Introduction to stochastic processes *

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1. Basic notions

1.1. Probability space

1.1.1 Definition A probability space is a triplet \((\Omega, \mathcal{A}, P)\) where

1. \(\Omega\) is the set of all possible results of an experiment;

2. \(\mathcal{A}\) is class of subsets of \(\Omega\) (called events) forming a \(\sigma\)–algebra, i.e.
   
   \begin{enumerate}
   
   \item \(\Omega \in \mathcal{A}\),
   \item \(A \in \mathcal{A} \Rightarrow A^c \in \mathcal{A}\),
   \item \(\bigcup_{j=1}^{\infty} A_j \in \mathcal{A}\), for any sequence \(\{A_1, A_2, \ldots\} \subseteq \mathcal{A}\);
   \end{enumerate}

3. \(P : \mathcal{A} \to [0, 1]\) is a function which assigns to each event \(A \in \mathcal{A}\) a number \(P(A) \in [0, 1]\), called the probability of \(A\) and such that

   \begin{enumerate}
   
   \item \(P(\Omega) = 1\),
   \item if \(\{A_j\}_{j=1}^{\infty}\) is a sequence of disjoint events, then \(P\left(\bigcup_{j=1}^{\infty} A_j\right) = \sum_{j=1}^{\infty} P(A_j)\).
   \end{enumerate}

1.2. Real random variable

1.2.1 Definition (heuristic) A real random variable \(X\) is a variable with real values whose behavior can be described by a probability distribution. Usually, this probability distribution is described by a distribution function:

\[
F_X(x) = P[X \leq x].
\] (1.1)

1.2.2 Definition (formal) A real random variable \(X\) is a function \(X : \Omega \to \mathbb{R}\) such that

\[
X^{-1}((\infty, x]) \equiv \{\omega \in \Omega : X(\omega) \leq x\} \in \mathcal{A}, \forall x \in \mathbb{R}, \quad \text{(measurable function)}.
\]

The probability law of \(X\) is defined by

\[
F_X(x) = P[X^{-1}((\infty, x])].
\] (1.2)
1.3. Stochastic process

1.3.1 Definition Let $T$ be a non-empty set. A stochastic process on $T$ is a collection of r.v.'s $X_t: \Omega \rightarrow \mathbb{R}$ such that to each element $t \in T$ is associated a r.v. $X_t$. The process can be written $\{X_t : t \in T\}$. If $T = \mathbb{R}$ (real numbers), we have a process in continuous time.

If $T = \mathbb{Z}$ (integers) or $T \subseteq \mathbb{Z}$, we have discrete time process.

The set $T$ can be finite or infinite, but usually it is assumed to be infinite. In the sequel, we shall be mainly interested by processes for which $T$ is a right-infinite interval of integers: i.e., $T = (n_0, \infty)$ where $n_0 \in \mathbb{Z}$ or $n_0 = -\infty$. We can also consider r.v.'s which take their values in more general spaces, i.e.

$$X_t: \Omega \rightarrow \Omega_0$$

where $\Omega_0$ is any non-empty set. Unless stated otherwise, we shall limit ourselves to the case where $\Omega_0 = \mathbb{R}$.

To observe a time series is equivalent to observing a realization of a process $\{X_t : t \in T\}$ or a portion of such a realization: given $\Omega, \mathcal{A}, P$, $\omega \in \Omega$ is first drawn and then the variables $X_t(\omega)$, $t \in T$, are associated with it. Each realization is determined in one shot by $\omega$.

The probability law of a stochastic process $\{X_t : t \in T\}$ where $T \subseteq \mathbb{R}$ can be described by specifying, for each subset $\{t_1, t_2, \ldots, t_n\} \subseteq T$ (where $n \geq 1$), the joint distribution function of $(X_{t_1}, \ldots, X_{t_n})$:

$$F(x_1, \ldots, x_n; t_1, \ldots, t_n) = P[X_{t_1} \leq x_1, \ldots, X_{t_n} \leq x_n]. \quad (1.1)$$

This follows from Kolmogorov's theorem [see Brockwell and Davis (1991, Chapter 1)].

1.4. $L_r$ spaces

1.4.1 Definition Let $r$ be a real number. $L_r$ is the set of real random variables $X$ defined on $(\Omega, \mathcal{A}, P)$ such that $E[|X|^r] < \infty$.

The space $L_r$ is always defined with respect to a probability space $(\Omega, \mathcal{A}, P)$. $L_2$ is the set of r.v.'s on $(\Omega, \mathcal{A}, P)$ whose second moments are finite (square-integrable variables). A stochastic process $\{X_t : t \in T\}$ is in $L_r$ iff $X_t \in L_r$, $\forall t \in T$, i.e.

$$E[|X_t|^r] < \infty, \forall t \in T. \quad (1.1)$$
The properties of moments of r.v.’s are summarized in Dufour (1999b).

2. Stationary processes

In general, the variables of a process \( \{X_t : t \in T\} \) are not identically distributed nor independent. In particular, if we suppose that \( E(X_t^2) < \infty \), we have

\[
E(X_t) = \mu_t, \quad (2.1)
\]

\[
\text{Cov}(X_{t_1}, X_{t_2}) = E[(X_{t_1} - \mu_{t_1})(X_{t_2} - \mu_{t_2})] = C(t_1, t_2). \quad (2.2)
\]

The means, variances and covariances of the variables of the process depend on their position in the series. The behavior of \( X_t \) can change with time. The function \( C : T \times T \to \mathbb{R} \) is called the \textit{covariance function} of the process \( \{X_t : t \in T\} \).

In this section, we will study the case where \( T \) is an right-infinite interval of integers.

2.1 Assumption (Process on an interval of integers).

\[
T = \{t \in \mathbb{Z} : t > n_0\}, \quad \text{where } n_0 \in \mathbb{Z} \cup \{-\infty\}. \quad (2.3)
\]

2.2 Definition (Strictly stationary process) : A stochastic process \( \{X_t : t \in T\} \) is strictly stationary (SS) iff the joint probability law of the vector \( (X_{t_1+k}, X_{t_2+k}, \ldots, X_{t_n+k})' \) is identical with the one of \( (X_{t_1}, X_{t_2}, \ldots, X_{t_n})' \), for any finite subset \( \{t_1, t_2, \ldots, t_n\} \subseteq T \) and for any integer \( k \geq 0 \). To indicate that \( \{X_t : t \in T\} \) is SS, we will write \( \{X_t : t \in T\} \sim \text{SS} \) or \( X_t \sim \text{SS} \).

2.3 Proposition If the process \( \{X_t : t \in T\} \) is SS, then the joint probability law of the vector \( (X_{t_1+k}, X_{t_2+k}, \ldots, X_{t_n+k})' \) is identical to the one of \( (X_{t_1}, X_{t_2}, \ldots, X_{t_n})' \), for any finite subset \( \{t_1, t_2, \ldots, t_n\} \subseteq T \) and any integer \( k > n_0 - \min\{t_1, \ldots, t_n\} \).

2.4 Proposition (Strict stationarity of a process on the integers) A process \( \{X_t : t \in \mathbb{Z}\} \) is SS iff the joint probability law of \( (X_{t_1+k}, X_{t_2+k}, \ldots, X_{t_n+k})' \) is identical with the law of \( (X_{t_1}, X_{t_2}, \ldots, X_{t_n})' \), for any subset \( \{t_1, t_2, \ldots, t_n\} \subseteq \mathbb{Z} \) and any integer \( k \).
Suppose $E(X_t^2) < \infty$, for any $t \in T$. If the process $\{X_t : t \in T\}$ is SS, we see easily that
\begin{align*}
E(X_s) &= E(X_t), \forall s, t \in T, \quad (2.4) \\
E(X_sX_t) &= E(X_{s+k}X_{t+k}), \forall s, t \in T, \forall k \geq 0. \quad (2.5)
\end{align*}
Furthermore, since
\begin{equation}
\text{Cov}(X_s, X_t) = E(X_sX_t) - E(X_s)E(X_t), \quad (2.6)
\end{equation}
we also have
\begin{equation}
\text{Cov}(X_s, X_t) = \text{Cov}(X_{s+k}, X_{t+k}), \forall s, t \in T, \forall k \geq 0. \quad (2.7)
\end{equation}
The conditions (2.4) and (2.5) are equivalent to the conditions (2.4) and (2.7). The mean of $X_t$ is constant and the covariance between any two variables of the process only depends on the distance between the variables, but not their position in the series.

**2.5 Definition** (Second-order stationary process). A stochastic process $\{X_t : t \in T\}$ is second-order stationary (S2) iff
\begin{enumerate}
  
  \item $E(X_t^2) < \infty$, $\forall t \in T$, 
  \item $E(X_s) = E(X_t), \forall s, t \in T$, 
  \item $\text{Cov}(X_s, X_t) = \text{Cov}(X_{s+k}, X_{t+k}), \forall s, t \in T, \forall k \geq 0$.
\end{enumerate}

If $\{X_t : t \in T\}$ is S2, we write $\{X_t : t \in T\} \sim S2$ or $X_t \sim S2$.

**2.6 Remark** Instead of second-order stationary, one also says weakly stationary (WS).

**2.7 Proposition** (Relation between strict stationarity and second-order stationarity). If the process $\{X_t : t \in T\}$ is strictly stationary and $E(X_t^2) < \infty$ for any $t \in T$, then the process $\{X_t : t \in T\}$ is second-order stationary.

