

Estimation of the mean and autocorrelations of a stationary process *

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1. General distributional results

1.1 Suppose we have T observations X_1, X_2, \dots, X_T from a realization of a second-order stationary process. The natural estimators of the first and second moments of the process are: for the mean,

$$\bar{X}_T = \frac{1}{T} \sum_{t=1}^T X_t,$$

for the autocovariances

$$c_k = \frac{1}{T} \sum_{t=1}^{T-k} (X_t - \bar{X}_T) (X_{t+k} - \bar{X}_T), \quad 1 \leq k \leq T-1,$$

and for the autocorrelations

$$r_k = c_k/c_0, \quad 1 \leq k \leq T-1.$$

1.2 Theorem DISTRIBUTION OF THE ARITHMETIC MEAN. Let $\{X_t : t \in \mathbb{Z}\}$ a second-order stationary process with mean μ , and let $\bar{X}_T = \sum_{t=1}^T X_t / T$. Then

(1) $E(\bar{X}_T) = \mu$ and \bar{X}_T is an unbiased estimator of μ ;

(2) $Var(\bar{X}_T) = \frac{1}{T} \sum_{k=-(T-1)}^{T-1} \left(1 - \frac{|k|}{T}\right) \gamma_x(k)$;

(3) if $\gamma_x(k) \xrightarrow[k \rightarrow \infty]{} 0$,

$$Var(\bar{X}_T) \xrightarrow[T \rightarrow \infty]{} 0 \text{ and } \bar{X}_T \xrightarrow[T \rightarrow \infty]{m.q.} \mu;$$

(4) if the series $\sum_{k=-\infty}^{\infty} \gamma_x(k)$ converges, then

$$\lim_{T \rightarrow \infty} T Var(\bar{X}_T) = \sum_{k=-\infty}^{\infty} \gamma_x(k);$$

(5) if the spectral density $f_x(\omega)$ exists and is continuous at $\omega = 0$, then

$$\lim_{T \rightarrow \infty} T Var(\bar{X}_T) = 2\pi f_x(0);$$

(6) if

$$X_t = \mu + \sum_{j=-\infty}^{\infty} \psi_j u_{t-j}, \text{ where } \{u_t : t \in \mathbb{Z}\} \sim IID(0, \sigma^2),$$

and

$$\sum_{j=-\infty}^{\infty} |\psi_j| < \infty,$$

then

$$\sqrt{T}(\bar{X}_T - \mu) \xrightarrow[T \rightarrow \infty]{L} N\left[0, \sum_{k=-\infty}^{\infty} \gamma_x(k)\right]$$

and

$$\sum_{k=-\infty}^{\infty} \gamma_x(k) = \sigma^2 \left(\sum_{j=-\infty}^{\infty} \psi_j \right)^2.$$

PROOF. See Anderson (1971, Sections 8.3.1 and 8.4.1) and Brockwell and Davis (1991, Section 7.1). \square

1.3 Theorem DISTRIBUTION OF SAMPLE AUTOCORRELATIONS FOR A LINEAR STATIONARY PROCESS. Let $X_t = \mu + \sum_{j=-\infty}^{\infty} \psi_j u_{t-j}$, where $\sum_{j=-\infty}^{\infty} |\psi_j| < \infty$ and $\{u_t : t \in \mathbb{Z}\} \sim IID(0, \sigma^2)$. If

$$(a) \sum_{j=-\infty}^{\infty} |j| \psi_j^2 < \infty$$

or

$$(b) E(u_t^4) < \infty, \forall t,$$

then the asymptotic distribution of the vector

$$\left[\sqrt{T}(r_1 - \rho_1), \sqrt{T}(r_2 - \rho_2), \dots, \sqrt{T}(r_m - \rho_m) \right]'$$

is $N[0, W_m]$ as $T \rightarrow \infty$, where $\rho_k = \gamma_x(k) / \gamma_x(0)$, $W_m = [w_{jk}]_{j,k=1,\dots,m}$ and

