Estimation of the mean and autocorrelations of a stationary process *

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### List of Definitions, Assumptions, Propositions and Theorems

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1. General distributional results

1.1 Suppose we have \(T\) observations \(X_1, X_2, \ldots, X_T\) from a realization of a second-order stationary process. The natural estimators of the first and second moments of the process are: for the mean,

\[
\overline{X}_T = \frac{1}{T} \sum_{t=1}^{T} X_t ,
\]

for the autocovariances

\[
c_k = \frac{1}{T} \sum_{t=1}^{T-k} (X_t - \overline{X}_T) (X_{t+k} - \overline{X}_T) , \quad 1 \leq k \leq T - 1 ,
\]

and for the autocorrelations

\[
r_k = c_k / c_0 , \quad 1 \leq k \leq T - 1 .
\]

1.2 Theorem Distribution of the Arithmetic Mean. Let \(\{X_t : t \in \mathbb{Z}\}\) a second-order stationary process with mean \(\mu\), and let \(\overline{X}_T = \sum_{t=1}^{T} X_t / T\). Then

1. \(E(\overline{X}_T) = \mu\) and \(\overline{X}_T\) is an unbiased estimator of \(\mu\);

2. \(\text{Var}(\overline{X}_T) = \frac{1}{T} \sum_{k=-\lfloor T/2 \rfloor}^{T-1} \Big( 1 - \frac{|k|}{T} \Big) \gamma_x(k) ;\)

3. if \(\gamma_x(k) \xrightarrow[k \to \infty]{} 0\),

\[
\text{Var}(\overline{X}_T) \xrightarrow{T \to \infty} 0 \text{ and } \overline{X}_T \xrightarrow{m.q.} \mu ;
\]

4. if the series \(\sum_{k=-\infty}^{\infty} \gamma_x(k)\) converges, then

\[
\lim_{T \to \infty} T \text{Var}(\overline{X}_T) = \sum_{k=-\infty}^{\infty} \gamma_x(k) ;
\]

5. if the spectral density \(f_x(\omega)\) exists and is continuous at \(\omega = 0\), then

\[
\lim_{T \to \infty} T \text{Var}(\overline{X}_T) = 2\pi f_x(0) ;
\]

6. if

\[
X_t = \mu + \sum_{j=-\infty}^{\infty} \psi_j u_{t-j} , \quad \text{where } \{u_t : t \in \mathbb{Z}\} \sim \text{IID } (0, \sigma^2) ,
\]

and

\[
\sum_{j=-\infty}^{\infty} |\psi_j| < \infty ,
\]

then

\[
\sqrt{T} (\overline{X}_T - \mu) \xrightarrow{T \to \infty} N \left[ 0, \sum_{k=-\infty}^{\infty} \gamma_x(k) \right]
\]
and
\[ \sum_{k=-\infty}^{\infty} \gamma_x(k) = \sigma^2 \left( \sum_{j=-\infty}^{\infty} \psi_j \right)^2. \]