**2.8 Proposition** (Existence of an autocovariance function). If the process $\{X_t : t \in T\}$ is second-order stationary, then there exists a function $\gamma : \mathbb{Z} \rightarrow \mathbb{R}$ such that
\begin{equation}
\text{Cov}(X_s, X_t) = \gamma(t - s), \forall s, t \in T. \quad (2.8)
\end{equation}
The function $\gamma$ is called the autocovariance function of the process $\{X_t : t \in T\}$ and $\gamma(k)$, for $k$ given, the lag-$k$ autocovariance of the process $\{X_t : t \in T\}$.

**PROOF:** Let $r \in T$ any element of $T$. Since the process $\{X_t : t \in T\}$ is S2, we have, for any $s, t \in T$ such that $s \leq t$,

$$
\text{Cov}(X_r, X_{r+t-s}) = \text{Cov}(X_{r+s-r}, X_{r+t-s+s-r}) = \text{Cov}(X_s, X_t), \text{ if } s \geq r,
$$

(2.9)

$$
\text{Cov}(X_s, X_t) = \text{Cov}(X_{s+r-s}, X_{t+r-s}) = \text{Cov}(X_r, X_{r+t-s}), \text{ if } s < r.
$$

(2.10)

Further, in the case where $s > t$, we have

$$
\text{Cov}(X_s, X_t) = \text{Cov}(X_r, X_{r+s-t}) = \gamma(t-s).
$$

(2.11)

Thus

$$
\text{Cov}(X_s, X_t) = \text{Cov}(X_r, X_{r+|t-s|}) = \gamma(t-s).
$$

(2.12)

Q.E.D.

### 2.9 Proposition (Properties of the autocovariance function)

Let $\{X_t : t \in T\}$ be a second-order stationary process. The autocovariance function $\gamma(k)$ of the process $\{X_t : t \in T\}$ satisfies the following properties:

1. $\gamma(0) = Var(X_t) \geq 0$, $\forall t \in T$;

2. $\gamma(k) = \gamma(-k)$, $\forall k \in \mathbb{Z}$ (i.e., $\gamma(k)$ is an even function of $k$);

3. $|\gamma(k)| \leq \gamma(0)$, $\forall k \in \mathbb{Z}$;

4. the function $\gamma(k)$ is positive semi-definite, i.e. $\sum_{i=1}^{N} \sum_{j=1}^{N} a_i a_j \gamma(t_i - t_j) \geq 0$, for any positive integer $N$ and for all the vectors $a = (a_1, ..., a_N)' \in \mathbb{R}^N$ and $\tau = (t_1, ..., t_N)' \in T^N$;

5. any $N \times N$ matrix of the form

$$
\Gamma_N = [\gamma(j-i)]_{i,j=1,...,N} = \begin{bmatrix}
\gamma_0 & \gamma_1 & \gamma_2 & \cdots & \gamma_{N-1} \\
\gamma_1 & \gamma_0 & \gamma_1 & \cdots & \gamma_{N-2} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\gamma_{N-1} & \gamma_{N-2} & \gamma_{N-3} & \cdots & \gamma_0
\end{bmatrix}
$$

(2.13)
is positive semi-definite, where $\gamma_k \equiv \gamma(k)$.

2.10 Proposition (Existence of an autocorrelation function). If the process $\{X_t : t \in T\}$ is second-order stationary, then there exists a function $\rho : \mathbb{Z} \to [-1, 1]$ such that

$$\rho(t - s) = Corr(X_s, X_t) = \frac{\gamma(t - s)}{\gamma(0)}, \forall s, t \in T,$$

(2.14)

where $0/0 \equiv 1$. The function $\rho$ is called the autocorrelation function of the process $\{X_t : t \in T\}$, and $\rho(k)$, for $k$ given, the lag-$k$ autocorrelation of the process $\{X_t : t \in T\}$.

2.11 Proposition (Properties of the autocorrelation function). Let $\{X_t : t \in T\}$ be a second-order stationary process. The autocorrelation function $\rho(k)$ of the process $\{X_t : t \in T\}$ satisfies the following properties:

(1) $\rho(0) = 1$;

(2) $\rho(k) = \rho(-k)$, $\forall k \in \mathbb{Z}$;

(3) $|\rho(k)| \leq 1$, $\forall k \in \mathbb{Z}$;

(4) the function $\rho(k)$ is positive semi-definite, i.e.

$$\sum_{i=1}^{N} \sum_{j=1}^{N} a_i a_j \rho(t_i - t_j) \geq 0$$

(2.15)

for any positive integer $N$ and for all the vectors $a = (a_1, ... , a_N)' \in \mathbb{R}^N$ and $\tau = (t_1, ... , t_N)' \in T^N$;

(5) any $N \times N$ matrix of the form

$$R_N = \frac{1}{\gamma_0} \Gamma_N = \begin{bmatrix} 1 & \rho_1 & \rho_2 & \cdots & \rho_{N-1} \\ \rho_1 & 1 & \rho_1 & \cdots & \rho_{N-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \rho_{N-1} & \rho_{N-2} & \rho_{N-3} & \cdots & 1 \end{bmatrix}$$

(2.16)

is positive semi-definite, where $\gamma_0 = Var(X_t)$ and $\rho_k \equiv \rho(k)$.
2.12 **Theorem** (Characterization of autocovariance functions): An even function \( \gamma : \mathbb{Z} \to \mathbb{R} \) is positive semi-definite iff \( \gamma(.) \) is the autocovariance function of a second-order stationary process \( \{X_t : t \in \mathbb{Z}\} \).

**PROOF:** See Brockwell and Davis (1991, Chapter 2).

2.13 **Corollary** (Characterization of autocorrelation functions). An even function \( \rho : \mathbb{Z} \to [-1, 1] \) is positive semi-definite iff \( \rho \) is the autocorrelation function of a second-order stationary process \( \{X_t : t \in \mathbb{Z}\} \).

2.14 **Definition** (Deterministic process). Let \( \{X_t : t \in T\} \) be a stochastic process, \( T_1 \subseteq T \) and \( I_t = \{X_s : s \leq t\} \). We say that the process \( \{X_t : t \in T\} \) is deterministic on \( T_1 \) iff there exists a collection of functions \( \{g_t(I_{t-1}) : t \in T_1\} \) such that \( X_t = g_t(I_{t-1}) \) with probability 1, \( \forall t \in T_1 \).

A deterministic process is a process which can be perfectly predicted from its own past (at points where it is deterministic).

2.15 **Proposition** (Criterion for a deterministic process). Let \( \{X_t : t \in T\} \) be a second-order stationary process, where \( T = \{t \in \mathbb{Z} : t > n_0\} \) and \( n_0 \in \mathbb{Z} \cup \{-\infty\} \), and let \( \gamma(k) \) its autocovariance function. If there exists an integer \( N \geq 1 \) such that the matrix \( \Gamma_N \) is singular [where \( \Gamma_N \) is defined in Proposition 2.9], then the process \( \{X_t : t \in T\} \) is deterministic for \( t > n_0 + N - 1 \). In particular, if \( \text{Var}(X_t) = \gamma(0) = 0 \), the process is deterministic for \( t \in T \).

For a second-order indeterministic stationary process en any \( t \in T \), all the matrices \( \Gamma_N, N \geq 1 \), are invertible.

2.16 **Definition** (Stationary of order \( m \)). Let \( m \) be a non-negative integer. A stochastic process \( \{X_t : t \in T\} \) is stationary of order \( m \) iff

1. \( E(|X_t|^m) < \infty, \forall t \in T \),

and

2. \( E[X_{t_1}^{m_1}X_{t_2}^{m_2} ... X_{t_n}^{m_n}] = E[X_{t_1+k}^{m_1}X_{t_2+k}^{m_2} ... X_{t_n+k}^{m_n}] \)

for any \( k \geq 0 \), any subset \( \{t_1, ..., t_n\} \in T^N \) and all the non-negative integers \( m_1, ..., m_n \) such that \( m_1 + m_2 + ... + m_n \leq m \).
If \( m = 1 \), the mean is constant, but not necessarily the other moments. If \( m = 2 \), the process is second-order stationary.

2.17 Definition (Asymptotically stationary process of order \( m \)). Let \( m \) a non-negative integer. A stochastic process \( \{X_t : t \in T\} \) is asymptotically stationary of order \( m \) iff

1. there exists an integer \( N \) such that \( (|X_t|^m) < \infty \) for \( t \geq N \),

and

2. \[ \lim_{t_1 \to \infty} \left\{ E\left(X_{t_1}^{m_1}X_{t_1+\Delta_2}^{m_2}...X_{t_1+\Delta_n}^{m_n}\right) - E\left(X_{t_1+k}^{m_1}X_{t_1+\Delta_2+k}^{m_2}...X_{t_1+\Delta_n+k}^{m_n}\right)\right\} = 0 \]

for any \( k \geq 0 \), \( t_1 \in T \), all the positive integers \( \Delta_2, \Delta_3, ..., \Delta_n \) such that \( \Delta_2 < \Delta_3 < ... < \Delta_n \), and all the non-negative integers \( m_1, ..., m_n \) such that \( m_1 + m_2 + ... + m_n \leq m \).

