Basic asymptotic theory *

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1. Stochastic convergence

1.1 Basic definitions

1.1 Definition Let \( \{X_n = X_n(\omega) : n = 1, 2, \ldots\} \) a sequence of real r.v.’s defined on a probability space \((\Omega, \mathcal{A}, P)\) and \(X = X(\omega)\) another real r.v. defined on the same space.

(a) \(X_n\) converges in probability to \(X\) as \(n \to \infty\) (denoted \(X_n \xrightarrow{p} X\)) iff
\[
\lim_{n \to \infty} P[|X_n - X| > \varepsilon] = 0, \quad \forall \varepsilon > 0. \tag{1.1}
\]

(b) \(X_n\) converges almost surely to \(X\) as \(n \to \infty\) (denoted \(X_n \xrightarrow{a.s.} X\)) iff
\[
P \left( \lim_{n \to \infty} X_n = X \right) = 1. \tag{1.2}
\]

(c) Suppose \(E|X_n|^r < \infty, \forall n\), where \(r > 0\). \(X_n\) converges in mean of order \(r\) to \(X\) (denoted \(X_n \xrightarrow{L_r} X\)) iff
\[
\lim_{n \to \infty} E[|X_n - X|^r] = 0. \tag{1.3}
\]

In this case, we also say that \(X_n\) converges to \(X\) in quadratic mean (q.m.).

(d) Let \(F_n(x)\) and \(F(x)\) be the distribution functions of \(X_n\) and \(X\) respectively. \(X_n\) converges in law (or in distribution) to \(X\) as \(n \to \infty\) (denoted \(X_n \xrightarrow{L} X\)) iff
\[
\lim_{n \to \infty} F_n(x) = F(x) \text{ at all continuity points of } F(x). \tag{1.4}
\]

An important special case of the above concepts is the one where \(X\) is a fixed real constant \(c\).

1.2 Definition Let \(\{X_n = X_n(\omega) : n = 1, 2, \ldots\}\) a sequence of real r.v.’s defined on a probability space \((\Omega, \mathcal{A}, P)\) and \(c\) a real constant.

(a) \(X_n\) converges in probability to \(c\) as \(n \to \infty\) (denoted \(X_n \xrightarrow{p} c\)) iff
\[
\lim_{n \to \infty} P[|X_n - c| > \varepsilon] = 0, \quad \forall \varepsilon > 0. \tag{1.5}
\]

(b) \(X_n\) converges almost surely to \(c\) as \(n \to \infty\) (denoted \(X_n \xrightarrow{a.s.} c\)) iff
\[
P \left( \lim_{n \to \infty} X_n = c \right) = 1. \tag{1.6}
\]

(c) Suppose \(E|X_n|^r < \infty, \forall n\), where \(r > 0\). \(X_n\) converges in mean of order \(r\) to \(c\) (denoted \(X_n \xrightarrow{L_r} c\)) iff
\[
\lim_{n \to \infty} E[|X_n - c|^r] = 0. \tag{1.7}
\]
In this case, we also say that $X_n$ converges to $c$ in $L_r$. If $r = 2$, we say $X_n$ converges to $c$ in quadratic mean (q.m.).

1.3 Proposition Unicity of probability limit. Let $\{X_n : n = 1, 2, \ldots\}$ be a sequence of real r.v.'s defined on a probability space $(\Omega, \mathcal{A}, P)$, and let $X$ and $Y$ be two real r.v.'s defined on the same probability space. Then

$$X_n \xrightarrow{P} X \text{ and } X_n \xrightarrow{P} Y \Rightarrow P[X \neq Y] = 0.$$  \hspace{1cm} (1.8)

1.2. Relations between convergence concepts

1.4 Assumption Let $\{X_n\} \equiv \{X_n : n = 1, 2, \ldots\}$ be a sequence of real r.v.'s defined on a probability space $(\Omega, \mathcal{A}, P)$ and $X$ another real r.v. defined on the same space.

Unless stated otherwise, this assumption will hold for all the definitions, propositions and theorems in this section.

