

Model selection criteria *

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1. Introduction

On using usual preliminary specification and residual-based diagnostics, several models often appear to be essentially equivalent for representing the behavior of a time series. In such cases, it can be quite useful to use model selection criteria.

Suppose X_t follows an *ARIMA* (p, d, q) process:

$$\varphi_p(B)(1-B)^d X_t = \bar{\mu} + \theta_q(B)u_t, \quad t \geq 1-d$$

where $\{u_t : t \in Z\} \sim BB(0, \sigma^2)$.

This model is estimated from the series differentiated d times: $W_t = (1-B)^d X_t$, $t = 1, \dots, T$. Let

$$\hat{\sigma}_W^2 = \sum_{t=1}^T (W_t - \bar{W})^2 / T$$

where $\bar{W} = \sum_{t=1}^T W_t / T$, the sample variance of W_t , and let $\hat{\sigma}_T^2$ the maximum likelihood (*ML*) estimator of σ^2 :

$$\hat{\sigma}_T^2 = \sum \hat{u}_t^2 / T.$$

2. Predictive performance criteria

Since σ^2 is the variance of the one-step ahead error prediction error, it is natural to

a) minimize $\hat{\sigma}_T^2$,

or

b) maximize $R^2 = 1 - (\hat{\sigma}_T^2 / \hat{\sigma}_W^2)$.

These two criteria are equivalent. However, $\hat{\sigma}_T^2$ automatically decreases when p or q increases. In order to penalize models which contain too many parameters, it is preferable to use statistics which involve a correction for the number of degrees of freedom:

c) minimize

$$s_T^2 = \frac{T}{T-p-q} \hat{\sigma}_T^2 = \sum_t \hat{u}_t^2 / (T-p-q)$$

or

d) maximize

$$\bar{R}^2 = 1 - \frac{s_T^2}{s_W^2} = 1 - \frac{T-1}{T-p-q} \frac{\hat{\sigma}_T^2}{\hat{\sigma}_W^2}$$

where $s_W^2 = \sum_{t=1}^T (W_t - \bar{W})^2 / (T-1)$.

3. Information criteria

Another approach consists in evaluating the “distance” between the selected model and the true (unknown) model. Let $f(W)$ the density associated with the postulated model and $f_o(W)$ the density of the true model, where $W = (W_1, \dots, W_T)'$. One such distance consists in using the *Kullback distance*:

$$\begin{aligned} I(f, f_o) &= \int \log [f_o(w) / f(w)] f_o(w) dw \\ &= E_{f_o} \{ \log [f_o(W) / f(W)] \} \\ &= E_{f_o} \{ \log [f_o(W)] \} - E_{f_o} \{ \log [f(W)] \} . \end{aligned}$$

Minimizing $I(f, f_o)$ with respect to f is equivalent to minimizing $-E \{ \log [f(W)] \}$. We obtain an information criterion by selecting an “estimator” of $-E_{f_o} \{ \log (f) \}$. These different criteria take the following general form (up to an additive constant):

$$IC^* = -\frac{1}{T} \log (f) + \alpha(T) (p + q)$$

where $\alpha(T)$ is a decreasing function of T . We then try to minimize IC^* .

In the case where f is a normal density, IC^* takes the equivalent form:

$$IC = \log (\hat{\sigma}_T^2) + \alpha(T) (p + q) .$$

Different criteria are obtained by selecting different functions $\alpha(T)$. The most important ones are:

- a) $\alpha(T) = 2/T$ [Akaike (1969)];
- b) $\alpha(T) = \log (T) / T$ [Schwarz (1978)];
- c) $\alpha(T) = c \log [\log (T)] / T$ where $c > 2$ [Hannan and Quinn (1979)].

The following criteria are then obtained:

- a) Akaike criterion [Akaike (1969)]:

$$AIC(p, q) = \log (\hat{\sigma}_T^2) + \frac{2(p + q)}{T} ;$$

- b) Schwarz criterion [Schwarz (1978)]:

$$BIC(p, q) = \log (\hat{\sigma}_T^2) + (p + q) \frac{\log (T)}{T} ;$$

c) Hannan-Quinn criterion [Hannan and Quinn (1979)]:

$$\varphi(p, q) = \log(\hat{\sigma}_T^2) + c(p + q) \frac{\log[\log(T)]}{T}, \text{ where } c > 2.$$

If we assume that the true values p_o and q_o satisfy $0 \leq p_o \leq P$ and $0 \leq q_o \leq Q$, and we minimize the information criterion over all the pairs $\{(p, q) : 0 \leq p \leq P, 0 \leq q \leq Q\}$, it is possible to show [see Shibata (1976, 1980), Taniguchi (1980), Hannan and Quinn (1979), Hannan and Rissanen (1982)] that:

1. the Akaike criterion tends to identify values of p and q which are too large, i.e., the values of p and q that minimize *AIC* converge (as $T \rightarrow \infty$) towards values which are larger than p_o and q_o ;
2. the values of p and q that minimize *BIC* converge towards p_o and q_o .

4. Bibliographic notes

For a general review of this topic, see Choi (1992, Chapter 4). For further discussion, see Brockwell and Davis (1991, Section 9.3), Lütkepohl (1991, Chapter 11) and Gouriéroux and Monfort (1997, Section 6.3). On the case of integrated series, see Paulsen (1984) and Toda and Yamamoto (1995).

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