3. Some important models

In this section, we will again assume that \( T \) is a right-infinite interval of integers (Assumption 2.1):

\[ T = \{t \in \mathbb{Z} : t > n_0\} , \text{ where } n_0 \in \mathbb{Z} \cup \{-\infty\} . \tag{3.1} \]

3.1. Noise models

3.1.1 Definition Sequence of independent r.v.’s: process \( \{X_t : t \in T\} \) such that the variables \( X_t \) are mutually independent. We write

\[ X_t : t \in T \} \sim IND \text{ or } \{X_t} \sim IND; \tag{3.2} \]

\[ \{X_t : t \in T\} \sim IND(\mu_t) \text{ or } E(X_t) = \mu_t; \tag{3.3} \]

\[ \{X_t : t \in T\} \sim IND(\mu_t, \sigma_t^2) , \text{ if } E(X_t) = \mu_t \text{ and } Var(X_t) = \sigma_t^2. \tag{3.4} \]

3.1.2 Definition Random sample: sequence of independent and identically distributed (i.i.d.) r.v.’s. We write

\[ \{X_t : t \in T\} \sim IID . \tag{3.5} \]

A random sample is a SS process. If \( E(X_t^2) < \infty \), for any \( t \in T \), the process is S2. In this case, we write

\[ \{X_t : t \in T\} \sim IID(\mu, \sigma^2) , \text{ if } E(X_t) = \mu \text{ and } V(X_t) = \sigma^2. \tag{3.6} \]
3.1.3 **Definition** White noise: sequence of r.v.'s in $L_2$ of mean zero, of same variance and mutually uncorrelated, i.e.

\[
E(X_t^2) < \infty, \forall t \in T, \quad (3.7)
\]

\[
E(X_t) = 0, \forall t \in T, \quad (3.8)
\]

\[
E(X_t^2) = \sigma^2, \forall t \in T, \quad (3.9)
\]

\[
\text{Cov}(X_s, X_t) = 0, \text{ if } s \neq t. \quad (3.10)
\]

We write:

\[
\{X_t : t \in T\} \sim BB(0, \sigma^2) \text{ or } \{X_t\} \sim BB(0, \sigma^2). \quad (3.11)
\]

3.1.4 **Definition** Heteroskedastic white noise: sequence of r.v.'s in $L_2$ with mean zero and mutually uncorrelated, i.e.

\[
E(X_t^2) < \infty, \forall t \in T, \quad (3.12)
\]

\[
E(X_t) = 0, \forall t \in T, \quad (3.13)
\]

\[
\text{Cov}(X_t, X_s) = 0, \text{ if } s \neq t, \quad (3.14)
\]

\[
E(X_t^2) = \sigma_t^2, \forall t \in T. \quad (3.15)
\]

We write:

\[
\{X_t : t \in \mathbb{Z}\} \sim BB(0, \sigma_t^2) \text{ or } \{X_t\} \sim BB(0, \sigma_t^2). \quad (3.16)
\]

Each one of these four models will be called a *noise* process.

### 3.2. Harmonic processes

Many time series exhibit apparent periodic behavior. This suggests one to use periodic functions to describe them.

3.2.1 **Definition** A function $f(t)$, $t \in \mathbb{R}$, is periodic of period $P$ if

\[
f(t + P) = f(t), \forall t.
\]

$\frac{1}{P}$ is the frequency associated with the function (number of cycles per unit of time).

3.2.2 Example

\[
\sin(t) = \sin(t + 2\pi) = \sin(t + 2\pi k), \forall k \in \mathbb{Z}. \quad (3.17)
\]

3.2.3 Example

\[
\cos(t) = \cos(t + 2\pi) = \cos(t + 2\pi k), \forall k \in \mathbb{Z}. \quad (3.18)
\]
3.2.4 Example

\[
\sin(\nu t) = \sin \left[ \nu \left( t + \frac{2\pi}{v} \right) \right] = \sin \left[ \nu \left( t + \frac{2\pi k}{v} \right) \right], \forall k \in \mathbb{Z}. \quad (3.19)
\]

3.2.5 Example

\[
\cos(\nu t) = \cos \left[ \nu \left( t + \frac{2\pi}{v} \right) \right] = \cos \left[ \nu \left( t + \frac{2\pi k}{v} \right) \right], \forall k \in \mathbb{Z}. \quad (3.20)
\]

For \( \sin(\nu t) \) and \( \cos(\nu t) \), the period is \( P = 2\pi/\nu \).

3.2.6 Example

\[
f(t) = C \cos(\nu t + \theta) = C[\cos(\nu t) \cos(\theta) - \sin(\nu t) \sin(\theta)]
\]

\[
= A \cos(\nu t) + B \sin(\nu t) \quad (3.21)
\]

where \( C \geq 0 \), \( A = C \cos(\theta) \) and \( B = -C \sin \theta \). Further,

\[
C = \sqrt{A^2 + B^2}, \quad \tan(\theta) = -B/A \quad (if \ C \neq 0). \quad (3.22)
\]

3.2.7 Definition  We call:

\[
C = \ \text{amplitude};
\]

\[
\nu = \ \text{angular frequency (radians/time unit)};
\]

\[
P = \frac{2\pi}{\nu} = \ \text{period};
\]

\[
\bar{v} = \frac{1}{P} = \frac{v}{2\pi} = \ \text{frequency (number of cycles per time unit)};
\]

\[
\theta = \ \text{phase angle (usually} \ 0 \leq \theta < 2\pi \text{ or } -\pi/2 < \theta \leq \pi/2). \]

3.2.8 Example

\[
f(t) = C \sin(\nu t + \theta) = C \cos(\nu t + \theta - \pi/2) \quad (3.23)
\]

\[
= C[\sin(\nu t) \cos(\theta) + \cos(\nu t) \sin(\theta)] \quad (3.24)
\]

\[
= A \cos(\nu t) + B \sin(\nu t) \quad (3.25)
\]

where

\[
0 \leq \nu < 2\pi, \quad (3.26)
\]

\[
A = C \sin(\theta) = C \cos \left( \theta - \frac{\pi}{2} \right), \quad (3.27)
\]
\[ B = C \cos(\theta) = -C \sin\left(\theta - \frac{\pi}{2}\right). \] (3.28)

Consider the model
\[ X_t = C \cos(\nu t + \theta) \] (3.29)
\[ = A \cos(\nu t) + B \sin(\nu t), \quad t \in \mathbb{Z}. \] (3.30)

If \( A \) and \( B \) are constants,
\[ E(X_t) = A \cos(\nu t) + B \sin(\nu t), \quad t \in \mathbb{Z}, \] (3.31)
and thus the process \( X_t \) is non-stationary (the mean is not constant). Suppose now \( A \) and \( B \) are r.v.'s such that
\[ E(A) = E(B) = 0, \quad E(A^2) = E(B^2) = \sigma^2, \quad E(AB) = 0. \] (3.32)

\( A \) and \( B \) do not depend on \( t \) but are fixed for each realization of the process \([A = A(\omega), B = B(\omega)]\). In this case,
\[ E(X_t) = 0, \] (3.33)
\[ E(X_sX_t) = E(A^2) \cos(\nu s) \cos(\nu t) + E(B^2) \sin(\nu s) \sin(\nu t) \]
\[ = \sigma^2 [\cos(\nu s) \cos(\nu t) + \sin(\nu s) \sin(\nu t)] = \sigma^2 \cos[\nu(t - s)]. \] (3.34)

The process \( X_t \) is stationary of order 2 with the following autocovariance and autocorrelation functions:
\[ \gamma_X(k) = \sigma^2 \cos(\nu k), \quad \rho_X(k) = \cos(\nu k). \] (3.35)

If we add \( m \) cyclic processes of the form (3.29), we obtain a harmonic process of order \( m \).

3.2.9 Definition (Harmonic process of order \( m \)). We say the process \( \{X_t : t \in T\} \) is a harmonic process of order \( m \) if it can written in the form
\[ X_t = \sum_{j=1}^{m} [A_j \cos(\nu_j t) + B_j \sin(\nu_j t)], \quad \forall t \in T, \] (3.36)
where \( \nu_1, \ldots, \nu_m \) are distinct constants in the interval \([0, 2\pi]\).
If we suppose $A_j, B_j, j = 1, \ldots, m$, are r.v.'s in $L_2$ such that
\begin{align*}
E(A_j) &= E(B_j) = 0, E(A_j^2) = E(B_j^2) = \sigma_j^2, j = 1, \ldots, m, \quad (3.37) \\
E(A_jA_k) &= E(B_jB_k) = 0, \text{ pour } j \neq k, \quad (3.38) \\
E(A_jB_k) &= 0, \forall j, k, \quad (3.39)
\end{align*}

the process $X_t$ can be considered second-order stationary:
\begin{align*}
E(X_t) &= 0, \quad (3.40) \\
E(X_sX_t) &= \sum_{j=1}^{m} \sigma_j^2 \cos[\nu_j(t - s)], \quad (3.41)
\end{align*}

hence
\begin{align*}
\gamma_X(k) &= \sum_{j=1}^{m} \sigma_j^2 \cos(\nu_jk), \quad (3.42) \\
\rho_X(k) &= \sum_{j=1}^{m} \sigma_j^2 \cos(\nu_jk) / \sum_{j=1}^{m} \sigma_j^2. \quad (3.43)
\end{align*}

If we add a white noise $u_t$ to $X_t$ in (3.36), we obtain again a second-order stationary process:
\begin{equation*}
X_t = \sum_{j=1}^{m} [A_j \cos(\nu_jt) + B_j \sin(\nu_jt)] + u_t, t \in T, \quad (3.44)
\end{equation*}

where the process $\{u_t : t \in T\} \sim BB(0, \sigma^2)$ is uncorrelated with $A_j, B_j, j = 1, \ldots, m$. In this case, $E(X_t) = 0$ and
\begin{equation*}
\gamma_X(k) = \sum_{j=1}^{m} \sigma_j^2 \cos(\nu_jk) + \sigma^2 \delta(k) \quad (3.45)
\end{equation*}

where $\delta(k) = 1$ for $k = 0$, and $\delta(k) = 0$ otherwise. If a series can be described by an equation of the form (3.44), we can view it as a realization of a second-order stationary process.

