Stochastic processes: generating functions and identification *

Jean-Marie Dufour †
McGill University

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† William Dow Professor of Economics, McGill University, Centre interuniversitaire de recherche en analyse des organisations (CIRANO), and Centre interuniversitaire de recherche en économie quantitative (CIREQ). Mailing address: Department of Economics, McGill University, Leacock Building, Room 414, 855 Sherbrooke Street West, Montréal, Québec H3A 2T7, Canada. TEL: (1) 514 398 4400 ext. 09156; FAX: (1) 514 398 4800; e-mail: `jean-marie.dufour@mcgill.ca` . Web page: `http://www.jeanmariedufour.com`

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List of Definitions, Assumptions, Propositions and Theorems

1. Generating functions and spectral density

Generating functions constitute a convenient technique for representing and determining the autocovariance structure of a stationary process.

Definition 1.1 GENERATING FUNCTION. *Let* $(a_k : k = 0,\ 1,\ 2,\dots)$ *and* $(b_k : k = \dots,\ -1,\ 0,\ 1,\dots)$ *two sequences of complex numbers. Let* $D(a) \subseteq \mathbf{C}$ *the set of points* $z \in \mathbf{C}$ *at which the series* $\sum_{k=0}^{\infty} a_k z^k$ *converges, and* $D(b) \subseteq \mathbf{C}$ *the set of points* z *for which where the series* $\sum_{k=-\infty}^{\infty} b_k z^k$ *converges. Then the functions*

$$a(z) = \sum_{k=0}^{\infty} a_k z^k, z \in D(a) \tag{1.1}$$

and

$$b(z) = \sum_{k=-\infty}^{\infty} b_k z^k, z \in D(b) \tag{1.2}$$

are called the generating functions of the sequences a_k *and* b_k *respectively.*

Proposition 1.1 CONVERGENCE ANNULUS OF A GENERATING FUNCTION. *Let* $(a_k : k \in \mathbb{Z})$ *be a sequence of complex numbers. Then the generating function*

$$a(z) = \sum_{k=-\infty}^{\infty} a_k z^k \tag{1.3}$$

converges for $R_1 < |z| < R_2$ *where*

$$R_1 = \limsup_{k \to \infty} |a_{-k}|^{1/k}, \tag{1.4}$$

$$R_2 = 1/\left[\limsup_{k \to \infty} |a_k|^{1/k}\right], \tag{1.5}$$

and diverges for $|z| < R_1$ *or* $|z| > R_2$*. If* $R_2 < R_1$*,* $a(z)$ *converges nowhere and, if* $R_1 = R_2$*,* $a(z)$ *diverges everywhere except possibly, for* $|z| = R_1 = R_2$*. Further, when* $R_1 < R_2$*, the coefficients* a_k *are uniquely defined, and*

$$a_k = \frac{1}{2\pi i} \int_C \frac{a(z)\, dz}{(z - z_0)^{k+1}}, \; k = 0, \pm 1, \pm 2, \ldots \tag{1.6}$$

where $C = \{z \in \mathbf{C} : |z - z_0| = R\}$ *and* $R_1 < R < R_2$ *.*

Proposition 1.2 SUMS AND PRODUCTS OF GENERATING FUNCTIONS. *Let* $(a_k : k \in \mathbb{Z})$ *and* $(b_k \in \mathbb{Z})$ *two sequences of complex numbers such that the generating functions* $a(z)$ *and* $b(z)$ *converge for* $R_1 < |z| < R_2$*, where* $0 \leq R_1 < R_2 \leq \infty$*. Then,*

1. *the generating function of the sum* $c_k = a_k + b_k$ *is* $c(z) = a(z) + b(z)$*;*
2. *if the product sequence*

$$d_k = \sum_{j=-\infty}^{\infty} a_j b_{k-j} \tag{1.7}$$

converges for any k, the generating function of the sequence d_k *is*

$$d(z) = a(z)b(z). \tag{1.8}$$

Further, the series $c(z)$ *and* $d(z)$ *converge for* $R_1 < |z| < R_2$*.*

We will be especially interested by generating functions of autocovariances γ_k and autocorrelations ρ_k of a second-order stationary process X_t:

$$\gamma_x(z) = \sum_{k=-\infty}^{\infty} \gamma_k z^k, \tag{1.9}$$

$$\rho_x(z) = \sum_{k=-\infty}^{\infty} \rho_k z^k = \gamma_x(z)/\gamma_0. \tag{1.10}$$

