) - R(\theta)]\}_{T=1}^{\infty}$ converges weakly in $C[0,\pi]$ to a given Gaussian process.

1. *Introduction.*

Let $\{X(P): P \in S\}$ be a real-valued process on the unit circle S of the two-dimensional space R^2, which has finite second-order moment and which is continuous in quadratic mean (q.m.). Under these conditions, the process $X(P)$ can be expanded in a Fourier series which is convergent in q.m.:

$$X(P) = C_{01} + \sum_{n=1}^{\infty}\{C_{n1}\cos(nP) + C_{n2}\sin(nP)\}\,, \tag{1.1}$$

where

$$\begin{cases} C_{01} = \frac{1}{2\pi}\int_0^{2\pi} X(P)\,dP\,, \\ C_{n1} = \frac{1}{\pi}\int_0^{2\pi} X(P)\cos(nP)\,dP\,, \\ C_{n2} = \frac{1}{\pi}\int_0^{2\pi} X(P)\sin(nP)\,dP,\ n \geq 1\,. \end{cases} \tag{1.2}$$

UTILITAS MATHEMATICA, Vol. 5 (1974), pp. 281-291.

The integrals in (1.2) are defined in the q.m. sense and the series (1.1) converges in q.m.

The process X(P) is said to be homogeneous if its first and second-order moments are invariant under the group of rotations of the circle. This is equivalent to say that the mean $E[X(P)] \equiv \mu$, a constant, and that the covariance function cov(X(P), X(Q)) depends only on the angular distance Θ_{PQ} between the points P and Q. Obviously, $E[X(P)] \equiv \mu$ implies that

$$(1.3) \qquad E[C_{ni}] = \begin{cases} \mu & \text{if } n = 0,\ i = 1, \\ 0 & \text{if } n > 0,\ i = 1, 2 . \end{cases}$$

Also, from Yaglom [9], theorem 5 (see Roy [6] for a more elementary proof), the homogeneity property implies that the coefficients C_{ni} are uncorrelated; i.e.,

$$(1.4) \qquad \text{cov}(C_{ni}, C_{mj}) = \delta_{ij}\ \delta_{nm}\ a_n \geq 0 ,$$

for all possible values of i, j, n, m, δ being the Kronecker delta. From (1.1) and (1.4), it is easily deduced that

$$(1.5) \qquad \text{cov}(X(P), X(Q)) = R(\Theta_{PQ}) = \sum_{n=0}^{\infty} a_n \cos(n\Theta_{PQ}),\ \Theta_{PQ} \varepsilon [0,\pi] ,$$

where the spectral parameters a_n are defined by (1.4) and satisfy

$$(1.6) \qquad \sum_{n=0}^{\infty} a_n < \infty .$$

An analysis of data from a process on the circle is presented in Benton and Kahn [1]. See also Hannan [4] for application in hydrology.

The purpose of the paper is to develop a spectral analysis when independent realizations of the process on the circle are available. In practice, the process is sampled at a finite number of points for each realization. However, if the observations are fairly evenly distributed over the circle, one will be able to evaluate numerically the Fourier coefficients and the results presented here will be applicable.

The case of realizations stationary in time has been studied by Roy [6]; however, all the results obtained there are asymptotic in nature. By taking advantage of the independence of the realizations, one will be able to deduce some exact results. In section 2, estimates of the mean and of the spectral parameters are proposed and some of their exact properties are deduced in the case of a Gaussian process. In section 3, a covariance function estimate is presented and some of its statistical properties are derived.

2. *Spectral estimates.*

In the following, we will say that the process $X(P)$ is Gaussian if, for any finite collection of points $P_1, P_2, \ldots, P_k \in S$, $X(P_1), \ldots, X(P_k)$ have a joint normal distribution. For a Gaussian process, the coefficients C_{ni} have a joint normal distribution since they are defined as q.m. integrals and since the q.m. limit of a sequence of finite linear combinations of jointly normal variables is normal. Using (1.3) and (1.4), we deduce that the coefficients C_{ni} are mutually independent with C_{01} being $N(\mu, a_0)$ and C_{ni} being $N(0, a_n)$ for $i = 1, 2$, $n = 1, 2, \ldots$.