### 3.3. Linear processes

Many stochastic processes with dependence are obtained as transformations of noise processes.
3.3.1 Definition The process \( \{X_t : t \in T\} \) is an autoregressive process of order \( p \) if it satisfies an equation of the form

\[
X_t = \bar{\mu} + \sum_{j=1}^{p} \varphi_j X_{t-j} + u_t, \forall t \in T, \tag{3.46}
\]

where \( \{u_t : t \in \mathbb{Z}\} \sim \text{BB}(0, \sigma^2) \). In this case, we denote

\[ \{X_t : t \in T\} \sim \text{AR}(p). \]

Usually, \( T = \mathbb{Z} \) or \( T = \mathbb{Z}_+ \) (positive integers). If \( \sum_{j=1}^{p} \varphi_j \neq 1 \), we can define \( \mu = \bar{\mu} / (1 - \sum_{j=1}^{p} \varphi_j) \) and write

\[ \tilde{X}_t = \sum_{j=1}^{p} \varphi_j \tilde{X}_{t-j} + u_t, \forall t \in T, \]

where \( \tilde{X}_t \equiv X_t - \mu \).

3.3.2 3.3.3 Definition The process \( \{X_t : t \in T\} \) is a moving average process of order \( q \) if it can be written in the form

\[
X_t = \bar{\mu} + \sum_{j=0}^{q} \psi_j u_{t-j}, \forall t \in T, \tag{3.47}
\]

where \( \{u_t : t \in \mathbb{Z}\} \sim \text{BB}(0, \sigma^2) \). In this case, we denote

\[ \{X_t : t \in T\} \sim \text{MA}(q). \tag{3.48} \]

Without loss of generality, we can set \( \psi_0 = 1 \) and \( \psi_j = -\theta_j, j = 1, \ldots, q : \)

\[ X_t = \bar{\mu} + u_t - \sum_{j=1}^{q} \theta_j u_{t-j}, t \in T \]

or, equivalently,

\[ \tilde{X}_t = u_t - \sum_{j=1}^{q} \theta_j u_{t-j} \]
where $\tilde{X}_t \equiv X_t - \bar{\mu}$.

### 3.3.4 Definition

The process $\{X_t : t \in T\}$ is an autoregressive-moving-average (ARMA) process of order $(p, q)$ if it can be written in the form

$$X_t = \bar{\mu} + \sum_{j=1}^{p} \varphi_j X_{t-j} + u_t - \sum_{j=1}^{q} \theta_j u_{t-j}, \forall t \in T,$$

(3.49)

where $\{u_t : t \in \mathbb{Z}\} \sim BB(0, \sigma^2)$. In this case, we denote

$$\{X_t : t \in T\} \sim ARMA(p, q).$$

(3.50)

If $\sum_{j=1}^{p} \varphi_j \neq 1$, we can also write

$$\tilde{X}_t = \sum_{j=1}^{p} \varphi_j \tilde{X}_{t-j} + u_t - \sum_{j=1}^{q} \theta_j u_{t-j}$$

(3.51)

where $\tilde{X}_t = X_t - \mu$ and $\mu = \bar{\mu} / (1 - \sum_{j=1}^{p} \varphi_j)$.

### 3.3.5 Definition

The process $\{X_t : t \in T\}$ is a moving-average process of infinite order if it can be written in the form

$$X_t = \bar{\mu} + \sum_{j=-\infty}^{+\infty} \psi_j u_{t-j}, \forall t \in \mathbb{Z},$$

(3.52)

where $\{u_t : t \in \mathbb{Z}\} \sim BB(0, \sigma^2)$. We also say that $X_t$ is a weakly linear process. In this case, we denote

$$\{X_t : t \in T\} \sim MA(\infty).$$

(3.53)

In particular, if $\psi_j = 0$ for $j < 0$, i.e.

$$X_t = \bar{\mu} + \sum_{j=0}^{\infty} \psi_j u_{t-j}, \forall t \in \mathbb{Z},$$

(3.54)

we say that $X_t$ is a causal function of $u_t$ (causal linear process). [Box and Jenkins (1976) speak about general linear processes.]
3.3.6 Definition The process \( \{ X_t : t \in T \} \) is an autoregressive process of infinite order if it can be written in the form

\[
X_t = \bar{\mu} + \sum_{j=1}^{\infty} \varphi_j X_{t-j} + u_t, t \in T,
\]

where \( \{ u_t : t \in \mathbb{Z} \} \sim BB(0, \sigma^2) \). In this case, we denote

\[
\{ X_t : t \in T \} \sim AR(\infty).
\]

3.3.7 Remark Generalization: We can generalize the notions defined above by assuming that \( \{ u_t : t \in \mathbb{Z} \} \) is a noise. Unless sated otherwise, we will suppose \( \{ u_t \} \) is a white noise.

3.3.8 QUESTIONS:

1. Under which conditions are the processes defined above stationary (strictly or in \( L_r \))?

2. Under which conditions are the processus \( MA(\infty) \) or \( AR(\infty) \) well defined (convergent series)?

3. What are the links between the different classes of processes defined above?

4. When a process is stationary, what are its autocovariance and autocorrelation functions?

3.4. Integrated processes

3.4.1 Definition The process \( \{ X_t : t \in T \} \) is a random walk if it satisfies an equation of the form

\[
X_t - X_{t-1} = v_t, \forall t \in T,
\]

where \( \{ v_t : t \in \mathbb{Z} \} \sim IID \). For such a process to be well defined, we must suppose that \( n_0 \neq -\infty \) (the process ne can start at \( -\infty \)). If \( n_0 = -1 \), we can write

\[
X_t = X_0 + \sum_{j=1}^{t} v_j
\]
hence the name “integrated process”. If $E(v_t) = \bar{\mu}$ or $Med(v_t) = \bar{\mu}$, one often writes
\[ X_t - X_{t-1} = \bar{\mu} + u_t \] (3.59)
where $u_t \equiv v_t - \bar{\mu} \sim IID$ and $E(u_t) = 0$ or $Med(u_t) = 0$ (depending on whether $E(u_t) = 0$ or $Med(u_t) = 0$). If $\bar{\mu} \neq 0$, the random walk has drift.

3.4.2 Definition The process $\{X_t : t \in T\}$ is a random walk generated by a white noise [or an heteroskedastic white noise, or a sequence of independent r.v.'s] If $X_t$ satisfies an equation of the form
\[ X_t - X_{t-1} = \bar{\mu} + u_t \] (3.60)
where $\{u_t : t \in T\} \sim BB(0, \sigma^2)$ [or $\{u_t : t \in T\} \sim BB(0, \sigma_t^2)$, or $\{u_t : t \in T\} \sim IND(0)$].

3.4.3 Definition The process $\{X_t : t \in T\}$ is integrated of order $d$ if it can be written in the form
\[ (1 - B)^d X_t = Z_t, \forall t \in T, \] (3.61)
where $\{Z_t : t \in T\}$ is a stationary process (usually stationary of order $2$) and $d$ is a non-negative integer ($d = 0, 1, 2, ...$). In particular, if $\{Z_t : t \in T\}$ is an $ARMA(p, q)$ stationary process, $\{X_t : t \in T\}$ is an $ARIMA(p, d, q)$ process: $\{X_t : t \in T\} \sim ARIMA(p, d, q)$. We note
\[
\begin{align*}
B X_t &= X_{t-1}, \quad (3.62) \\
(1 - B) X_t &= X_t - X_{t-1}, \quad (3.63) \\
(1 - B)^2 X_t &= (1 - B)(1 - B)X_t = (1 - B)(X_t - X_{t-1}) \quad (3.64) \\
&= X_t - 2X_{t-1} + X_{t-2}, \quad (3.65) \\
(1 - B)^d X_t &= (1 - B)(1 - B)^{d-1}X_t, d = 1, 2, ... \quad (3.66)
\end{align*}
\]
where $(1 - B)^0 = 1.$

3.5. Models of deterministic tendency

3.5.1 Definition The process $\{X_t : t \in T\}$ follows a deterministic tendency if it can be written in the form
\[ X_t = f(t) + Z_t, \forall t \in T, \] (3.67)
where \( f(t) \) is a deterministic function of time and \( \{Z_t : t \in T\} \) is a noise or a stationary process.

3.5.2 Important cases of deterministic tendency:

\[
X_t = \beta_0 + \beta_1 t + u_t, \quad (3.68)
\]

\[
X_t = \sum_{j=0}^{k} \beta_j t^j + u_t, \quad (3.69)
\]

where \( \{u_t : t \in T\} \sim BB(0, \sigma^2) \).

4. Transformations of stationary processes

4.1 Theorem Let \( \{X_t : t \in \mathbb{Z}\} \) be a stochastic process on the integers, \( r \geq 1 \) and \( \{a_j : j \in \mathbb{Z}\} \) a sequence of real numbers. If \( \sum_{j=-\infty}^{\infty} |a_j|E(|X_{t-j}|^r)^{1/r} < \infty \), then, for any \( t \), the random series \( \sum_{j=-\infty}^{\infty} a_j X_{t-j} \) converges absolutely a.s. and in mean of order \( r \) to a r.v. \( Y_t \) such that \( E(|Y_t|^r) < \infty \).

PROOF: See Dufour (1999a).