We see immediately that the generating function with a white noise $\{u_t : t \in \mathbb{Z}\} \sim WN(0, \sigma^2)$ is constant::

$$\gamma_u(z) = \sigma^2,\ \rho_u(z) = 1. \tag{1.11}$$

Proposition 1.3 CONVERGENCE OF AUTOCOVARIANCE GENERATING FUNCTIONS. *Let $\gamma_k, k \in \mathbb{Z}$, the autocovariances of a second-order stationary process X_t, and ρ_k, $k \in \mathbb{Z}$, the corresponding autocorrelations.*

1. *If $R \equiv \limsup_{k\to\infty} |\rho_k|^{1/k} < 1$, the generating functions $\gamma_x(z)$ and $\rho_x(z)$ converge for $R < |z| < 1/R$.*
2. *If $R = 1$, the functions $\gamma_x(z)$ and $\rho_x(z)$ diverge everywhere, except possibly on the circle $|z| = 1$.*
3. *If $\sum_{k=0}^{\infty} |\rho_k| < \infty$, the functions $\gamma_x(z)$ and $\rho_x(z)$ converge absolutely and uniformly on the circle $|z| = 1$.*

Proposition 1.4 IDENTIFIABILITY OF AUTOCOVARIANCES AND AUTOCORRELATIONS BY GENERATING FUNCTIONS. *Let γ_k and ρ_k, $k \in \mathbb{Z}$, autocovariance and autocorrelation sequences such that*

$$\gamma(z) = \sum_{k=-\infty}^{\infty} \gamma_k z^k = \sum_{k=-\infty}^{\infty} \gamma'_k z^k, \tag{1.12}$$

$$\rho(z) = \sum_{k=-\infty}^{\infty} \rho_k z^k = \sum_{k=-\infty}^{\infty} \rho'_k z^k \tag{1.13}$$

where the series considered converge for $R < |z| < 1/R$, where $R \geq 0$. Then $\gamma_k = \gamma'_k$ and $\rho_k = \rho'_k$ for any $k \in \mathbb{Z}$.

Proposition 1.5 GENERATING FUNCTION OF THE AUTOCOVARIANCES OF A MA(∞) PROCESS. *Let* $\{X_t : t \in \mathbb{Z}\}$ *a second-order stationary process such that*

$$X_t = \sum_{j=-\infty}^{\infty} \psi_j u_{t-j} \tag{1.14}$$

where $\{u_t : t \in \mathbb{Z}\} \sim WN(0, \sigma^2)$. *If the series*

$$\psi(z) = \sum_{j=-\infty}^{\infty} \psi_j z^j \tag{1.15}$$

and $\psi(z^{-1})$ *converge absolutely, then*

$$\gamma_x(z) = \sigma^2 \psi(z) \psi(z^{-1}). \tag{1.16}$$

Corollary 1.6 GENERATING FUNCTION OF THE AUTOCOVARIANCES OF AN ARMA PROCESS. *Let* $\{X_t : t \in \mathbb{Z}\}$ *a second-order stationary and causal ARMA*(p,q) *process, such that*

$$\varphi(B)X_t = \bar{\mu} + \theta(B)u_t \tag{1.17}$$

where $\{u_t : t \in \mathbb{Z}\} \sim WN(0,\sigma^2)$, $\varphi(z) = 1 - \varphi_1 z - \cdots - \varphi_p z^p$ *and* $\theta(z) = 1 - \theta_1 z - \cdots - \theta_q z^q$. *Then the generating function of the autocovariances of* X_t *is*

$$\gamma_x(z) = \sigma^2 \frac{\theta(z)\ \theta\left(z^{-1}\right)}{\varphi(z)\ \varphi\left(z^{-1}\right)} \tag{1.18}$$

for $R < |z| < 1/R$, *where*

$$0 < R = \max\{|G_1|, |G_2|, ..., |G_p|\} < 1 \tag{1.19}$$

and $G_1^{-1}, G_2^{-1}, ..., G_p^{-1}$ *are the roots of the polynomial* $\varphi(z)$.

