28-03-10

ECON 469 – ECONOMETRICS

MID-TERM EXAM

SOLUTIONS

28-03-16

(20) 1. (a) This is a MA(1) model, so that

$$\gamma(0) = (1+\theta^2)\sigma^2,$$

$$\gamma(1) = -\theta\sigma^2,$$

$$\rho(1) = -\theta/(1+\theta^2).$$

The derivative of $\rho(1)$ with respect to θ is:

$$\frac{\partial \rho(1)}{\partial \theta} = \frac{-(1+\theta^2)+\theta(2\theta)}{(1+\theta^2)^2} = \frac{-1+\theta^2}{(1+\theta^2)^2}$$

hence

$$\frac{\partial \rho(1)}{\partial \theta} \begin{cases} >0 & \text{if } \theta<-1 \\ =0 & \text{if } \theta=-1 \\ <0 & \text{if } |\theta|<1 \\ =0 & \text{if } \theta=1 \\ >0 & \text{if } \theta>1 \end{cases}$$

This entails that $\rho(1)$ has a minimum at $\theta=-1$ and a maximum at $\theta=1$

Since

$$\rho(1) = 0.5 \quad \text{if} \quad \theta = -1,$$

$$\rho(1) = -0.5 \quad \text{if} \quad \theta = 1,$$

$$\rho(1) \longrightarrow 0 \quad \text{as} \quad \theta \to -\infty,$$

$$\rho(1) \longrightarrow 0 \quad \text{as} \quad \theta \to +\infty,$$

it follows that

$$|\rho(1)| \leq 0.5.$$

(5) The upper bound of $|\rho(1)|$ is attained at $\theta = -1$ and $\theta = 1$ with

$$\rho(1) = -0.5 \quad \text{if} \quad \theta = -1$$

$$\rho(1) = 0.5 \quad \text{if} \quad \theta = +1.$$

2. We can consider the model:

$$X_t = 10 + u_t - 0.75\, u_{t-1} + 0.125\, u_{t-1}$$

$$\{u_t : t \in \mathbb{Z}\} \overset{i.i.d}{\sim} N(0,1)$$

(a) Yes. This process is an MA(2) and consequently it is stationary.

(b) Yes.

We have:

$$X_t = 10 + \theta(B)\, u_t$$

where

$$\theta(B) = 1 - 0.75B + 0.125B^2$$

$$= (1 - 0.5B)(1 - 0.25B)$$

The roots of the equation

$$\theta(z) = 0$$

are $z_1 = 1/0.5 = 2.0$, $z_2 = 1/0.25 = 4$

Since $|z_1| > 1$ and $|z_2| > 1$, both the roots are outside the unit circle.

2.(c) i) $E(X_t) = 10$

ii) $\gamma(0) = Var(X_t) = [1 + (0.75)^2 + (0.125)^2] = 1.578125$

$\gamma(1) = E[(X_t - 10)(X_{t-1} - 10)]$

$= E[(u_t - 0.75u_{t-1} + 0.125u_{t-2})(u_{t-1} - 0.75u_{t-2} + 0.125u_{t-3})]$

$= 0.75 - (0.125)(0.75) = -0.84375$

$\gamma(2) = E[(X_t - 10)(X_{t-2} - 10)]$

$= E[(u_t - 0.75u_{t-1} + 0.125u_{t-2})(u_{t-2} - 0.75u_{t-3} + 0.125u_{t-4})]$

$= 0.125$

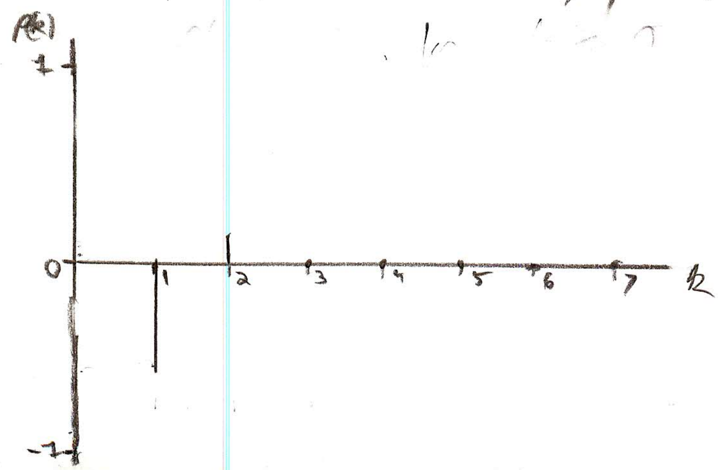
$\gamma(k) = 0$, for $k \geq 3$

Thus: $\gamma(0) = 1.578125$, $\gamma(1) = -0.84375$, $\gamma(2) = 0.125$
$\gamma(k) = 0$, for $k \geq 3$

iii) $\rho(k) = \gamma(k)/\gamma(0)$

$\rho(1) = -0.53465$, $\rho(2) = 0.07921$, $\rho(k) = 0$, for $k \geq 0$

(d)



2. (e) Since the model is already in MA form, the coefficients of the MA representation are the same:

$$X_t = 10 + \sum_{k=0}^{\infty} \psi_k u_{t-k}$$

where

$\psi_0 = 1$, $\psi_1 = -0.75$, $\psi_2 = 0.125$,

$\psi_k = 0$, for $k \geq 3$.

(f) The first partial autocorrelation is

$r_{11} = \rho(1) = -0.53465$

The second partial autocorrelation is obtained by solving the equations:

$$\rho(1) = \rho(0) r_{21} + \rho(1) r_{22} = r_{21} + \rho(1) r_{22}$$

$$\rho(2) = \rho(1) r_{21} + \rho(0) r_{22} = r_{21}\rho(1) + r_{22}$$

or

$$\begin{bmatrix} \rho(1) \\ \rho(2) \end{bmatrix} = \begin{bmatrix} 1 & \rho(1) \\ \rho(1) & 1 \end{bmatrix} \begin{bmatrix} r_{21} \\ r_{22} \end{bmatrix}$$

hence

$$r_{22} = \begin{vmatrix} 1 & \rho(1) \\ \rho(1) & \rho(2) \end{vmatrix} \Big/ \begin{vmatrix} 1 & \rho(1) \\ \rho(1) & 1 \end{vmatrix} = \frac{\rho(2) - \rho(1)^2}{1 - \rho(1)^2}$$

$$= \frac{-0.20664}{0.71415} = -0.28935$$