4.2 Theorem Let \( \{X_t : t \in \mathbb{Z}\} \) be a second-order stationary process and \( \{a_j : j \in \mathbb{Z}\} \) an sequence of real numbers absolutely convergent sequence of real numbers, i.e. \( \sum_{j=-\infty}^{\infty} |a_j| < \infty \). Then the random series \( \sum_{j=-\infty}^{\infty} a_j X_{t-j} \) converges absolutely p.s. and in mean of order 2 to a r.v. \( Y_t \in L_2, \forall t \), and the process \( \{Y_t : t \in \mathbb{Z}\} \) is second-order stationary.

PROOF : See Gouriéroux and Monfort (1997, Property 5.6).
4.3 Theorem If \( \{X_t : t \in \mathbb{Z}\} \) be a second-order stationary process with autocovariance function \( \gamma_X(k) \), the autocovariance function of the transformed process

\[
Y_t = \sum_{j=-\infty}^{\infty} a_j X_{t-j},
\]

where \( \sum_{j=-\infty}^{\infty} |a_j| < \infty \), is given by

\[
\gamma_Y(k) = \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} a_i a_j \gamma_X(k - i + j).
\]

4.4 Theorem The series \( \sum_{j=-\infty}^{\infty} a_j X_{t-j} \) converges absolutely p.s. for any second-order stationary process \( \{X_t : t \in \mathbb{Z}\} \) iff

\[
\sum_{j=-\infty}^{\infty} |a_j| < \infty.
\]

5. Infinite order moving averages

Consider the random series

\[
\sum_{j=-\infty}^{\infty} \psi_j u_{t-j}, t \in \mathbb{Z}
\]

where \( \{u_t : t \in \mathbb{Z}\} \sim BB(0, \sigma^2) \).

5.1. Convergence conditions

We can write

\[
\sum_{j=-\infty}^{\infty} \psi_j u_{t-j} = \sum_{j=-\infty}^{\infty} Y_j(t) = \sum_{j=-\infty}^{-1} Y_j(t) + \sum_{j=0}^{\infty} Y_j(t)
\]

where \( Y_j(t) \equiv \psi_j u_{t-j} \) and

\[
E[|Y_j(t)|] = |\psi_j| E[|u_{t-j}|] \leq |\psi_j| [E(u_{t-j}^2)]^{1/2} = |\psi_j| \sigma < \infty,
\]

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\[ \sum_{j=-\infty}^{\infty} \psi_j u_{t-j} \] is a series of orthogonal variables.

Suppose \( \sum_{j=-\infty}^{-1} \psi_j^2 < \infty \). Then

\[ Y_{m}^1(t) \equiv \sum_{j=-m}^{-1} \psi_j u_{t-j} \xrightarrow{m \to \infty} \sum_{j=-\infty}^{-1} \psi_j u_{t-j}, \]

\[ Y_{n}^2(t) \equiv \sum_{j=0}^{n} \psi_j u_{t-j} \xrightarrow{n \to \infty} \sum_{j=1}^{\infty} \psi_j u_{t-j} \]

[see Dufour (1999a)], and thus

\[ Y_{m,n}(t) \equiv Y_{m}^1(t) + Y_{n}^2(t) \xrightarrow{m \to \infty, n \to \infty} \sum_{j=-\infty}^{\infty} \psi_j u_{t-j}, \forall t \in \mathbb{Z}. \]

It is also clear that

\[ X_{n}(t) \equiv Y_{n}^1(t) + Y_{n}^2(t) = \sum_{j=-n}^{-1} \psi_j u_{t-j} + \sum_{j=0}^{n} \psi_j u_{t-j} \xrightarrow{n \to \infty} \sum_{j=-\infty}^{\infty} \psi_j u_{t-j}, \forall t \in \mathbb{Z}. \]

Thus,

\[ \sum_{j=-\infty}^{+\infty} \psi_j^2 < \infty \Rightarrow \sum_{j=-\infty}^{\infty} \psi_j u_{t-j} \text{ converges in } q.m. \text{ to a r.v. } \hat{X}_t \]

[see Dufour (1999a)]. Further

\[ \sum_{j=-\infty}^{+\infty} \psi_j^2 < \infty \Rightarrow \sum_{j=-\infty}^{\infty} \psi_j u_{t-j} \text{ converges in } q.m. \text{ to a r.v. } \hat{X}_t \]

[see Dufour (1999a)],

\[ \sum_{j=-\infty}^{\infty} |\psi_j| < \infty \Rightarrow \sum_{j=-\infty}^{\infty} \psi_j^2 < \infty \]

\[ \Rightarrow \sum_{j=-\infty}^{\infty} \psi_j u_{t-j} \text{ converges in } q.m. \text{ to a } \hat{X}_t. \]
If the variables \( \{u_t : t \in \mathbb{Z}\} \) are mutually independent,

\[
\sum_{j=\infty}^{+\infty} \psi_j^2 < \infty \Rightarrow \sum_{j=\infty}^{+\infty} \psi_j u_{t-j} \text{ converges in a.s. to a r.v. } \tilde{X}_t
\]

[see Dufour (1999a)]. The variable \( \tilde{X}_t \) is called the limit (in q.m. or a.s.) of the series \( \sum_{j=\infty}^{\infty} \psi_j u_{t-j} \), and we write

\[
\tilde{X}_t = \sum_{j=-\infty}^{\infty} \psi_j u_{t-j}.
\]

on defining \( X_t \equiv \mu + \tilde{X}_t \), we obtain the linear process

\[
X_t = \mu + \sum_{j=-\infty}^{\infty} \psi_j u_{t-j}
\]

where it is assumed that the series converges.

### 5.2. Mean, variance and covariances

By (5.3), we have:

\[
E[X_n(t)] \rightarrow_{n \to \infty} E(\tilde{X}_t),
\]

\[
E[X_n(t)^2] \rightarrow_{n \to \infty} E(\tilde{X}_t^2),
\]

\[
E[X_n(t)X_n(t+k)] \rightarrow_{n \to \infty} E(\tilde{X}_t \tilde{X}_{t+k});
\]

see Dufour (1999a). Consequently,

\[
E(\tilde{X}_t) = 0,
\]

\[
Var(\tilde{X}_t) = E(\tilde{X}_t^2) = \lim_{n \to \infty} \sum_{j=-n}^{n} \psi_j^2 \sigma^2 = \sigma^2 \sum_{j=-\infty}^{\infty} \psi_j^2,
\]

\[
Cov(\tilde{X}_t, \tilde{X}_{t+k}) = E(\tilde{X}_t \tilde{X}_{t+k}) = \lim_{n \to \infty} \left[ \left( \sum_{j=-n}^{n} \psi_j u_{t-i} \right) \left( \sum_{j=-n}^{n} \psi_j u_{t+k-j} \right) \right].
\]
\[
\lim_{n \to \infty} \sum_{i=-n}^{n} \sum_{j=-n}^{n} \psi_i \psi_j E(u_{t-i} u_{t+k-j})
\]

\[
= \begin{cases} 
\lim_{n \to \infty} \sum_{i=-n}^{n-k} \sum_{j=-n}^{n} \psi_i \psi_{i+k} \sigma^2 = \sigma^2 \sum_{i=-\infty}^{\infty} \psi_i \psi_{i+k}, & \text{if } k \geq 1, \\
\lim_{n \to \infty} \sum_{j=-n}^{n} \sum_{j=-n}^{n} \psi_j \psi_{j+|k|} \sigma^2 = \sigma^2 \sum_{j=-\infty}^{\infty} \psi_j \psi_{j+|k|}, & \text{if } k \leq -1,
\end{cases}
\] (5.6)

since \( t-i = t+k-j \Rightarrow j = i+k \text{ and } i = j-k \). For any \( k \in \mathbb{Z} \), we can write

\[
Cov(\tilde{X}_t, \tilde{X}_{t+k}) = \sigma^2 \sum_{j=-\infty}^{\infty} \psi_j \psi_{j+|k|},
\] (5.7)

\[
Corr(\tilde{X}_t, \tilde{X}_{t+k}) = \sum_{j=-\infty}^{\infty} \psi_j \psi_{j+|k|} / \sum_{j=-\infty}^{\infty} \psi_j^2.
\] (5.8)

The series \( \sum_{j=-\infty}^{\infty} \psi_j \psi_{j+k} \) converges absolutely, for

\[
\left| \sum_{j=-\infty}^{\infty} \psi_j \psi_{j+k} \right| \leq \sum_{j=-\infty}^{\infty} |\psi_j \psi_{j+k}| \leq \left[ \sum_{j=-\infty}^{\infty} \psi_j^2 \right]^\frac{1}{2} \left[ \sum_{j=-\infty}^{\infty} \psi_{j+k}^2 \right]^\frac{1}{2} < \infty.
\] (5.9)

If \( X_t = \mu + \tilde{X}_t = \mu + \sum_{j=-\infty}^{\infty} \psi_j u_{t-j} \), then

\[
E(X_t) = \mu, \quad Cov(X_t, X_{t+k}) = Cov(\tilde{X}_t, \tilde{X}_{t+k}).
\] (5.10)

In the case of a causal \( MA(\infty) \) process causal, we have

\[
X_t = \mu + \sum_{j=0}^{\infty} \psi_j u_{t-j}
\] (5.11)

where \( \{u_t : t \in \mathbb{Z}\} \sim BB(0, \sigma^2) \),

\[
Cov(X_t, X_{t+k}) = \sigma^2 \sum_{j=0}^{\infty} \psi_j \psi_{j+|k|},
\] (5.12)

\[
Corr(X_t, X_{t+k}) = \sum_{j=0}^{\infty} \psi_j \psi_{j+|k|} / \sum_{j=0}^{\infty} \psi_j^2.
\] (5.13)
5.3. Stationarity

The process

\[ X_t = \mu + \sum_{j=-\infty}^{\infty} \psi_j u_{t-j}, t \in \mathbb{Z}, \quad (5.14) \]

where \( \{u_t : t \in \mathbb{Z}\} \sim BB(0, \sigma^2) \) and \( \sum_{j=-\infty}^{\infty} \psi_j^2 < \infty \), is second-order stationary, for \( E(X_t) \) and \( Cov(X_t, X_{t+k}) \) do not depend on \( t \). If we suppose that \( \{u_t : t \in \mathbb{Z}\} \sim \text{IID} \), with \( E|u_t| < \infty \) and \( \sum_{j=-\infty}^{\infty} \psi_j^2 < \infty \), the process is strictly stationary.