Proposition 1.7 GENERATING FUNCTION OF THE AUTOCOVARIANCES OF A FILTERED PROCESS. *Let* $\{X_t : t \in \mathbb{Z}\}$ *a second-order stationary process and*

$$Y_t = \sum_{j=-\infty}^{\infty} c_j X_{t-j}, t \in \mathbb{Z}, \tag{1.20}$$

where $(c_j : j \in \mathbb{Z})$ *is a sequence of real constants such that* $\sum_{j=-\infty}^{\infty} |c_j| < \infty$. *If the series* $\gamma_x(z)$ *and* $c(z) = \sum_{j=-\infty}^{\infty} c_j z^j$ *converge absolutely, then*

$$\gamma_y(z) = c(z)c(z^{-1})\gamma_x(z). \tag{1.21}$$

Definition 1.2 SPECTRAL DENSITY. *Let X_t a second-order stationary process such that the generating function of the autocovariances $\gamma_x(z)$ converge for $|z| = 1$. The spectral density of the process X_t is the function*

$$\begin{aligned} f_x(\omega) &= \frac{1}{2\pi}\left[\gamma_0 + 2\sum_{k=1}^{\infty}\gamma_k \cos(\omega k)\right] \\ &= \frac{\gamma_0}{2\pi} + \frac{1}{\pi}\sum_{k=1}^{\infty}\gamma_k \cos(\omega k) \end{aligned} \tag{1.22}$$

where the coefficients γ_k are the autocovariances of the process X_t. The function $f_x(\omega)$ is defined for all the values of ω such that the series $\sum_{k=1}^{\infty}\gamma_k \cos(\omega k)$ converges.

Remark 1.1 If the series $\sum_{k=1}^{\infty}\gamma_k \cos(\omega k)$ converges, it is immediate that $\gamma_x(e^{-i\omega})$ converge and

$$f_x(\omega) = \frac{1}{2\pi}\gamma_x(e^{-i\omega}) = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty}\gamma_k e^{-i\omega k} \tag{1.23}$$

where $i = \sqrt{-1}$.

Proposition 1.8 CONVERGENCE AND PROPERTIES OF THE SPECTRAL DENSITY. *Let γ_k, $k \in \mathbb{Z}$, be an autocovariance function such that* $\sum_{k=0}^{\infty} |\gamma_k| < \infty$. *Then*

1. *the series*

$$f_x(\omega) = \frac{\gamma_0}{2\pi} + \frac{1}{\pi}\sum_{k=1}^{\infty}\gamma_k \cos(\omega k) \tag{1.24}$$

   converges absolutely and uniformly in ω ;

2. *the function $f_x(\omega)$ is continuous ;*

3. $f_x(\omega + 2\pi) = f_x(\omega)$ *and* $f_x(-\omega) = f_x(\omega)$, $\forall \omega$ *;*

4. $\gamma_k = \int_{-\pi}^{\pi} f_x(\omega)\cos(\omega k) d\omega$, $\forall k$ *;*

5. $f_x(\omega) \geq 0$ *;*

6. (6) $\gamma_0 = \int_{-\pi}^{\pi} f_x(\omega) d\omega$.

Proposition 1.9 SPECTRAL DENSITIES OF SPECIAL PROCESSES. *Let $\{X_t : t \in \mathbb{Z}\}$ be a second-order stationary process with autocovariances γ_k, $k \in \mathbb{Z}$.*

1. *If $X_t = \mu + \sum_{j=-\infty}^{\infty} \psi_j u_{t-j}$ where $\{u_t : t \in \mathbb{Z}\} \sim WN(0, \sigma^2)$ and $\sum_{j=-\infty}^{\infty} |\psi_j| < \infty$, then*

$$f_x(\omega) = \frac{\sigma^2}{2\pi} \psi(e^{i\omega})\psi(e^{-i\omega}) = \frac{\sigma^2}{2\pi} |\psi(e^{i\omega})|^2. \tag{1.25}$$

2. *If $\varphi(B)X_t = \bar{\mu} + \theta(B)u_t$, where $\varphi(B) = 1 - \varphi_1 B - \cdots - \varphi_p B^p$, $\theta(B) = 1 - \theta_1 B - \cdots - \theta_q B^q$ and $\{u_t : t \in \mathbb{Z}\} \sim WN(0, \sigma^2)$, then*

$$f_x(\omega) = \frac{\sigma^2}{2\pi} \left| \frac{\theta\left(e^{i\omega}\right)}{\varphi\left(e^{i\omega}\right)} \right|^2 \tag{1.26}$$