Given $T (T > 1)$ independent realizations of the process on the circle: $\{X(P,t): P \in S\}$, $t = 1, \ldots, T$, we can compute the coefficients $C_{ni}(t)$ corresponding to the t^{th} realization, $t = 1, \ldots, T$. By the previous remarks, we see that the random variables $C_{01}(t)$, $t = 1, \ldots, T$ are independent and identically distributed $N(\mu, a_0)$. So, the usual estimates of μ and a_0 are given respectively by

$$\mu^{(T)} = \frac{1}{T}\sum_{t=1}^{T} C_{01}(t), \qquad a_0^{(T)} = \frac{1}{T-1}\sum_{t=1}^{T}\left(C_{01}(t) - \mu^{(T)}\right)^2 . \tag{2.1}$$

When $n > 0$, we have $E\left[C_{ni}^2(t)\right] = a_n$ for $i = 1, 2$ and a simple unbiased estimate of a_n is given by

$$(2.2) \qquad a_n^{(T)} = \frac{1}{2T}\sum_{t=1}^{T}\left\{C_{n1}^2(t) + C_{n2}^2(t)\right\}, \quad n > 0 .$$

The properties of the proposed estimates are summarized in the following theorem.

THEOREM 2.1. *Let* X(P) *be a Gaussian homogeneous process whose mean is* μ *and let* $\mu^{(T)}$, $a_n^{(T)}$, $n \geq 0$, *be the estimates defined by* (2.1) *and* (2.2). *Then, the estimates* $\mu^{(T)}$, $a_n^{(T)}$, $n \geq 0$, *are mutually independent with* $\mu^{(T)}$ *being* $N\left(\mu, \frac{a_0}{T}\right)$ *and* $a_n^{(T)}$ *being* $\frac{a_0}{T-1}\ \chi_{T-1}^2$ *if* $n = 0$, $\frac{a_n}{2T}\ \chi_{2T}^2$ *if* $n > 0$ (χ_n^2 *denotes a chi-square variable with* n *degrees of freedom).*

From the previous theorem, we see that

$$(2.3) \qquad \operatorname{var}\left(a_n^{(T)}\right) = \begin{cases} 2a_0^2 \big/ (T-1) & \text{if } n = 0 , \\ a_n^2 \big/ T & \text{if } n > 0 , \end{cases}$$

which means that $a_n^{(T)}$ is consistent for a_n .

If only one realization $\{X(P): P \in S\}$ is available, then the spectral estimates for $n \geq 1$ are given by

$$a_n^{(1)} = \frac{1}{2}\left\{C_{n1}^2 + C_{n2}^2\right\}.$$

Replacing C_{ni} by its definition (1.2) and using the fact that

$$\cos n(P - Q) = \cos n\ \Theta_{PQ} ,$$

one can write

$$a_n^{(1)} = \frac{1}{2\pi^2} \int_S \int_S \cos (n\, \Theta_{PQ})\, X(P)\, X(Q)\, dP\, dQ \quad , \qquad n \geq 1 .$$

By an argument analogous to the one used in Jones [5], one can show that $a_n^{(1)}$ is the unbiased estimate of minimum variance in the class of quadratic estimates of the form

$$\int_S \int_S W_n(P,Q)\, X(P)\, X(Q)\, dP\, dQ$$

where $W_n(P,Q)$ is real-valued, symmetric and square integrable.