$$\begin{aligned} w_{jk} &= \sum_{h=-\infty}^{\infty} (\rho_{h+j} \rho_{h+k} + \rho_{h-j} \rho_{h+k} - 2 \rho_k \rho_h \rho_{h+j} - 2 \rho_j \rho_h \rho_{h+k} + 2 \rho_j \rho_k \rho_h^2) \\ &= \sum_{h=1}^{\infty} (\rho_{h+j} + \rho_{h-j} - 2 \rho_j \rho_h) (\rho_{h+k} + \rho_{h-k} - 2 \rho_k \rho_h) \\ &= \frac{4\pi}{\gamma_x(0)^2} \int_{-\pi}^{\pi} [\cos(\omega j) - \rho_j] [\cos(\omega k) - \rho_k] f_x(\omega)^2 d\omega. \end{aligned}$$

PROOF. See Anderson (1971, Theorem 8.4.6, p. 489) and Brockwell and Davis (1991, Theorems 7.2.1 and 7.2.2). \square

1.4 The expressions w_{jk} are called Bartlett's formula for the covariances of the autocorrelations. The formula w_{jk} may also be written

$$\begin{aligned} w_{jk} &= (\lambda_{j+k} + \lambda_{j-k} - 2\rho_j \lambda_k - 2\rho_k \lambda_j + 2\rho_j \rho_k \lambda_0) / \gamma_0^2 \\ &= \bar{\lambda}_{j+k} + \bar{\lambda}_{j-k} - 2\rho_j \bar{\lambda}_k - 2\rho_k \bar{\lambda}_j - 2\rho_j \rho_k \bar{\lambda}_0 \end{aligned}$$

where

$$\lambda_i = \sum_{h=-\infty}^{\infty} \gamma_h \gamma_{h+i}, \quad \bar{\lambda}_i \equiv \lambda_i / \gamma_0^2 = \sum_{h=-\infty}^{\infty} \rho_h \rho_{h+i}.$$

2. Special cases

2.1 ASYMPTOTIC VARIANCE. Under the conditions of Theorem **1.3**, the asymptotic distribution of $\sqrt{T}(r_k - \rho_k)$ is $N[0, w_{kk}]$, where

$$\begin{aligned} w_{kk} &= \sum_{h=-\infty}^{\infty} (\rho_{h+k}^2 + \rho_{h-k}\rho_{h+k} - 4\rho_k\rho_h\rho_{h+k} + 2\rho_k^2\rho_h^2) \\ &= \sum_{h=-\infty}^{\infty} (\rho_h^2 + \rho_h\rho_{h+2k} - 4\rho_h\rho_k\rho_{h+k} + 2\rho_h^2\rho_k^2) \\ &= \sum_{h=1}^{\infty} (\rho_{h+k} + \rho_{h-k} - 2\rho_h\rho_k)^2 . \end{aligned}$$

For T large, $\sqrt{T}(r_k - \rho_k) \stackrel{a}{\sim} N[0, w_{kk}]$.

2.2 WHITE NOISE. If

$$\begin{aligned} \rho_k &= 1, \text{ for } k = 0, \\ &= 0, \text{ for } k \neq 0, \end{aligned}$$

we find

$$\begin{aligned} w_{jk} &= 1, \text{ if } j = k \\ &= 0, \text{ if } j \neq k . \end{aligned}$$

For T large, the sampling autocorrelations are mutually uncorrelated and

$$\sqrt{T}r_k \stackrel{a}{\sim} N[0, 1], \text{ for } k \geq 1 .$$

2.3 MA(q) PROCESS. If $\rho_k = 0$, for $|k| \geq q + 1$, we find

$$\begin{aligned} w_{jk} &= \sum_{h=1}^{\infty} \rho_{h-j}\rho_{h-k} = \sum_{h=1}^{\infty} \rho_{j-h}\rho_{k-h} = \sum_{h=1}^{\infty} \rho_{k-h+(j-k)}\rho_{k-h} \\ &= \sum_{h=-\infty}^{k-1} \rho_h\rho_{h+(j-k)} = \sum_{h=-q}^{q-(j-k)} \rho_h\rho_{h+(j-k)}, \text{ for } j \geq k \geq q + 1, \end{aligned}$$

hence

$$\begin{aligned} w_{jk} &= 0, && \text{if } k \geq q + 1 \text{ and } j \geq k + 2q + 1 \\ &= \sum_{h=-q}^{q-(j-k)} \rho_h\rho_{h+(j-k)}, && \text{if } q + 1 \leq k \leq j \leq k + 2q . \end{aligned} \tag{2.1}$$

