Solution to Econ 763 Assignment 3 (Winter 2017)
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Question 1

(a) Because X_t has a continuous distribution, we can define $\text{Med}(X_t)$ as the unique real number such that

$$\mathsf{P}\left[X_t \geq \text{Med}(X_t)\right] = \mathsf{P}\left[X_t < \text{Med}(X_t)\right] = \frac{1}{2}.$$

(b) Let $p_t = \mathsf{P}[X_t \geq 0]$, then $u(X_t)$ clearly follows a Bernoulli distribution where p_t is the probability of getting 1. Because $X_1, \ldots, X_T$ are mutually independent, the same is true for $u(X_1), \ldots, u(X_T)$. Therefore, the joint distribution of the random variables $u(X_1), \ldots, u(X_T)$ can be described as:

$$\mathsf{P}\left[u(X_1) = \theta_1, \ldots, u(X_T) = \theta_T\right] = \prod_{t=1}^{T} p_t^{\theta_t}(1-p_t)^{1-\theta_t}.$$

In particular, this joint distribution is a discrete distribution on E^T where $E \equiv \{0, 1\}$. Moreover, if $p_t = \frac{1}{2}$ for all $t = 1, \ldots, T$, the joint distribution of $u(X_1), \ldots, u(X_T)$ is the uniform distribution on E^T.

(c) Consider the sign statistic

$$S \equiv \sum_{t=1}^{T} u(Z_t) \tag{1}$$

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where $Z_t = X_t,\ t = 1, \ldots, T$. The distribution of S is binomial $\mathrm{Bin}(T, 0.5)$, and the mean of S is $E(S) = T/2$. This result does not require that the distributions of $X_1, \ldots, X_T$ be symmetric. We reject H_0 when $|S-(T/2)| \geq c_T(\alpha/2)$ where $c_T(\alpha/2)$ is chosen so that $\mathsf{P}[|S-(T/2)| \geq c_T(\alpha/2)] \leq \alpha/2$ according to the $\mathrm{Bin}(T, 0.5)$ distribution.

(d) Repeat (c) while modifying (1) with the following definition of Z_k:

$$Z_t = (X_t - m_0),\ t = 1, \ldots, T. \tag{2}$$

(e) Repeat (c) while modifying (1) with the following definition of Z_k:

$$Z_k = (X_t - a - bt),\ t = 1, \ldots, T. \tag{3}$$

Question 2

(a) Define

$$Z_t \equiv X_t X_{t+k},\ t = 1, \ldots, T-k,$$
$$S_k \equiv \sum_{t=1}^{T-k} u(Z_t).$$

Under the null hypothesis of independence H_0, the characteristic function of S_k is

$$E[\exp(i\tau S_k)] = \left(\frac{1}{2}\right)^{T-k} \prod_{r=1}^{T-k} [1 + e^{i\tau}], \quad i \equiv \sqrt{-1}, \quad \tau \in \mathbb{R}. \tag{4}$$

This is the characteristic function of a $\mathrm{Bin}(T-k, 0.5)$ variable, so $S_k \sim \mathrm{Bin}(T-k, 0.5)$. This result does not require that the distributions of $X_1, \ldots, X_T$ be symmetric. Against the alternative of positive dependence

$$\mathrm{Med}(X_t X_{t+k}) > 0\,,\ t = 1, \ldots, T-k, \tag{5}$$

it is then natural to reject H_0 when $S_k \geq c^{+}_{T-k}(\alpha)$ where $c^{+}_{T-k}(\alpha)$ is chosen so that $\mathsf{P}[S_k \geq c^{+}_{T-k}(\alpha)] \leq \alpha$ according to the $\mathrm{Bin}(T-k, 0.5)$ distribution.

(b) Against the alternative

$$\mathrm{Med}(X_t X_{t+k}) < 0\,,\ t = 1, \ldots, T-k, \tag{6}$$

we reject H_0 when $S_k \leq c^{-}_{T-k}(\alpha)$ where $c^{-}_{T-k}(\alpha)$ is chosen so that $\mathsf{P}[S_k \leq c^{-}_{T-k}(\alpha)] \leq \alpha$ according to the $\mathrm{Bin}(T-k, 0.5)$ distribution.

(c) Against the two-sided alternative

$$\text{Med}(X_t X_{t+k}) \neq 0\,,\ t = 1, \ldots, T-k \tag{7}$$

we reject H_0 when $S_k \geq c^+_{T-k}(\alpha/2)$ or $S_k \leq c^-_{T-k}(\alpha/2)$, where $c^+_{T-k}(\alpha/2)$ and $c^-_{T-k}(\alpha/2)$ are chosen so that $\mathsf{P}[S_k \geq c^+_{T-k}(\alpha/2)] \leq \alpha/2$ and $\mathsf{P}[S_k \leq c^-_{T-k}(\alpha/2)] \leq \alpha/2$.

(d) When $k = 1$, an interpretation in terms of "runs" can be seen as follows. We have:

$$S_1 = \sum_{t=1}^{T-1} u(Z_t).$$

S_1 is the number of times consecutive X_t's have the same sign; thus $(T-1)-S_1$ is the number of times changes of sign occur in the sequence $X_1, \ldots, X_T$, and $T-S_1$ is (with probability 1) the total number of runs in the sequence $u(X_1), \ldots, u(X_T)$.

(e) The procedures above remain valid because they no assumption is made on the variances of $X_1, \ldots, X_T$.

Question 3

(a) Consider the test statistic

$$W \equiv \sum_{t=1}^{T} u(X_t)\, a_N(R^+_t)\,.$$

Due to the symmetry assumption, the sign vector $\big(u(X_1), \ldots,\ u(X_T)\big)$ and the rank vector $\big(R^+_1, \ldots,\ R^+_T\big)$ are independent. Thus the characteristic function of W can be established as in Theorem 1 of Dufour (1981): under the null H_0, W has a distribution which is completely specified by the characteristic function

$$E[\exp(i\tau W)] = \left(\frac{1}{2}\right)^N \prod_{r=1}^{N} [1 + e^{\tau i a_N(r)}], \quad i \equiv \sqrt{-1}, \quad \tau \in \mathbb{R}. \tag{8}$$

When $a_N(r) = r$, we get the Wilcoxon signed-rank statistic, which has been tabulated. More generally, we can use the technique of Monte Carlo tests to perform these tests [see Dufour (2006)].
Signed-rank tests use information on the size of the observations as well as their signs, which may yield more power. On the other hand, they require the additional assumption of symmetry.

(b) Define

$$Z_t \equiv X_t X_{t+k}, \quad t = 1, \ldots, T-k,$$

$$R_t^+ \equiv \sum_{j=1}^{N} u(|Z_t| - |Z_j|), \quad t = 1, \ldots, T-k,$$

$$W_k \equiv \sum_{t=1}^{N} u(Z_t) a_N(R_t^+)\,.$$

Under the null hypothesis of independence, the characteristic function of W_k is

$$E[\exp(i\tau W_k)] = \left(\frac{1}{2}\right)^N \prod_{r=1}^{N} [1 + e^{\tau i a_N(r)}], \quad i \equiv \sqrt{-1}, \quad \tau \in \mathbb{R}. \tag{9}$$

see Theorem 1 in Dufour (1981). So the distribution of W_k is the same as the one of the corresponding signed-rank statistic for testing the hypothesis of zero median. Tests can be performed in the same way.

References

Jean-Marie Dufour. Rank Tests for Serial Dependence. *Journal of Time Series Analysis*, 2(3):117–128, 1981.

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