3. Estimation of the covariance function.

Given T realizations of the process on the circle, as an estimate of the covariance function $R(\Theta)$, we consider

$$R^{(T)}(\Theta) = \sum_{n=0}^{N_T} a_n^{(T)} \cos n\Theta, \qquad 0 \leq \Theta \leq \pi \, , \tag{3.1}$$

where for each T, N_T is a positive integer. An analogous estimate for the covariance function of a process on the sphere has been studied by Roy [7]. From Schoenberg [8], $R^{(T)}(\Theta)$ represents a positive definite function on the circle since $a_n^{(T)} \geq 0$ for all $n \geq 0$. We have

$$E[R^{(T)}(\Theta)] = \sum_{n=0}^{N_T} a_n \cos n\Theta \, , \tag{3.2}$$

which means that $R^{(T)}(\Theta)$ is asymptotically unbiased for $R(\Theta)$ if $N_T \to \infty$ as $T \to \infty$. Furthermore, for a Gaussian process, using the independence of the $a_n^{(T)}$'s and equation (2.3), we obtain

$$\operatorname{cov}\left(R^{(T)}(\Theta_1),\, R^{(T)}(\Theta_2)\right) = \frac{2a_0^2}{T-1} + \frac{1}{T} \sum_{n=1}^{N_T} a_n^2 \cos(n\Theta_1) \cos(n\Theta_2) \, . \tag{3.3}$$

So,

$$\lim_{T\to\infty} T \operatorname{cov}\left(R^{(T)}(\Theta_1), R^{(T)}(\Theta_2)\right) = 2a_0^2 + \sum_{n=1}^{\infty} a_n^2 \cos(n\Theta_1) \cos(n\Theta_2) ,$$

which is well defined since $\sum_{n=1}^{\infty} a_n^2 < \infty$ by (1.6) .

The bias of our estimate is given by

$$R(\Theta) - E[R^{(T)}(\Theta)] = \sum_{n>N_T} a_n \cos n\Theta , \tag{3.4}$$

which implies that

$$|R(\Theta) - E[R^{(T)}(\Theta)]| \le \sum_{n>N_T} a_n ,$$

and by a suitable choice of the N_T's, the bias can be uniformly reduced to the order of magnitude we want.

A useful technique to choose N_T would be to compare $R^{(T)}(0) = \sum_{n=0}^{N_T} a_n^{(T)}$ for different values of N_T with the usual estimate of the variance $s^2 = \frac{1}{nT} \sum_{t=1}^{T} \sum_{j=1}^{n} \left(X(P_j,t) - \mu^{(T)}\right)^2$ for a given choice of $P_1, \ldots, P_n \in S$ and take the value of N_T for which we have nearly equality. This would allow us to conclude that the bias of $R^{(T)}(\Theta)$ is negligible.

From (3.3) and (3.4), we see that the mean square error of $R^{(T)}(\Theta)$ is given by

$$(3.5)\quad E\left[\left(R^{(T)}(\Theta)-R(\Theta)\right)^2\right]=\frac{2a_0}{t-1}+\frac{1}{T}\sum_{n=1}^{N_T}a_n^2\cos^2(n\Theta)+\left(\sum_{n>N_T}a_n\cos(n\Theta)\right)^2.$$

For example, if the N_T's are such that

$$(3.6)\qquad T^{1/2}\sum_{n>N_T}a_n\to 0\qquad \text{as } T\to\infty,$$

then

$$\lim_{T\to\infty} T\ E\left[\left(R^{(T)}(\Theta)-R(\Theta)\right)^2\right]=2a_0^2+\sum_{n=1}^{\infty}a_n^2\cos^2(n\Theta).$$

For the following, let us define the processes

$$Y_T(\Theta)=T^{1/2}\left\{R^{(T)}(\Theta)-E\left[R^{(T)}(\Theta)\right]\right\},\qquad 0\le\Theta\le\pi.$$

If the N_T's satisfy the condition (3.6), the results obtained for the processes $Y_T(\Theta)$ will be valid also for the processes $T^{1/2}\left\{R^{(T)}(\Theta)-R(\Theta)\right\}$ since $\sup_{0\le\Theta\le\pi} T^{1/2}\left\{R(\Theta)-E\left[R^{(T)}(\Theta)\right]\right\}\to 0$ as $T\to\infty$.

