Linear models with nonscalar covariance matrix and generalized least squares [*]

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1. Generalized least squares

1.1. Best linear unbiased estimator

$$y = X\beta + u \tag{1.1}$$

where y is a $T \times 1$ vector of observations on a dependent variable, X is a $T \times k$ nonstochastic matrix of rank k, and u is a $T \times 1$ vector of disturbances (errors) such that

$$\begin{aligned} \mathsf{E}(u) &= 0 \\ \mathsf{V}(u) &= \sigma^2 V \end{aligned} \tag{1.2}$$

and V is a known $T \times T$ positive definite matrix. Then the least-squares estimator

$$\hat{\beta} = (X'X)^{-1}X'y \tag{1.3}$$

is unbiased but does not have minimal variance. The covariance matrix of $\hat{\beta}$ is

$$\mathsf{V}(\hat{\beta}) = \sigma^2(X'X)^{-1}X'VX(X'X)^{-1} \tag{1.4}$$

so that the usual formula

$$\mathsf{V}(\hat{\beta}) = \sigma^2(X'X)^{-1} \tag{1.5}$$

is not valid.

The fact V is positive definite entails that $|V| \neq 0$, so there is no perfect correlation between the disturbances. Further, there exists a nonsingular $T \times T$ matrix P such that

$$PVP' = I_T\,, \tag{1.6}$$

$$(P')^{-1}V^{-1}P^{-1} = (PVP')^{-1} = I_T\,. \tag{1.7}$$

Multiply both sides of (1.1) by P :

$$Py = PX\beta + Pu\,. \tag{1.8}$$

We get in this way the transformed model

$$y_* = X_*\beta + u_* \tag{1.9}$$

where

$$y_* = Py\,, \quad X_* = PX\,, \quad u_* = Pu \tag{1.10}$$

$$\mathsf{E}(u_*) = 0\,, \tag{1.11}$$

$$\mathsf{V}(u_*) = \mathsf{E}\left[Puu'P'\right] = \sigma^2 PVP' = \sigma^2 I_T\,. \tag{1.12}$$

Then

$$\hat{\beta}_G = \left(X_*' X_*\right)^{-1} X_*' y_* \tag{1.13}$$

is the best linear unbiased estimator of β :

$$\begin{aligned} \mathsf{E}(\hat{\beta}_G) &= \beta \\ \mathsf{V}(\hat{\beta}_G) &= \sigma^2 (X_*' X_*)^{-1}. \end{aligned} \tag{1.14}$$

We can also write:

$$\hat{\beta}_G = \left(X'P'PX\right)^{-1} X'P'Py = \left(X'V^{-1}X\right)^{-1} X'V^{-1}y \tag{1.15}$$

for

$$\begin{aligned} PVP' = I_T \quad &\Rightarrow V = P^{-1}(P')^{-1} = (P'P)^{-1} \\ &\Rightarrow V^{-1} = P'P\,. \end{aligned} \tag{1.16}$$

$\hat{\beta}_G$ is called the generalized least squares estimator of β :

$$\begin{aligned} \mathsf{E}\left(\hat{\beta}_G\right) &= \beta\,, \\ \mathsf{V}\left(\hat{\beta}_G\right) &= \sigma^2 \left(X_*' X_*\right) = \sigma^2 \left(X'V^{-1}X\right)^{-1}. \end{aligned} \tag{1.17}$$

We know that $\hat{\beta}$ minimizes

$$(y - X\beta)' (y - X\beta)\,. \tag{1.18}$$

Similarly, $\hat{\beta}_G$ minimizes

$$\begin{aligned} (y_* - X_*\beta)' (y_* - X_*\beta) &= (Py - PX\beta)' (Py - PX\beta) \\ &= (y - X\beta)' P'P (y - X\beta) \\ &= (y - X\beta)' V^{-1} (y - X\beta) \end{aligned}$$

This is why $\hat{\beta}_G$ is also called a *weighted least squares* estimator of β.

1.2. Gaussian case

Suppose

$$u \sim N\left[0, \sigma^2 V\right] \tag{1.19}$$

Then

$$\hat{\beta}_G \sim N\left[\beta, \sigma^2\left(X'V^{-1}X\right)^{-1}\right] \tag{1.20}$$

is the best mean squares unbiased estimator of β.

We can build tests and confidence intervals in the usual manner by using the transformed model

$$(Py) = (PX)\beta + (Pu) \tag{1.21}$$

instead of

$$y = X\beta + u\,. \tag{1.22}$$

2. Estimation with heteroskedasticity

2.1. Known variance structure

Suppose

$$\mathsf{E}\left[uu'\right]=\sigma^2\begin{bmatrix} d_1^2 & 0 & \cdots & 0 \\ 0 & d_2^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_T^2 \end{bmatrix}=\sigma^2 V\,. \tag{2.1}$$

The variance of each element of u is then

$$\mathsf{V}(u_t)=\sigma_t^2=d_t^2\sigma^2 \tag{2.2}$$

and we have:

$$y_t = x_t'\beta + u_t\;, \quad t=1,\ldots,T$$

$$\frac{y_t}{d_t}=\frac{1}{d_t}x_t'\beta+\frac{u_t}{d_t}\;, \quad t=1,\ldots,T \tag{2.3}$$

$$y_{*t}=x_{*t}'\beta+u_{*t}\;,\; t=1,\ldots,T \tag{2.4}$$

$$\mathsf{V}(u_{*t}^2)=\mathsf{V}\left(\frac{u_t}{d_t}\right)=\sigma^2\frac{d_t^2}{d_t^2}=\sigma^2 \tag{2.5}$$

$$P=\begin{bmatrix} 1/d_1 & 0 & \cdots & 0 \\ 0 & 1/d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1/d_T \end{bmatrix} \tag{2.6}$$

2.2. Unknown variance structure

It is rare that $d_1,\ldots,d_T$ are known.
It is impossible to estimate $T+k$ parameters with T observations (incodental parameter problem)..
One must make hypotheses on the form of the variance structure.

1. $d_t^2=c\,(x_{tk})^2$
   where x_k is one of the explanatory variables or another variable. Then

$$\begin{aligned} \frac{y_t}{x_{tk}} &= \frac{1}{x_{tk}}x_t'\beta+\frac{u_t}{x_{tk}}\;, \quad t=1,\ldots,T \\ \mathsf{V}\left(\frac{u_t}{x_{tk}}\right) &= \sigma^2 c = c\sigma^2 \end{aligned} \tag{2.7}$$

2. $\sigma_t^2=c\,(\mathsf{E}y_t)^2=c\,(x_t'\beta)^2$

Then

$$\frac{y_t}{\mathsf{E}(y_t)} = \frac{1}{\mathsf{E}(y_t)} x_t' \beta + \frac{u_t}{\mathsf{E}(y_t)} \,, \quad t = 1, \ldots, T \tag{2.8}$$

A difficulty here is that $\mathsf{E}(y_t) = x_t'\beta$ is unknown. This suggests a two-step procedure.

1. Estimate β par OLS. This is reasonable because $\hat{\beta}$ is unbiased.
2. The model is then transformed according to:

$$\frac{y_t}{x_t'\hat{\beta}} = \left(\frac{1}{x_t'\hat{\beta}} x_t'\right) \beta + \frac{u_t}{x_t'\hat{\beta}} \,. \tag{2.9}$$

   In this way, the model becomes "approximately homoskedastic". For T large this leads to efficient estimators and valid tests and confidence intervals.