1.5 Proposition Relations between convergence concepts.

(a) $X_n \xrightarrow{a.s.} X \Rightarrow X_n \xrightarrow{P} X \Rightarrow X_n \xrightarrow{L} X$.

(b) $X_n \xrightarrow{r} X \Rightarrow X_n \xrightarrow{s} X$ for all $s$ such that $0 < s \leq r \Rightarrow X_n \xrightarrow{P} X \Rightarrow X_n \xrightarrow{L} X$.

1.6 Remark In general, the implications in 1.5 (b) and (c) cannot be reversed.

1.3. Convergence of expectations and functions of random variables

1.1 Assumption Let $\{X_n : n = 1, 2, \ldots\}$ be a sequence of real r.v.'s, $X$ a real r.v. and $g : \mathbb{R} \rightarrow \mathbb{R}$ a function such that $g(X)$ and $g(X_n), n = 1, 2, \ldots$, are real r.v.'s.

Unless stated otherwise, this assumption will hold for all the definitions, propositions and theorems in this section.

1.2 Proposition Let $g : \mathbb{R} \rightarrow \mathbb{R}$ a continuous function everywhere on $\mathbb{R}$, except possibly in a set $A \subseteq \mathbb{R}$, and let $X$ a r.v. such that $P[X \in A] = 0$. Then

(a) $X_n \xrightarrow{P} X \Rightarrow g(X_n) \xrightarrow{P} g(X)$;

(b) $X_n \xrightarrow{a.s.} X \Rightarrow g(X_n) \xrightarrow{a.s.} g(X)$;
(c) $X_n \overset{L}{\rightarrow} X \Rightarrow g(X_n) \overset{L}{\rightarrow} g(X)$.

1.3 Proposition Let $\{X_n\}$ and $\{Y_n\}$ two sequences of random variables. Then

(a) $X_n \overset{p}{\rightarrow} X$ and $Y_n \overset{p}{\rightarrow} Y \Rightarrow X_n + Y_n \overset{p}{\rightarrow} X + Y$;

(b) $X_n \overset{a.s.}{\rightarrow} X$ and $Y_n \overset{a.s.}{\rightarrow} Y \Rightarrow X_n + Y_n \overset{a.s.}{\rightarrow} X + Y$;

(c) $X_n \overset{p}{\rightarrow} X$ and $Y_n \overset{p}{\rightarrow} Y \Rightarrow g(X_n, Y_n) \overset{p}{\rightarrow} g(X, Y)$ for any continuous function $g(x, y)$.

1.4 Proposition Let $\{X_n\}$ and $\{Y_n\}$ two sequences of r.v.’s such that $X_n \overset{L}{\rightarrow} X$ and $Y_n \overset{p}{\rightarrow} c$, where $X$ is a r.v. and $c$ is a real constant $(-\infty < c < +\infty)$. Then

(a) $X_n + Y_n \overset{L}{\rightarrow} X + c$;

(b) $X_n Y_n \overset{L}{\rightarrow} Xc$;

(c) $X_n / Y_n \overset{L}{\rightarrow} X / c$ if $c \neq 0$;

(d) $(X_n, Y_n) \overset{L}{\rightarrow} (X, c)$.

1.5 Proposition Let $\{X_n\}$ and $\{Y_n\}$ two sequences of r.v.’s such that $X_n - Y_n \overset{p}{\rightarrow} 0$ and $Y_n \overset{L}{\rightarrow} Y$, and let $g : \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function. Then

(a) $X_n \overset{L}{\rightarrow} Y$;

(b) $g(X_n) - g(Y_n) \overset{p}{\rightarrow} 0$;

(c) $g(X_n) \overset{L}{\rightarrow} g(Y)$.

1.4. Random series

1.6 Definition Let $\{X_t : t \in \mathbb{N}\}$ be a real-valued stochastic process and consider the series $\sum_{t=1}^{\infty} X_t$.

1.7 Definition We say $\sum_{t=1}^{\infty} X_t$ converges (according to given mode of convergence) iff there exists a real r.v. $Y$ such that

$$\sum_{t=1}^{N} X_t \overset{N \rightarrow \infty}{\rightarrow} Y$$ (according to the same mode of convergence).
1.8 Remark The mode of convergence: a.s., in probability or in mean of order r.