THEOREM 3.1. *Under the assumption of* Theorem 2.1 *and if* $N_T\to\infty$ *as* $T\to\infty$, *then for any* $\Theta_1, \ldots, \Theta_k\varepsilon[0,\pi]$, $\left(Y_T(\Theta_1), \ldots, Y_T(\Theta_k)\right)$ *is asymptotically normal with mean zero and covariance matrix*

$$\left[2a_0^2+\sum_{n=1}^{\infty}a_n^2\cos(n\Theta_i)\cos(n\Theta_j)\right]_{i,j=1}^{k}.$$

Proof. If cum $(X_1, \ldots, X_h)$ denotes the joint cumulant of order h of $X_1, \ldots, X_h$, we have only to show that cum $\left\{Y_T\left(\Theta_{i_1}\right), \ldots, Y_T\left(\Theta_{i_h}\right)\right\}\to 0$ for $i_1, \ldots, i_h\ \varepsilon\{1, \ldots, k\}$ and $h>2$. By elementary properties of cumulants (see for example Brillinger [3],section 1), for $h>2$,

$$\text{cum}\left\{Y_T\left(\Theta_{i_1}\right), \ldots, Y_T\left(\Theta_{i_h}\right)\right\}$$

$$= T^{h/2} \sum_{n=0}^{N_T} \cos\left(n\Theta_{i_1}\right) \ldots \cos\left(n\Theta_{i_h}\right) \text{cum}_h \left(a_n^{(T)}\right),$$

where $\text{cum}_h(X)$ denotes the cumulant of order h of the random variable X. Furthermore, $\text{cum}_h \left(X_n^2\right) = (h-1)!\, 2^{h-1} n$ for $h \geq 1$ and from theorem 2.1, we have that

$$\text{cum}_h \left(a_n^{(T)}\right) = \begin{cases} a_0^h (h-1)!\ (2/(t-1))^{h-1} & \text{if } n = 0, \\ a_n^h (h-1)!\ T^{-h+1} & \text{if } n > 0. \end{cases}$$

Then, we obtain

$$\left|\text{cum}\left\{Y_T\left(\Theta_{i_1}\right), \ldots, Y_T\left(\Theta_{i_h}\right)\right\}\right| \leq M\, T^{(1-h/2)} \sum_{n=0}^{\infty} a_n^h ,$$

where the constant M is independent of T and the proof is complete since $\sum_{n=0}^{\infty} a_n^h < \infty$ by (1.6).

From the previous theorem, we have the convergence of the finite dimensional distributions of $\{Y_T(\Theta):\ 0 \leq \Theta \leq \pi\}$. Under a stronger assumption on the process on the circle; i.e.,

$$\sum_{n=0}^{\infty} n^2 a_n^2 < \infty, \tag{3.7}$$

we can establish the weak convergence of the processes $Y_T(\Theta)$. Condition (3.7) is a regularity condition and means that the spectral parameters a_n decrease faster to zero than under the existence of the second moment of X(P).

THEOREM 3.2. *Under the assumptions of* Theorem 3.1 *and if* (3.7) *is satisfied, the processes* $\{Y_T(\Theta):\ 0 \le \Theta \le \pi\}$ *converge weakly in* $C[0,\pi]$ *to a Gaussian process* $\{Y(\Theta):\ 0 \le \Theta \le \pi\}$ *with mean* 0 *and*

$$\operatorname{cov}\left(Y\left(\Theta_1\right),\ Y\left(\Theta_2\right)\right) = 2a_0^2 + \sum_{n=1}^{\infty} a_n^2 \cos\left(n\Theta_1\right)\cos\left(n\Theta_2\right) .$$