5.4. Operational notation

We can denote the process \( MA(\infty) \)

\[ X_t = \mu + \psi(B)u_t = \mu + \left( \sum_{j=-\infty}^{\infty} \psi_j B^j \right) u_t \quad (5.15) \]

where \( \psi(B) = \sum_{j=-\infty}^{\infty} \psi_j B^j \) and \( B^j u_t = u_{t-j} \).

6. Finite order moving averages

6.1 The \( MA(q) \) process can be written

\[ X_t = \mu + u_t - \sum_{j=1}^{q} \theta_j u_{t-j} \quad (6.1) \]

where \( \theta(B) = 1 - \theta_1 B - \ldots - \theta_q B^q \). This process is a special case of the \( MA(\infty) \) process with

\[
\begin{align*}
\psi_0 &= 1, \psi_j = -\theta_j, \text{ for } 1 \leq j \leq q, \\
\psi_j &= 0, \text{ for } j < 0 \text{ or } j > q. 
\end{align*}
\]

6.2 This process is clearly second-order stationary, with

\[ E(X_t) = \mu, \quad (6.3) \]
\[ V(X_t) = \sigma^2 \left( 1 + \sum_{j=1}^{q} \theta_j^2 \right), \quad (6.4) \]

\[ \gamma(k) \equiv \text{Cov}(X_t, X_{t+k}) = \sigma^2 \sum_{j=-\infty}^{\infty} \psi_j \psi_{j+k}, \quad (6.5) \]

On defining \( \theta_0 \equiv -1 \), we then see that

\[ \gamma(k) = \sigma^2 \sum_{j=0}^{q-k} \theta_j \theta_{j+k} \]

\[ = \sigma^2 \left[ -\theta_k + \sum_{j=1}^{q-k} \theta_j \theta_{j+k} \right] \]

\[ = \sigma^2 \left[ -\theta_k + \theta_1 \theta_{k+1} + \ldots + \theta_{q-k} \theta_q \right], \quad \text{for } 1 \leq k \leq q, \quad (6.6) \]

\[ \gamma(k) = 0, \quad \text{for } k \geq q + 1, \]

\[ \gamma(-k) = \gamma(k), \quad \text{for } k < 0. \quad (6.7) \]

The autocorrelation function of \( X_t \) is thus

\[ \rho(k) = \left( -\theta_k + \sum_{j=1}^{q-k} \theta_j \theta_{j+k} \right) / \left( 1 + \sum_{j=1}^{q} \theta_j^2 \right), \quad 1 \leq k \leq q \]

\[ = 0, \quad k \geq q + 1 \]

(6.8)

The autocorrelations are zero for \( k \geq q + 1 \).

**6.3** For \( q = 1 \),

\[ \rho(k) = -\theta_1 / (1 + \theta_1^2), \quad k = 1, \]

\[ = 0, \quad k \geq 2, \quad (6.9) \]

hence \( |\rho(1)| \leq 0.5 \).

**6.4** For \( q = 2 \),

\[ \rho(k) = (-\theta_1 + \theta_1 \theta_2) / (1 + \theta_1^2 + \theta_2^2), \quad k = 1, \]

\[ = -\theta_2 / (1 + \theta_1^2 + \theta_2^2), \quad k = 2, \]

\[ = 0, \quad k \geq 3, \quad (6.10) \]

hence \( |\rho(2)| \leq 0.5 \).
6.5 For any $MA(q)$ process,
\[ \rho(q) = -\theta_q/(1 + \theta_1^2 + \ldots + \theta_q^2), \tag{6.11} \]
hence $|\rho(q)| \leq 0.5$.

6.6 There are general constraints on the autocorrelations of an $MA(q)$ process:
\[ |\rho(k)| \leq \cos(\pi/\{[q/k] + 2\}) \tag{6.12} \]
where $[x]$ is the largest integer less than or equal to $x$. From the latter formula, we see:

for $q = 1$, $|\rho(1)| \leq \cos(\pi/3) = 0.5$,
for $q = 2$, $|\rho(1)| \leq \cos(\pi/4) = 0.7071$,
$|\rho(2)| \leq \cos(\pi/3) = 0.5$,
for $q = 3$, $|\rho(1)| \leq \cos(\pi/5) = 0.809$,
$|\rho(2)| \leq \cos(\pi/3) = 0.5$,
$|\rho(3)| \leq \cos(\pi/3) = 0.5$. \tag{6.13}

See Chanda (1962), and Kendall, Stuart, and Ord (1983, p. 519).

7. Autoregressive processes

7.1 Consider a process \{\(X_t : t \in \mathbb{Z}\)\} which satisfies the equation:
\[ X_t = \bar{\mu} + \sum_{j=1}^{p} \varphi_j X_{t-j} + u_t, \forall t \in \mathbb{Z}, \tag{7.1} \]
where \{\(u_t : t \in \mathbb{Z}\)\} ~ $BB(0, \sigma^2)$. In symbolic notation,
\[ \varphi(B)X_t = \bar{\mu} + u_t, t \in \mathbb{Z}, \tag{7.2} \]
where $\varphi(B) = 1 - \varphi_1 B - \ldots - \varphi_p B^p$.

7.2 Stationarity
Consider the process AR(1)

\[ X_t = \varphi_1 X_{t-1} + u_t, \varphi_1 \neq 0. \]  

(7.3)

If \( X_t \) is S2

\[ E(X_t) = \varphi_1 E(X_{t-1}) = \varphi_1 E(X_t), \]  

(7.4)

hence \( E(X_t) = 0 \). By successive substitutions,

\[
X_t = \varphi_1 [\varphi_1 X_{t-2} + u_{t-1}] + u_t \\
= u_t + \varphi_1 u_{t-1} + \varphi_1^2 X_{t-2} \\
= \sum_{j=0}^{N-1} \varphi_1^j u_{t-j} + \varphi_1^N X_{t-N}.
\]  

(7.5)

If we suppose that \( X_t \) is S2 with \( E(X_t^2) \neq 0 \), we see that

\[
E \left[ (X_t - \sum_{j=0}^{N-1} \varphi_1^j u_{t-j})^2 \right] = \varphi_1^{2N} E(X_{t-N}^2) = \varphi_1^{2N} E(X_t^2) \to 0 \Leftrightarrow |\varphi_1| < 1.
\]  

(7.6)

The series \( \sum_{j=0}^{\infty} \varphi_1^j u_{t-j} \) converges in q.m. to \( X_t \):

\[
X_t = \sum_{j=0}^{\infty} \varphi_1^j u_{t-j} \equiv (1 - \varphi_1 B)^{-1} u_t = \frac{1}{1 - \varphi_1 B} u_t
\]  

(7.7)

where

\[
(1 - \varphi_1 B)^{-1} = \sum_{j=0}^{\infty} \varphi_1^j B^j.
\]  

(7.8)

Since

\[
\sum_{j=0}^{\infty} E|\varphi_1^j u_{t-j}| \leq \sigma \sum_{j=0}^{\infty} |\varphi_1|^j = \frac{\sigma}{1 - |\varphi_1|} < \infty
\]  

(7.9)

when \( |\varphi_1| < 1 \), the convergence is also a.s. The process \( X_t = \sum_{j=0}^{\infty} \varphi_1^j u_{t-j} \) is S2.

When \( |\varphi_1| < 1 \), the difference equation

\[
(1 - \varphi_1 B)X_t = u_t
\]  

(7.10)
has a unique stationary solution which can be written

$$X_t = \sum_{j=0}^{\infty} \varphi_1^j u_{t-j} = (1 - \varphi_1 B)^{-1} u_t. \quad (7.11)$$

The latter is thus a causal MA($\infty$) process.

This condition is sufficient (but non necessary) for the existence of a unique stationary solution. The stationarity condition is often expressed by saying that the polynome $\varphi(z) = 1 - \varphi_1 z$ has all its roots outside the unit circle $|z| = 1$:

$$1 - \varphi_1 z_s = 0 \leftrightarrow z_s = \frac{1}{\varphi_1}, \quad (7.12)$$

where $|z_s| = 1/|\varphi_1| > 1$. In this case, we also have $E(X_{t-k} u_t) = 0, \forall k \geq 1$. The same conclusion holds if we consider the general process

$$X_t = \bar{\mu} + \varphi_1 X_{t-1} + u_t. \quad (7.13)$$

For the AR($p$) process,

$$X_t = \bar{\mu} + \sum_{j=1}^{p} \varphi_j X_{t-j} + u_t \quad (7.14)$$

or

$$\varphi(B)X_t = \bar{\mu} + u_t, \quad (7.15)$$

the stationarity condition is the following:

if the polynome $\varphi(z) = 1 - \varphi_1 z - \ldots - \varphi_p z^p$ has all its roots outside the unit circle, the equation (7.14) has one and only one weakly stationary solution.