In particular,

$$w_{kk} = \sum_{h=-q}^q \rho_h^2 = 1 + 2 \sum_{h=1}^q \rho_h^2, \text{ if } k \geq q + 1 .$$

3. Exact tests of randomness

3.1 Theorem EXACT MOMENTS OF AUTOCORRELATIONS FOR AN *i.i.d.* SAMPLE. *Let the random variables X_1, \dots, X_T be independent and identically distributed (*i.i.d.*) according to a continuous distribution. Then*

$$E(r_k) = -\frac{T-k}{T(T-1)}, \text{ for } 1 \leq k \leq T-1,$$

and

$$\text{Var}(r_k) \leq \bar{V}_k,$$

where

$$\bar{V}_k \equiv \frac{T^4 - (k+7)T^3 + (7k+16)T^2 + 2(k^2 - 9k - 6)T - 4k(k-4)}{T(T-1)^2(T-2)(T-3)}$$

if $1 \leq k < T/2$ and $T > 3$, and

$$\bar{V}_k \equiv \frac{(T-k)[T^2 - 3T - 2(k-2)]}{T(T-1)^2(T-3)}$$

if $T/2 \leq k < T$ and $T > 3$.

PROOF. See Dufour and Roy (1985). □

3.2 For $k = 1$, we find

$$E(r_1) = -1/T,$$

$$\text{Var}(r_1) \leq \frac{T-2}{T(T-1)}.$$

By Chebyshev's inequality,

$$P[|r_k - E(r_k)| \geq \lambda] \leq \frac{\text{Var}(r_k)}{\lambda^2} \leq \frac{\bar{V}_k}{\lambda^2}.$$

3.3 Theorem EXACT MOMENTS OF AUTOCORRELATIONS FOR A GAUSSIAN *i.i.d.* SAMPLE. *Let X_1, \dots, X_T be *i.i.d.* random variables following a distribution $N[\mu, \sigma^2]$ distribution. Then*

$$E(r_k) = -\frac{(T-k)}{T(T-1)}, \text{ for } 1 \leq k \leq T-1,$$

$$\text{Var}(r_k) = \frac{T^4 - (k+3)T^3 + 3kT^2 + 2k(k+1)T - 4k^2}{(T+1)T^2(T-1)^2}$$

for $1 \leq k < T/2$ and $T > 3$, and

$$\text{Var}(r_k) = \frac{(T-k)(T-2)(T^2 + T - 2k)}{(T+1)T^2(T-1)^2}$$

for $T/2 \leq k < T$ and $T > 3$. Furthermore, for $1 \leq k < h \leq T - 1$,

$$Cov(r_k, r_h) = \frac{2[kh(T-1) - (T-h)(T^2-k)]}{(T+1)T^2(T-1)^2}$$

if $l < h + k < T$, and

$$Cov(r_k, r_h) = \frac{2(T-h)[2k - (k+1)T]}{(T+1)T^2(T-1)^2}$$

if $h + k \geq T$.

PROOF. See Dufour and Roy (1985). □

3.4 For T large, we have

$$\frac{r_k - E(r_k)}{[Var(r_k)]^{1/2}} \stackrel{a}{\sim} N(0, 1) .$$

In small or moderately large samples, the normal approximation is much more accurate when the formulae for $E(r_k)$ et $Var(r_k)$ given by Theorem 3.3 are used, rather than $E(r_k) = 0$ et $Var(r_k) = 1/T$; see Dufour and Roy (1985).

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