2. Laws of large numbers

2.1 Proposition Let \( \{X_t\}_{t=1}^{\infty} \) a sequence of r.v.’s such that \( X_t \in L_2 \) and \( \text{Cov}(X_s, X_t) = 0 \) for \( s \neq t \), and let \( \mu_t = E(X_t) \). Then

(a) \[ \sum_{n=1}^{\infty} \frac{1}{n^2} \text{Var}(X_n) < \infty \Rightarrow \overline{X}_n - \overline{\mu}_n \xrightarrow{n \to \infty} 0 \] (Chebychev law)

where \( \overline{X}_n = \frac{1}{n} \sum_{t=1}^{n} X_t / n \) and \( \overline{\mu}_n = \frac{1}{n} \sum_{t=1}^{n} \mu_t / n \), and

(b) \[ \sum_{n=1}^{\infty} \left( \frac{\log n}{n} \right)^2 \text{Var}(X_n) < \infty \Rightarrow \overline{X}_n - \overline{\mu}_n \xrightarrow{a.s.} 0. \]

In particular, if \( \text{Var}(X_t) = \sigma^2 < \infty \) and \( E(X_t) = \mu \) for all \( t \), then

\[ \frac{1}{n} \sum_{t=1}^{n} X_t \xrightarrow{a.s.} \mu \text{ and } \frac{1}{n} \sum_{t=1}^{n} X_t^2 \xrightarrow{n \to \infty} \mu. \]

2.2 Theorem Khintchine weak law of large numbers. Let \( \{X_t\}_{t=1}^{\infty} \) a sequence of independent and identically distributed r.v.’s whose mean \( E(X_t) \) exists. Then

\[ E(X_t) = \mu \Rightarrow \overline{X}_n \xrightarrow{n \to \infty} \mu. \]

2.3 Theorem First Kolmogorov’s strong law of large numbers. Let \( \{X_t\}_{t=1}^{\infty} \) a sequence of independent r.v.’s such that \( E(X_t) = \mu_t \text{ and } \text{Var}(X_t) = \sigma_t^2 \) exist for all \( t \). Then

\[ \sum_{n=1}^{\infty} \left( \frac{\sigma_t}{\sqrt{n}} \right)^2 < \infty \Rightarrow \overline{X}_n - \overline{\mu}_n \xrightarrow{n \to \infty} 0. \]

2.4 Theorem Second Kolmogorov’s strong law of large numbers. Let \( \{X_t\}_{t=1}^{\infty} \) a sequence of independent and identically distributed r.v.’s. Then

\[ E(X_t) \text{ exists and is equal to } \mu \Leftrightarrow \overline{X}_n - \overline{\mu}_n \xrightarrow{n \to \infty} 0. \]
3. Central limit theorems

3.1 Theorem  LINDEBERG-LÉVY CENTRAL LIMIT THEOREM. Let \( \{X_t\}_{t=1}^{\infty} \) a sequence of independent and identically distributed r.v.’s in \( L_2 \) such that \( E(X_t) = \mu \) and \( \text{Var}(X_t) = \sigma^2 > 0 \). Then

\[
\frac{1}{\sqrt{n}} \sum_{t=1}^{n} (X_t - \mu) / \sigma = \sqrt{n}(\bar{X}_t - \mu) / \sigma \xrightarrow{L} Z
\]

where \( Z \sim N(0, 1) \).

3.2 Theorem  LIAPUNOV CENTRAL LIMIT THEOREM. Let \( \{X_t\}_{t=1}^{\infty} \) a sequence of independent r.v.’s in \( L_3 \) such that \( E(X_t) = \mu_t \), \( \text{Var}(X_t) = \sigma_t^2 \neq 0 \), \( E[|X_t - \mu_t|^3] = \beta_t \) for all \( t \). Moreover,

\[
B_n = \left( \sum_{t=1}^{n} \beta_t \right)^{1/3}, C_n = \left( \sum_{t=1}^{n} \sigma_t^2 \right)^{1/2}.
\]

If \( \lim_{n \to \infty} (B_n / C_n) = 0 \), then

\[
\frac{1}{\sqrt{n}} \sum_{t=1}^{n} (X_t - \mu_t) / C_n = \sqrt{n}(\bar{X}_t - \beta_n) / \sigma \xrightarrow{L} Z
\]

where \( Z \sim N(0, 1) \).