**Proof.** See Anderson (1971, Sections 8.3.1 and 8.4.1) and Brockwell and Davis (1991, Section 7.1). \(\square\)

### 1.3 Theorem  Distribution of Sample Autocorrelations for a Linear Stationary Process

Let \(X_t = \mu + \sum_{j=-\infty}^{\infty} \psi_j u_{t-j}, \) where \(\sum_{j=-\infty}^{\infty} j|\psi_j| < \infty\) and \(\{u_t : t \in \mathbb{Z}\} \sim \text{IID} \left(0, \sigma^2\right)\). If
\[
(a) \sum_{j=-\infty}^{\infty} |j| \psi_j^2 < \infty \]
or
\[
(b) \mathbb{E}(u_t^4) < \infty, \forall t, \]
then the asymptotic distribution of the vector
\[ \left[ \sqrt{T} (r_1 - \rho_1), \sqrt{T} (r_2 - \rho_2), \ldots, \sqrt{T} (r_m - \rho_m) \right] \]
is \(N[0, W_m]\) as \(T \to \infty\), where \(\rho_k = \gamma_x(k) / \gamma_x(0)\), \(W_m = [w_{jk}]_{j,k=1,\ldots,m}\) and
\[
w_{jk} = \sum_{h=-\infty}^{\infty} \left( \rho_{h+j}\rho_{h+k} + \rho_{h-j}\rho_{h+k} - 2 \rho_k \rho_{h+j} - 2 \rho_j \rho_h \rho_{h+k} + 2 \rho_j \rho_k \rho_h^2 \right) \\
= \sum_{h=-1}^{\infty} \left( \rho_{h+j} + \rho_{h-j} - 2 \rho_j \rho_h \right) \left( \rho_{h+k} + \rho_{h-k} - 2 \rho_k \rho_h \right) \\
= \frac{4\pi}{\gamma_x(0)^2} \int_{-\pi}^{\pi} \left[ \cos(\omega j) - \rho_j \right] \left[ \cos(\omega k) - \rho_k \right] f_x(\omega)^2 d\omega.
\]

**Proof.** See Anderson (1971, Theorem 8.4.6, p. 489) and Brockwell and Davis (1991, Theorems 7.2.1 and 7.2.2). \(\square\)

### 1.4 The expressions \(w_{jk}\) are called Bartlett’s formula for the covariances of the autocorrelations. The formula \(w_{jk}\) may also be written
\[
w_{jk} = \left( \lambda_{j+k} + \lambda_{j-k} - 2 \rho_j \lambda_k - 2 \rho_k \lambda_j + 2 \rho_j \rho_k \lambda_0 \right) / \gamma_0^2 \]
\[
= \bar{\lambda}_{j+k} + \bar{\lambda}_{j-k} - 2 \rho_j \bar{\lambda}_k - 2 \rho_k \bar{\lambda}_j - 2 \rho_j \rho_k \bar{\lambda}_0
\]
where
\[
\lambda_i = \sum_{h=-\infty}^{\infty} \gamma_h \gamma_{h+i}, \quad \bar{\lambda}_i = \lambda_i / \gamma_0^2 = \sum_{h=-\infty}^{\infty} \rho_h \rho_{h+i}.
\]

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2. Special cases

2.1 Asymptotic variance. Under the conditions of Theorem 1.3, the asymptotic distribution of \( \sqrt{T} (r_k - \rho_k) \) is \( N[0, w_{kk}] \), where

\[
{w}_{kk} = \sum_{h=-\infty}^{\infty} (\rho_h^2 + \rho_{h-k} \rho_{h+k} - 4 \rho_k \rho_h \rho_{h+k} + 2 \rho_k^2 \rho_h^2)
\]

\[
= \sum_{h=-\infty}^{\infty} (\rho_h^2 + \rho_h \rho_{h+2k} - 4 \rho_h \rho_k \rho_{h+k} + 2 \rho_h^2 \rho_k^2)
\]

\[
= \sum_{h=1}^{\infty} (\rho_{h+k} + \rho_{h-k} - 2 \rho_h \rho_k)^2.
\]

For \( T \) large, \( \sqrt{T} (r_k - \rho_k) \overset{a}{\sim} N[0, w_{kk}] \).

2.2 White noise. If

\[
\rho_k = 1, \text{ for } k = 0,
\]

\[
= 0, \text{ for } k \neq 0,
\]

we find

\[
{w}_{jk} = 1, \text{ if } j = k
\]

\[
= 0, \text{ if } j \neq k.
\]

For \( T \) large, the sampling autocorrelations are mutually uncorrelated and

\( \sqrt{T} r_k \overset{a}{\sim} N[0, 1] \), for \( k \geq 1 \).