3. *If $Y_t = \sum_{j=-\infty}^{\infty} c_j X_{t-j}$ where $(c_j : j \in \mathbb{Z})$ is a sequence of real constants such that $\sum_{j=-\infty}^{\infty} |c_j| < \infty$, and if $\sum_{k=0}^{\infty} |\gamma_k| < \infty$, then*

$$f_y(\omega) = |c(e^{i\omega})|^2 f_x(\omega). \tag{1.27}$$

2. Inverse autocorrelations

Definition 2.1 INVERSE AUTOCORRELATIONS. *Let $f_x(\omega)$ the spectral density of a second-order stationary process $\{X_t : t \in \mathbb{Z}\}$. If the function $1/f_x(\omega)$ is also a spectral density, the autocovariances $\gamma_x^{(I)}(k)$, $k \in \mathbb{Z}$, associated with the inverse spectrum inverse $1/f_x(\omega)$ are called the* inverse autocovariances *of the process X_t*, i.e.

$$\gamma_x^{(I)}(k) = \int_{-\pi}^{\pi} \frac{1}{f_x(\omega)} \cos(\omega k) d\omega, k \in \mathbb{Z}. \tag{2.1}$$

The inverse autocovariances satisfy the equation

$$\frac{1}{f_x(\omega)} = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \gamma_x^{(I)}(k) \cos(\omega k) = \frac{1}{2\pi} \gamma_x^{(I)}(0) + \frac{1}{\pi} \sum_{k=1}^{\infty} \gamma_x^{(I)} \cos(\omega k). \tag{2.2}$$

The inverse autocorrelations are

$$\rho_x^{(I)}(k) = \gamma_x^{(I)}(k) / \gamma_x^{(I)}(0), k \in \mathbb{Z}. \tag{2.3}$$

A sufficient condition for the function $1/f_x(\omega)$ to be a spectral density is that the function $1/f_x(\omega)$ be continuous on the interval $-\pi \leq \omega \leq \pi$, which entails that $f_x(\omega) > 0$, $\forall \omega$.

If the process X_t is a second-order stationary $ARMA(p,q)$ process such that

$$\varphi_p(B)X_t = \bar{\mu} + \theta_q(B)u_t \tag{2.4}$$

where $\varphi_p(B) = 1 - \varphi_1 B - \cdots - \varphi_p B^p$ and $\theta_q(B) = 1 - \theta_1 B - \cdots - \theta_q B^q$ are polynomials whose roots are all outside the unit circle and $\{u_t : t \in \mathbb{Z}\} \sim WN(0, \sigma^2)$, then

$$f_x(\omega) = \frac{\sigma^2}{2\pi}\left|\frac{\theta_q\left(e^{i\omega}\right)}{\varphi_p\left(e^{i\omega}\right)}\right|^2, \tag{2.5}$$

$$\frac{1}{f_x(\omega)} = \frac{2\pi}{\sigma^2}\left|\frac{\varphi_p\left(e^{i\omega}\right)}{\theta_q\left(e^{i\omega}\right)}\right|^2. \tag{2.6}$$

The inverse autocovariances $\gamma_x^{(I)}(k)$ are the autocovariances associated with the model

$$\theta_q(B)X_t = \bar{\bar{\mu}} + \varphi_p(B)v_t \tag{2.7}$$

where $\{v_t : t \in \mathbb{Z}\} \sim WN(0\ ,\ 1/\sigma^2)$ and $\bar{\bar{\mu}}$ is some constant. Consequently, the inverse autocorrelations of an $ARMA(p,q)$ process behave like the autocorrelations of an $ARMA(q,p)$. For an process $AR(p)$ process,

$$\rho_x^{(I)}(k) = 0\,,\ \text{for } k > p. \tag{2.8}$$

For a $MA(q)$ process, the inverse partial autocorrelations (*i.e.* the partial autocorrelations associated with the inverse autocorrelations) are equal to zero for $k > q$. These properties can be used for identifying the order of a process.