3.3 Theorem  LINDEBERG-FELLER CENTRAL LIMIT THEOREM. Let \( \{X_t\}_{t=1}^{\infty} \) a sequence of independent r.v.’s in \( L_2 \) such that

\[
P[X_t \leq x] = G_t(x) , \ E(X_t) = \mu_t , \ \text{Var}(X_t) = \sigma_t^2 \neq 0 ,
\]

for all \( t \). Then

\[
\frac{1}{C_n} \sum_{t=1}^{n} \frac{X_t - \mu_t}{\sigma} \xrightarrow{L} Z \quad \text{and} \quad \lim_{n \to \infty} \max_{1 \leq i \leq n} \frac{\sigma_t}{C_n} = 0
\]

iff

\[
\lim_{n \to \infty} \frac{1}{C_n^2} \sum_{i=1}^{n} \int_{|x-\mu_t| \geq \epsilon C_n} (x-\mu_t)^2 dG_t(x) = 0, \forall \epsilon > 0.
\]

3.1. Extension to random vectors

3.1 Definition  STOCHASTIC CONVERGENCE FOR VECTORS. Let \( \{X_n\}_{n=1}^{\infty} \) a sequence of vectors of dimension \( k \),

\[
X_n = (X_{1n}, X_{2n}, \ldots, X_{kn})', n = 1, 2, \ldots
\]
whose components are real random variables all defined on the same probability space \((\Omega, Q, P)\), and

\[ X = (X_1, X_2, \ldots, X_k)' \]

another random vector of dimension \(k\) whose components are defined on the same space.

(a) We say \(X_n\) converges to \(X\) in probability (almost surely, in mean of order \(r\)) as \(n \to \infty\) if each component of \(X_n\) converges to the corresponding component of \(X\) in probability (almost surely, in mean of order \(r\)) as \(n \to \infty\). Depending on the case considered, we then write \(X_n \xrightarrow{P} X\), \(X_n \xrightarrow{a.s.} X\) or \(X_n \xrightarrow{r} X\).

(b) We say \(X_n\) converges in law to \(X\) \((X_n \xrightarrow{L} X)\) iff

\[ \lim_{n \to \infty} F_{X_n}(x) = F_X(x) \text{ at all continuity points of } F_X(x). \]

where \(x = (x_1, x_2, \ldots, x_k)' \in \mathbb{R}^k\),

\[ F_{X_n}(x) = P[X_{1n} \leq x_1, \ldots, X_{kn} \leq x_k], \quad n = 1, 2, \ldots \]

and

\[ F_X(x) = P[X_1 \leq x_1, \ldots, X_k \leq x_k]. \]

3.2 Theorem  **Univariate Characterization of Convergence in Law for a Sequence of Vectors.** Let \(\{X_n\}_{n=1}^{\infty}\) be a sequence of random vectors of dimension \(k \times 1\) and let \(X\) be another random vector of dimension \(k \times 1\). Then

\[ X_n \xrightarrow{L} X \iff \lambda' X_n \xrightarrow{L} \lambda' X, \forall \lambda \in \mathbb{R}^k. \]

In particular, if \(X \sim N[\mu, \Sigma]\),

\[ X_n \xrightarrow{L} N[\mu, \Sigma] \iff \lambda' X_n \xrightarrow{L} N[\lambda' \mu, \lambda' \Sigma \lambda], \forall \lambda \in \mathbb{R}^k. \]
3.2. Proofs and additional references

Proofs and further discussions of the results presented above may be found in the following references: Rao (1973), Lukacs (1975), Stout (1974), Loève (1977).
References