Proof. $\{Y_T(\Theta)\}$ is a sequence of random elements of $C[0,\pi]$. Since we already have the convergence of the finite dimensional distributions, by Theorems 8.1, 12.3 and 12.4 of Billingsley [2], it is sufficient to show that

$$E\left[\left|Y_T\left(\Theta_2\right) - Y_T\left(\Theta_1\right)\right|^{\gamma}\right] \le \left|F\left(\Theta_2\right) - F\left(\Theta_1\right)\right|^{\alpha} ,$$

for $\Theta_1 \le \Theta_2$ and $T \ge 1$, where $\gamma \ge 0$, $\alpha > 1$ and F is a non-decreasing, continuous function on $[0,\pi]$.

Using the independence of the $a_n^{(T)}$'s and equation (2.3), we obtain

$$E\left[\left|Y_T\left(\Theta_2\right) - Y_T\left(\Theta_1\right)\right|^2\right] = \sum_{n=1}^{N_T} a_n^2 \left(\cos n\Theta_2 - \cos n\Theta_1\right)^2 .$$

By the inequality

$$\left|\cos \alpha - \cos \beta\right| \le \left|\alpha - \beta\right| ,$$

we can write

$$E\left[\left|Y_T\left(\Theta_2\right) - Y_T\left(\Theta_1\right)\right|^2\right] \le \left|\Theta_2 - \Theta_1\right|^2 \sum_{n=1}^{N_T} n^2 a_n^2$$

$$\le M \left|\Theta_2 - \Theta_1\right|^2 ,$$

where $M = \sum_{n=0}^{\infty} n^2 a_n^2 < \infty$ by (3.7) and is independent of T. Thus, the proof is complete.

If (3.6) is satisfied, the previous theorem is also valid for the sequence of processes $T^{\frac{1}{2}}\left\{R^{(T)}(\Theta) - R(\Theta)\right\}$. This allows us to assert the convergence in distribution of functionals such as

$$T^{\frac{1}{2}} \sup_{0 \leq \Theta \leq \pi} \left| R^{(T)}(\Theta) - R(\Theta) \right| ,$$

$$T \int_0^{\pi} \left| R^{(T)}(\Theta) - R(\Theta) \right|^2 d\Theta ,$$

to corresponding functionals based on the Gaussian process $Y(\Theta)$ of the Theorem.

Acknowledgment. The authors wish to thank the referee for some helpful suggestions which led to an improved presentation.

REFERENCES

[1] G. S. Benton and A. B. Kahn, *Spectra of large-scale atmospheric flow at 300 millibars*, J. Met. *15* (1958), 404-410.

[2] P. Billingsley, *Convergence of Probability Measures*, Wiley, New York, 1968.

[3] D. R. Brillinger, "The frequency analysis of relations between stationary spatial series", *Proc. Twelfth Biennal Sem. Canad. Math. Congress on Time Series and Stochastic Processes; Convexity and Combinatorics*, ed. R. Pike, Canad. Math. Congress, Montreal, 1970, 39-81.

[4] E. J. Hannan, "The statistical analysis of hydrological time series", *Proc. National Symposium on Water Resources, Use and Management*, Melbourne University Press, 1963, 233-243.

[5] R. H. Jones, *Stochastic processes on a sphere*, Ann. Math. Statist. *34* (1963), 213-218.

[6] R. Roy, *Spectral analysis for a random process on the circle*, J. Appl. Prob. *9* (1972), 745-757.

[7] R. Roy, *Estimation of the covariance function of a homogeneous process on the sphere*, Ann. Statist. *1* (1973), 780-785.

[8] I. J. Schoenberg, *Positive definite functions on spheres*, Duke Math. J. *9* (1942), 96-108.

[9] A. M. Yaglom, *Second-order homogeneous random fields*, Proc. Fourth Berkeley Sympos. Vol. 2, 1961, pp. 593-622.

Université du Québec à Montréal
Université de Montréal

Received June 5, 1973; revised November 1, 1973.