2.3 MA(q) process. If \( \rho_k = 0 \), for \( |k| \geq q + 1 \), we find

\[
{w}_{jk} = \sum_{h=1}^{\infty} \rho_{h-j} \rho_{h-k} = \sum_{h=1}^{\infty} \rho_{j-h} \rho_{k-h} = \sum_{h=1}^{\infty} \rho_{k-h+(j-k)} \rho_{k-h}
\]

\[
= \sum_{h=-\infty}^{k-1} \rho_h \rho_{k+(j-k)} = \sum_{h=-q}^{q-(j-k)} \rho_h \rho_{k+(j-k)} , \text{ for } j \geq k \geq q + 1,
\]

hence

\[
{w}_{jk} = 0, \text{ if } k \geq q + 1 \text{ and } j \geq k + 2q + 1
\]

\[
= \sum_{h=-q}^{q-(j-k)} \rho_h \rho_{k+(j-k)} , \text{ if } q + 1 \leq k \leq j \leq k + 2q.
\] (2.1)

In particular,

\[
{w}_{kk} = \sum_{h=-q}^{q} \rho_h^2 = 1 + 2 \sum_{h=1}^{q} \rho_h^2 , \text{ if } k \geq q + 1.
\]
3. Exact tests of randomness

3.1 Theorem  Exact moments of autocorrelations for an i.i.d. sample.  Let the random variables $X_1, \ldots, X_T$ be independent and identically distributed (i.i.d.) according to a continuous distribution. Then

$$E(r_k) = -\frac{T-k}{T(T-1)}, \text{ for } 1 \leq k \leq T-1,$$

and

$$\text{Var}(r_k) \leq \overline{V}_k,$$

where

$$\overline{V}_k \equiv \frac{T^4 - (k+7)T^3 + (7k+16)T^2 + 2(k^2-9k-6)T - 4k(k-4)}{T(T-1)^2(T-2)(T-3)}$$

if $1 \leq k < T/2$ and $T > 3$, and

$$\overline{V}_k \equiv \frac{(T-k)\left[T^2 - 3T - 2(k-2)\right]}{T(T-1)^2(T-3)}$$

if $T/2 \leq k < T$ and $T > 3$.

Proof. See Dufour and Roy (1985).

3.2 For $k = 1$, we find

$$E(r_1) = -1/T,$$

$$\text{Var}(r_1) \leq \frac{T-2}{T(T-1)}.$$

By Chebyshev’s inequality,

$$P[|r_k - E(r_k)| \geq \lambda] \leq \frac{\text{Var}(r_k)}{\lambda^2} \leq \frac{\overline{V}_k}{\lambda^2}.$$  

3.3 Theorem  Exact moments of autocorrelations for a Gaussian i.i.d. sample.  Let $X_1, \ldots, X_T$ be i.i.d. random variables following a distribution $N(\mu, \sigma^2)$ distribution. Then

$$E(r_k) = -\frac{(T-k)}{T(T-1)}, \text{ for } 1 \leq k \leq T-1,$$

$$\text{Var}(r_k) = \frac{T^4 - (k+3)T^3 + 3kT^2 + 2(k+1)T - 4k^2}{(T+1)T^2(T-1)^2},$$

for $1 \leq k < T/2$ and $T > 3$, and

$$\text{Var}(r_k) = \frac{(T-k)(T-2)(T^2 + T - 2k)}{(T+1)T^2(T-1)^2}.$$
for $T/2 \leq k < T$ and $T > 3$. Furthermore, for $1 \leq k < h \leq T - 1$,

$$
Cov(r_k, r_h) = \frac{2[kh(T-1)-(T-h)(T^2-k)]}{(T+1)T^2(T-1)^2}
$$

if $l < h + k < T$, and

$$
Cov(r_k, r_h) = \frac{2(T-h)[2k-(k+1)T]}{(T+1)T^2(T-1)^2}
$$

if $h + k \geq T$.

PROOF. See Dufour and Roy (1985).

3.4 For $T$ large, we have

$$
\frac{r_k - E(r_k)}{\sqrt{Var(r_k)}} \sim N(0,1)
$$

In small or moderately large samples, the normal approximation is much more accurate when the formulae for $E(r_k)$ et $Var(r_k)$ given by Theorem 3.3 are used, rather than $E(r_k) = 0$ et $Var(r_k) = 1/T$; see Dufour and Roy (1985).
References