The order $p$ polynome $\varphi(z)$ can be written

$$\varphi(z) = (1 - G_1 z)(1 - G_2 z)\ldots(1 - G_p z) \quad (7.17)$$

and has the roots

$$z_s^1 = 1/G_1, \ldots, z_s^p = 1/G_p. \quad (7.18)$$

The stationarity condition may then be written:

$$|G_j| < 1, j = 1, \ldots, p. \quad (7.19)$$
The solution stationary can be written

\[ X_t = \varphi(B)^{-1}\bar{\mu} + \varphi(B)^{-1}u_t = \mu + \varphi(B)^{-1}u_t \]  \hspace{1cm} (7.20)

where

\[ \mu = \bar{\mu}/(1 - \sum_{j=1}^{p}\varphi_j), \]  \hspace{1cm} (7.21)

\[ \varphi(B)^{-1} = \prod_{j=1}^{p}(1 - G_jB)^{-1} = \prod_{j=1}^{p}\left(\sum_{k=0}^{\infty}G_j^kB^k\right) = \sum_{j=1}^{p}\frac{K_j}{1 - G_jB} \]  \hspace{1cm} (7.22)

and \( K_1, \ldots, K_p \) are constants (expansion in partial fractions). Consequently,

\[ X_t = \mu + \sum_{j=1}^{p}\frac{K_j}{1 - G_jB} u_t \]  \hspace{1cm} (7.23)

where \( \psi_k = \sum_{j=1}^{p}K_jG_j^k \). Thus

\[ E(X_{t-j}u_t) = 0, \forall j \geq 1. \]  \hspace{1cm} (7.24)

For the process AR(1) and AR(2), the stationarity conditions can be written as follows.

(a) AR(1) : \((1 - \varphi_1B)X_t = \bar{\mu} + u_t \)

| \( \varphi_1 \) | < 1 \hspace{1cm} (7.25)

(b) AR(2) : \((1 - \varphi_1B - \varphi_2B^2)X_t = \bar{\mu} + u_t \)

\[ \varphi_2 + \varphi_1 < 1 \]  \hspace{1cm} (7.26)

\[ \varphi_2 - \varphi_1 < 1 \]  \hspace{1cm} (7.27)

\[-1 < \varphi_2 < 1 \]  \hspace{1cm} (7.28)

7.3 Mean, variance and autocovariances
Suppose:

a) the autoregressive process $X_t$ is second-order stationary with $\sum_{j=1}^{p} \varphi_j \neq 1$ and

b) $E(X_{t-j}u_t) = 0$, $\forall j \geq 1$,

i.e. we assume $X_t$ is a weakly stationary solution of the equation (7.14) such that $E(X_{t-j}u_t) = 0$, $\forall j \geq 1$.

By the stationarity assumption,

$$E(X_t) = \mu, \forall t \Rightarrow \mu = \bar{\mu} + \sum_{j=1}^{p} \varphi_j \mu = \bar{\mu}j \left(1 - \sum_{j=1}^{p} \varphi_j\right) \quad (7.30)$$

For stationarity to hold, it is necessary that $\sum_{j=1}^{p} \varphi_j \neq 1$. Let us rewrite the process in the form

$$\tilde{X}_t = \sum_{j=1}^{p} \varphi_j \tilde{X}_{t-j} + u_t \quad (7.31)$$

where $\tilde{X}_t = X_t - \mu$, $E(\tilde{X}_t) = 0$. Then, for $k \geq 0$,

$$\tilde{X}_{t+k} = \sum_{j=1}^{p} \varphi_j \tilde{X}_{t+j} + u_{t+k}, \quad (7.32)$$

$$E(\tilde{X}_{t+k} \tilde{X}_t) = \sum_{j=1}^{p} \varphi_j E(\tilde{X}_{t+k-j} \tilde{X}_t) + E(u_{t+k} \tilde{X}_t), \quad (7.33)$$

$$\gamma(k) = \sum_{j=1}^{p} \varphi_j \gamma(k-j) + E(u_{t+k} \tilde{X}_t), \quad (7.34)$$

where

$$E(u_{t+k} \tilde{X}_t) = \sigma^2, \quad \text{if } k = 0,$$

$$= 0, \quad \text{if } k \geq 1. \quad (7.35)$$

Thus

$$\rho(k) = \sum_{j=1}^{p} \varphi_j \rho(k-j), k \geq 1. \quad (7.36)$$

These formulae are called the “Yule-Walker equations”. If we know $\rho(0), \ldots, \rho(p-1)$, we can easily compute $\rho(k)$ for $k \geq p + 1$. We can also write the Yule-Walker equations in the
form:

$$\varphi(B)\rho(k) = 0, k \geq 1,$$  \hspace{1cm} (7.37)

where $$B^j\rho(k) \equiv \rho(k-j)$$. To obtain $$\rho(1), \ldots, \rho(p-1)$$ when $$p > 1$$, it is sufficient to solve the linear equation system:

$$
\begin{align*}
\rho(1) &= \varphi_1 + \varphi_2\rho(1) + \ldots + \varphi_p\rho(p-1) \\
\rho(2) &= \varphi_1\rho(1) + \varphi_2 + \ldots + \varphi_p\rho(p-2) \\
&\vdots \\
\rho(p-1) &= \varphi_1\rho(p-2) + \varphi_2\rho(p-3) + \ldots + \varphi_p\rho(1)
\end{align*}
$$  \hspace{1cm} (7.38)

where we use the identity $$\rho(-j) = \rho(j)$$. The other autocorrelations may then be obtained by recurrence:

$$\rho(k) = \sum_{j=1}^{p} \varphi_j\rho(k-j), k \geq p.$$  \hspace{1cm} (7.39)

To compute $$\gamma(0) = Var(X_t)$$, we solve the equation

$$
\gamma(0) = \sum_{j=1}^{p} \varphi_j\gamma(-j) + E(u_t \tilde{X}_t)
$$

$$
= \sum_{j=1}^{p} \varphi_j\gamma(j) + \sigma^2,
$$  \hspace{1cm} (7.40)

hence, using $$\gamma(j) = \rho(j)\gamma(0)$$,

$$
\gamma(0) \left[1 - \sum_{j=1}^{p} \varphi_j\rho(j)\right] = \sigma^2
$$  \hspace{1cm} (7.41)

and

$$
\gamma(0) = \frac{\sigma^2}{1 - \sum_{j=1}^{p} \varphi_j\rho(j)}.
$$  \hspace{1cm} (7.42)

### 7.4 Special cases

1. **AR(1)**: $$\tilde{X}_t = \varphi_1 \tilde{X}_{t-1} + u_t$$

$$\rho(1) = \varphi_1.$$  \hspace{1cm} (7.43)
\[
\rho(k) = \varphi_1 \rho(k - 1), \; k \geq 1 \quad (7.44)
\]

\[
\rho(2) = \varphi_1 \rho(1) = \varphi_1^2 \quad (7.45)
\]

\[
\rho(k) = \varphi_1^k, \; k \geq 1 \quad (7.46)
\]

\[
\gamma(0) = \text{Var}(X_t) = \frac{\sigma^2}{1 - \varphi_1^2} \quad (7.47)
\]

These is no constraint on \(\rho(1)\), but there are constraints on \(\rho(k)\) for \(k \geq 2\).

2. AR(2): \(X_t = \varphi_1 \tilde{X}_{t-1} + \varphi_2 \tilde{X}_{t-2} + u_t \)

\[
\rho(1) = \varphi_1 + \varphi_2 \rho(1) \Rightarrow \rho(1) = \frac{\varphi_1}{1 - \varphi_2} \quad (7.48)
\]

\[
\rho(2) = \frac{\varphi_1^2}{1 - \varphi_2} + \varphi_2 = \frac{\varphi_1^2 + \varphi_2 (1 - \varphi_2)}{1 - \varphi_2} \quad (7.50)
\]

\[
\rho(k) = \varphi_1 \rho(k - 1) + \varphi_2 \rho(k - 2), \; k \geq 2. \quad (7.51)
\]

Constraints on \(\rho(1)\) and \(\rho(2)\) entailed by stationarity:

\[
|\rho(1)| < 1, |\rho(2)| < 1 \quad (7.52)
\]

\[
\rho(1)^2 < \frac{1}{2}[1 + \rho(2)]; \quad (7.53)
\]

see Box and Jenkins (1976, p. 61).

7.5 Explicit form for the autocorrelations

The autocorrelations of an \(AR(p)\) process satisfy the equation

\[
\rho(k) = \sum_{j=1}^{p} \varphi_j \rho(k - j), \; k \geq 1, \quad (7.54)
\]

where \(\rho(0) = 1\) and \(\rho(-k) = \rho(k)\), or equivalently

\[
\varphi(B)\rho(k) = 0, \; k \geq 1. \quad (7.55)
\]

The autocorrelations can be obtained by solving the homogeneous difference equation (7.54).
The polynome \( \varphi(z) \) has \( m \) distinct non-zero roots \( z_1^*, \ldots, z_m^* \) (where \( 1 \leq m \leq p \)) with multiplicities \( p_1, \ldots, p_m \) (where \( \sum_j p_j = p \)), so that \( \varphi(z) \) can be written

\[
\varphi(z) = (1 - G_1 z)^{p_1} (1 - G_2 z)^{p_2} \cdots (1 - G_m z)^{p_m}
\]  

(7.56)

where \( G_j = 1/z_j^* \), \( j = 1, \ldots, m \). The roots are real or complex numbers. If \( z_j^* \) is a complex (non real) root, its conjugate \( \bar{z}_j^* \) is also a root. Consequently, the solutions of equation (7.54) have the general form

\[
\rho(k) = \sum_{j=1}^{m} \left( \sum_{\ell=0}^{p_j-1} A_{j\ell} k^\ell \right) G_j^k, \quad k \geq 1,
\]  

(7.57)

where the \( A_{j\ell} \) are (possibly complex) constants which can be determined from the values \( p \) autocorrelations. We can easily find \( \rho(1), \ldots, \rho(p) \) from the Yule-Walker equations.