3. Multiplicity of representations

3.1. Backward representation ARMA models

By the backward Wold theorem, we know that any strictly indeterministic second-order stationary process $X_t : t \in \mathbb{Z}\}$ can be written in the form

$$X_t = \mu + \sum_{j=0}^{\infty} \bar{\psi}_j \bar{u}_{t+j} \tag{3.1}$$

where $\bar{u}_t$ is a white noise such that $E(X_{t-j}\bar{u}_t) = 0$, $\forall j \geq 1$. In particular, if

$$\varphi_p(B)(X_t - \mu) = \theta_q(B)u_t \tag{3.2}$$

where the polynomials $\varphi_p(B) = 1 - \varphi_1 B - \cdots - \varphi_p B^p$ and $\theta_q(B) = 1 - \theta_1 B - \cdots - \theta_q B^q$ have all their roots outside the unit circle and $\{u_t : t \in \mathbb{Z}\} \sim WN(0, \sigma^2)$, the spectral density of X_t is

$$f_x(\omega) = \frac{\sigma^2}{2\pi} \left| \frac{\theta_q\left(e^{i\omega}\right)}{\varphi_p\left(e^{i\omega}\right)} \right|^2 . \tag{3.3}$$

Consider the process

$$Y_t = \frac{\varphi_p\left(B^{-1}\right)}{\theta_q\left(B^{-1}\right)}\left(X_t - \mu\right) = \sum_{j=0}^{\infty} c_j (X_{t+j} - \mu). \tag{3.4}$$

By Proposition **1.9**, the spectral density of Y_t is

$$f_y(\omega) = \left|\frac{\varphi_p\left(e^{i\omega}\right)}{\theta_q\left(e^{i\omega}\right)}\right|^2 f_x(\omega) = \frac{\sigma^2}{2\pi} \tag{3.5}$$

and thus $\{Y_t : t \in \mathbb{Z}\} \sim WN(0, \sigma^2)$. If we define $\bar{u}_t = Y_t$, we see that

$$\frac{\varphi_p\left(B^{-1}\right)}{\theta_q\left(B^{-1}\right)}\left(X_t - \mu\right) = \bar{u}_t \tag{3.6}$$

or

$$\varphi_p(B^{-1})X_t = \bar{\mu} + \theta_q(B^{-1})\bar{u}_t, \tag{3.7}$$

and

$$X_t - \varphi_1 X_{t+1} - \cdots - \varphi_p X_{t+p} = \bar{\mu} + \bar{u}_t - \theta_1 \bar{u}_{t+1} - \cdots - \theta_q \bar{u}_{t+q} \tag{3.8}$$

where $(1 - \varphi_1 - \cdots - \varphi_p)\mu = \bar{\mu}$. We call (3.6) or (3.8) the backward representation of the X_t process.

3.2. Multiple moving-average representations

Let $\{X_t\} \sim \text{ARIMA}(p, d, q)$. Then

$$W_t = (1-B)^d X_t \sim ARMA(p,q). \tag{3.9}$$

If we suppose that $E(W_t) = 0$, W_t satisfies an equation of the form

$$\varphi_p(B)W_t = \theta_q(B)u_t \tag{3.10}$$

or

$$W_t = \frac{\theta_q(B)}{\varphi_p(B)}\, u_t = \psi(B)u_t. \tag{3.11}$$

To determine an appropriate *ARMA* model, one typically estimates the autocorrelations ρ_k. The latter are uniquely determined by the generating function of the autocovariances:

$$\gamma_x(z) = \sigma^2\psi(z)\psi(z^{-1}) = \sigma^2\frac{\theta_q(z)}{\varphi_p(z)}\ \frac{\theta_q\left(z^{-1}\right)}{\varphi_p\left(z^{-1}\right)}. \tag{3.12}$$

If

$$\theta_q(z) = 1-\theta_1 z\ -\cdots-\theta_q z^q = (1-H_1 z)\cdots(1-H_q z) = \prod_{j=1}^{q}(1-H_j z), \tag{3.13}$$

then

$$\gamma_x(z) = \frac{\sigma^2}{\varphi_p(z)\,\varphi_p\left(z^{-1}\right)}\prod_{j=1}^{q}(1-H_j z)(1-H_j z^{-1}). \tag{3.14}$$

