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# Econ 763: Financial Econometrics
## Exercises 1 Stochastic Processes: Answers

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1.

(a) A probability space is a triplet $(\Omega, A, P)$ where the sample space $\Omega$ consists of all possible outcomes of an experiment and the $\sigma$-algebra $A$ is a collection of all possible events on which the countably additive probability measure $P$ could be defined.

(b) A real-valued stochastic process on a non-empty index set $T$ is a collection of real-valued random variables $\{X_t, t \in T\}$ jointly defined on a probability space $(\Omega, A, P)$.

2.

(1) False. $\{X_t, t \in T\} \sim iid$ Cauchy is not in $L_2$ since it has no second moment, however it is a strictly stationary process.

(2) False. The same example as above.

(3) True. By Definition 2.16(1), a stochastic process $\{X_t, t \in T\}$ satisfies $E(|X_t|^3) < \infty, \forall t \in T$. Note that the function $f(x) = x^{3/2}$ is convex on $\mathbb{R}^+ \cup \{0\}$ as $f''(x) = \frac{3}{4}x^{-1/2} \geq 0$ for $x \geq 0$. By Jensen's inequality we have

$$E[f(X_t^2)] \geq fE[X_t^2] \Leftrightarrow E[|X_t^2|^{3/2}] \geq E[X_t^2]^{3/2}$$
$$\Leftrightarrow E[|X_t|^3]^{2/3} \geq E[X_t^2],$$

Thus $E[X_t^2] < \infty$ for all t. Next setting $n = 2$ and $m_1 = m_2 = 1$ in Definition 2.16(2) gives $E[X_{t_1}X_{t_2}] = E[X_{t_1+k}X_{t_2+k}]$ for any $k \geq 0$. Further, setting $n = 1$ and $m_1 = 1$ we obtain $E[X_{t_1}] = E[X_{t_1+k}]$ to conclude that $Cov[X_{t_1}X_{t_2}] = Cov[X_{t_1+k}X_{t_2+k}]$ for all $t_1, t_2 \in T$ and $k \geq 0$.

(4) True. All we need to do is to add a limiting argument to (3).

Setting $n = 2$ with $m_1 = m_2 = 1$ and $n = 1$ with $m_1 = 1$ we need

$$\lim_{t_1 \to \infty} E(X_{t_1}X_{t_1+\Delta_2}) = \lim_{t_1 \to \infty} E(X_{t_1+k}X_{t_1+\Delta_2+k}), \quad and \quad \lim_{t_1 \to \infty} E(X_{t_1}) = \lim_{t_1 \to \infty} E(X_{t_1+k}),$$

for any $k \geq 0$, $t_1 \in T$ and $\Delta_2$ is any positive integer.

(5) Uncertain. Let $\{u_t, t \in \mathbb{Z}\} \sim NID(0, 1)$. This process is a white noise and stationary of order 4. Now let $\{u_t, t \in \mathbb{Z}\}$ be a collection of independent $t(3)$ ( i.e., t distribution with degrees of freedom 3) distributed random variables. Then, since $u_t$ does not possess moments greater than 2, this is not stationary process of order 4.

3. (a) By definition $\gamma(k) = Cov[X_t, X_{t+k}]$, so setting $k = 0$ we have $\gamma(0) = Cov[X_t, X_t] = Var[X_t]$. Also, since $\gamma(k) = Cov[X_t, X_{t+k}] = Cov[X_t, X_{t-k}]$ and $\gamma(-k) = Cov[X_t, X_{t-k}]$, we

obtain $\gamma(k) = \gamma(-k)$. For vector valued second-order stationary process, $\gamma(k) = \gamma(-k)'$ and for complex valued scalar series $\gamma(k) = \overline{\gamma(-k)}$.

(b) $|\gamma(k)| \leq \gamma(0), \ \forall \ \ k \in \mathbb{Z}$. This is an immediate consequences of Cauchy-Schwarz inequality.

$$\begin{aligned}|Cov(X_t X_{t+k})| &\leq \sqrt{Var(X_t)Var(X_{t+k})} \\ &= \sqrt{Var(X_t)^2} = Var(X_t),\end{aligned}$$

or, equivalently $|\gamma(k)| \leq \gamma(0)$.

(c) For any $a = (a_1, ..., a_k)' \in \mathbb{R}^k$, we expand $Var(\sum_{i=1}^{k} a_i X_{t_i}) \geq 0$ and find the desired result.

$$\sum_{i=1}^{k}\sum_{j=1}^{k} a_i a_j Cov[X_{t_i} X_{t_j}] = \sum_{i=1}^{k}\sum_{j=1}^{k} a_i a_j \gamma(t_i - t_j) = a'\gamma(k)a \geq 0.$$

Therefore $\gamma(k)$ is positive semi-definite.

4.

(a) Note that

$$E[X_t] = \sum_{j=1}^{m}[E(A_j)\cos(v_j t) + E(B_j)\sin(v_j t)] = 0,$$

and

$$\begin{aligned}Cov[X_t, X_{t+k}] &= E[X_t, X_{t+k}] \\ &= \sum_{j=1}^{m} E(A_j^2)\cos(v_j t)\cos(v_j(t+k)) + E(B_j^2)\sin(v_j t)\sin(v_j(t+k)) \\ &= \sum_{j=1}^{m} \sigma_j^2 \cos(v_j(t+k-t)) = \sum_{j=1}^{m} \sigma_j^2 \cos(v_j k),\end{aligned}$$

for all $k \in \mathbb{N} \cup \{0\}$. As the above quantities are independent of t and finite, this process is second order stationary.

(b) For $m = 1$, the series becomes $X_t = A_1\cos(v_1 t) + B_1 \sin(v_1 t)$. Once $A_1$ and $B_1$ are realized, $X_t$ behaves as a deterministic function of $t$. Indeed two observations are sufficient to determine $A_1$ and $B_1$:

$$\begin{bmatrix} A_1 \\ B_1 \end{bmatrix} = \begin{bmatrix} \cos(v_j t_1) & \sin(v_j t_1) \\ \cos(v_j t_2) & \sin(v_j t_2) \end{bmatrix}^{-1} \begin{bmatrix} X_{t_1} \\ X_{t_1} \end{bmatrix},$$

provided $v_j(t_1 + t_2) \neq \pi\mathbb{Z}$. Based on the values of $A_1$ and $B_1$ all past and future values of the process can be predicted perfectly, so this process is deterministic.