However

$$\begin{aligned}(1-H_jz)(1-H_jz^{-1}) &= 1-H_jz-H_jz^{-1}+H_j^2 = H_j^2(1-H_j^{-1}z-H_j^{-1}z^{-1}+H_j^{-2}) \\ &= H_j^2(1-H_j^{-1}z)(1-H_j^{-1}z^{-1}) \end{aligned} \tag{3.15}$$

hence

$$\gamma_x(z) = \frac{\left[\sigma^2 \prod_{j=1}^{q} H_j^2\right]}{\varphi_p(z)\,\varphi_p(z^{-1})} \prod_{j=1}^{q} \left(1-H_j^{-1}z\right)\left(1-H_j^{-1}z^{-1}\right) = \bar{\sigma}^2 \frac{\theta_q'(z)\,\theta_q'\left(z^{-1}\right)}{\varphi_p(z)\,\varphi_p(z^{-1})} \tag{3.16}$$

where

$$\bar{\sigma}^2 = \sigma^2 \prod_{j=1}^{q} H_j^2\,, \quad \theta_q'(z) = \prod_{j=1}^{q} (1-H_j^{-1}z)\,. \tag{3.17}$$

$\gamma_x(z)$ in (3.16) can be viewed as the generating function of a process of the form

$$\varphi_p(B)W_t = \theta_q'(B)\bar{u}_t = [\prod_{j=1}^{q}(1-H_j^{-1}B)]\bar{u}_t \tag{3.18}$$

while $\gamma_x(z)$ in (3.14) is the generating function of

$$\varphi_p(B)W_t = \theta_q(B)u_t = [\prod_{j=1}^{q}(1-H_jB)]u_t. \tag{3.19}$$

The processes (3.18) and (3.19) have the same autocovariance function and thus cannot be distinguished by looking at their seconds moments.

Example 3.1 Identification of an $\text{ARMA}(1,1)$ model

$$(1-0.5B)W_t = (1-0.2B)(1+0.1B)u_t \tag{3.20}$$

$$(1-0.5B)W_t = (1-5B)(1+10B)\bar{u}_t \tag{3.21}$$

have the same autocorrelation function.

In general, the models

$$\varphi_p(B)W_t = \left[\prod_{j=1}^{q}(1-H_j^{\pm 1}B)\right]\bar{u}_t \tag{3.22}$$

all have the same autocovariance function (and are thus indistinguishable). Since it is easier with an invertible model, we select

$$H_j^* = \begin{cases} H_j & \text{if } H_j < 1 \\ H_j^{-1} & \text{if } H_j > 1 \end{cases} \tag{3.23}$$

where $|H_j| \leq 1$, in order to have an invertible model.

3.3. Redundant parameters

Suppose $\varphi_p(B)$ and $\theta_q(B)$ have a common factor, say $G(B)$:

$$\varphi_p(B) = G(B)\varphi_{p_1}(B)\,, \quad \theta_q(B) = G(B)\theta_{q_1}(B)\,. \tag{3.24}$$

Consider the models

$$\varphi_p(B)W_t = \theta_q(B)u_t \tag{3.25}$$

$$\varphi_{p_1}(B)W_t = \theta_{q_1}(B)u_t. \tag{3.26}$$

The MA(∞) representations of these two models are

$$W_t = \psi(B)u_t, \tag{3.27}$$

where

$$\psi(B) = \frac{\theta_q(B)}{\varphi_p(B)} = \frac{\theta_{q_1}(B)\,G(B)}{\varphi_{p_1}(B)\,G(B)} = \frac{\theta_{q_1}(B)}{\varphi_{p_1}(B)} \equiv \psi_1(B)\,, \tag{3.28}$$

$$W_t = \psi_1(B)u_t\,. \tag{3.29}$$

(3.25) and (3.26) have the same MA(∞) representation, hence the same autocovariance generating functions:

$$\gamma_x(z) = \sigma^2\psi(z)\psi(z^{-1}) = \sigma^2\psi_1(z)\psi_1(z^{-1})\,. \tag{3.30}$$

It is not possible to distinguish a series generated by (3.25) form one produced with (3.26). Among these two models, we will select the simpler one, *i.e.* (3.26). Further, if we tried to estimate (3.25) rather than (3.26), we would meet singularity problems (in the covariance matrix of the estimators).

4. Proofs and references

A general overview of the technique of generating functions is available in Wilf (1994).

References

WILF, H. S. (1994): *Generatingfunctionology*. Academic Press, New York, second edn.