If we write \( G_j = r_j e^{i \theta_j} \), where \( i = \sqrt{-1} \) while \( r_j \) and \( \theta_j \) are real numbers \( (r_j > 0) \), we see that

\[
\rho(k) = \sum_{j=1}^{m} \left( \sum_{\ell=0}^{p_j-1} A_{j\ell} k^\ell \right) r_j^k e^{i \theta_j k}
\]

\[
= \sum_{j=1}^{m} \left( \sum_{\ell=0}^{p_j-1} A_{j\ell} k^\ell \right) r_j^k [\cos(\theta_j k) + i \sin(\theta_j k)]
\]

\[
= \sum_{j=1}^{m} \left( \sum_{\ell=0}^{p_j-1} A_{j\ell} k^\ell \right) r_j^k \cos(\theta_j k).
\]  

(7.58)

By stationarity, \( 0 < |G_j| = r_j < 1 \) so that \( \rho(k) \to 0 \) when \( k \to \infty \). The autocorrelations decrease at an exponential rate with oscillations.

### 7.6 MA(\( \infty \)) representation of an AR(\( p \)) process

We have seen that a weakly stationary process

\[
\varphi(B) \tilde{X}_t = u_t
\]  

(7.59)

where \( \varphi(B) = 1 - \varphi_1 B - \cdots - \varphi_p B^p \), can be written

\[
\tilde{X}_t = \psi(B) u_t
\]  

(7.60)
with
\[ \psi(B) = \varphi(B)^{-1} = \sum_{j=0}^{\infty} \psi_j B^j \]  
(7.61)

To compute the coefficients \( \psi_j \), it is sufficient to note that
\[ \varphi(B)\psi(B) = 1. \]  
(7.62)

Defining \( \psi_j = 0 \) for \( j < 0 \), we see that
\[ \left(1 - \sum_{k=1}^{p} \varphi_k B^k\right) \left(\sum_{j=-\infty}^{\infty} \psi_j B^j\right) = \sum_{j=-\infty}^{\infty} \psi_j \left(B^j - \sum_{k=1}^{p} \varphi_k B^{j+k}\right) \]
\[ = \sum_{j=-\infty}^{\infty} \left(\psi_j - \sum_{k=1}^{p} \varphi_k \psi_{j-k}\right) B^j \]
\[ = \sum_{j=-\infty}^{\infty} \tilde{\psi}_j B^j = 1. \]  
(7.63)

Thus \( \tilde{\psi}_j = 1 \), if \( j = 0 \), and \( \tilde{\psi}_j = 0 \), if \( j \neq 0 \). Consequently,
\[ \varphi(B)\psi_j = \psi_j - \sum_{k=1}^{p} \varphi_k \psi_{j-k} = 1, \text{ if } j = 0 \]
\[ = 0, \text{ if } j \neq 0, \]
(7.64)

where \( B^k \psi_j \equiv \psi_{j-k} \). Since \( \psi_j = 0 \) for \( j < 0 \), we see that:
\[ \psi_0 = 1 \]
\[ \psi_j = \sum_{k=1}^{p} \varphi_k \psi_{j-k}, j \geq 1. \]  
(7.65)

More explicitly,
\[ \psi_0 = 1, \]
\[ \psi_1 = \varphi_1 \psi_0 = \varphi_1, \]
\[ \psi_2 = \varphi_1 \psi_1 + \varphi_2 \psi_0 = \varphi_1^2 + \varphi_2, \]
\[ \psi_3 = \varphi_1 \psi_2 + \varphi_2 \psi_1 + \varphi_3 = \varphi_1^3 + 2 \varphi_2 \varphi_1 + \varphi_3, \]
\[ \vdots \]
\[ \psi_p = \sum_{k=1}^{p} \phi_k \psi_{j-k}, \]
\[ \psi_j = \sum_{k=1}^{p} \phi_k \psi_{j-k}, j \geq p + 1. \]  
(7.66)

Under the stationarity condition [roots of \( \varphi(z) = 0 \) outside the unit circle], the coefficients \( \psi_j \) decline at an exponential rate as \( j \to \infty \), possibly with oscillations.

Given the representation
\[ \tilde{X}_t = \psi(B)u_t = \sum_{j=0}^{\infty} \psi_j u_{t-j} \]  
(7.67)
we can easily compute the autocovariances and autocorrelations of \( X_t \):
\[ \text{Cov}(X_t, X_{t+k}) = \sigma^2 \sum_{j=0}^{\infty} \psi_j \psi_{j+|k|}, \]  
(7.68)
\[ \text{Corr}(X_t, X_{t+k}) = \sum_{j=0}^{\infty} \psi_j \psi_{j+|k|} / \sum_{j=0}^{\infty} \psi_j^2. \]  
(7.69)
However, this has the inconvenient of requiring one to compute limits of series.

**7.7 Partial autocorrelations**

The Yule-Walker equations allow one to determine the autocorrelations from the coefficients \( \varphi_1, \ldots, \varphi_p \). In the same way we can determine \( \varphi_1, \ldots, \varphi_p \) from the autocorrelations
\[ \rho(k) = \sum_{j=1}^{n} \varphi_j \rho(k-j), k = 1, 2, 3, \ldots \]  
(7.70)
Taking into account the fact that \( \rho(0) = 1 \) and \( \rho(-k) = \rho(k) \), we find an AR(\( p \)) process:
\[ \begin{bmatrix} 1 & \rho(1) & \rho(2) & \ldots & \rho(p-1) \\ \rho(1) & 1 & \rho(1) & \ldots & \rho(p-2) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \rho(p-1) & \rho(p-2) & \rho(p-3) & \ldots & 1 \end{bmatrix} \begin{bmatrix} \varphi_1 \\ \varphi_2 \\ \vdots \\ \varphi_p \end{bmatrix} = \begin{bmatrix} \rho(1) \\ \rho(2) \\ \vdots \\ \rho(p) \end{bmatrix} \]  
(7.71)
or, in more compact notation,
\[ P_p \tilde{\phi}_p = \tilde{\rho}_p. \] (7.72)

It follows that
\[ P_k \tilde{\phi}_k = \tilde{\rho}_k, \quad k = 1, 2, 3, \ldots \] (7.73)

where \( \tilde{\phi}_k = (\varphi_{k1}, \varphi_{k2}, \ldots, \varphi_{kk})' \), so that we can solve for \( \tilde{\phi}_k \):
\[ \tilde{\phi}_k = P_k^{-1} \tilde{\rho}_k. \] (7.74)

[If \( \sigma^2 > 0 \), we can show that \( P_k^{-1} \) exists, \( \forall k \geq 1 \).]

The coefficients \( \varphi_{kk} \) are called the lag- \( k \) partial autocorrelations.

Particular values of \( \varphi_{kk} \) [setting \( \rho_k = \rho(k) \)]:

\[ \varphi_{11} = \rho_1, \] (7.76)

\[ \varphi_{22} = \begin{vmatrix} 1 & \rho_1 \\ \rho_1 & 1 \end{vmatrix} = \frac{\rho_2 - \rho_1^2}{1 - \rho_1^2}, \] (7.77)

\[ \varphi_{33} = \begin{vmatrix} 1 & \rho_1 & \rho_2 \\ \rho_1 & 1 & \rho_2 \\ \rho_2 & \rho_1 & 1 \end{vmatrix}. \] (7.78)

### 7.8 Durbin-Levinson recurrence formula

The partial autocorrelations may be computed using the following recursive formulae:

\[ \varphi_{k+1, k+1} = \frac{\rho (k + 1) - \sum_{j=1}^{k} \varphi_{kj} \rho (k + 1 - j)}{1 - \sum_{j=1}^{k} \varphi_{kj} \rho (j)}, \] (7.79)

\[ \varphi_{k+1, j} = \varphi_{kj} - \varphi_{k+1, k+1} \varphi_{k, k-j+1}, \quad j = 1, 2, \ldots, k. \] (7.80)

Given \( \rho(1), \ldots, \rho(k+1) \) and \( \varphi_{k1}, \ldots, \varphi_{kk} \), we can compute \( \varphi_{k+1,j} \), \( j = 1, \ldots, k + 1 \). See Durbin (1960) and Box and Jenkins (1976, pp. 82-84).
8. Mixed processes

Consider a process \( \{X_t : t \in \mathbb{Z}\} \) which satisfies the equation:

\[
X_t = \bar{\mu} + \sum_{j=1}^{p} \varphi_j X_{t-j} + u_t - \sum_{j=1}^{q} \theta_j u_{t-j}
\]  
(8.1)

where \( \{u_t : t \in \mathbb{Z}\} \sim BB(0, \sigma^2) \). Using operational notation,

\[
\varphi(B)X_t = \bar{\mu} + \theta(B)u_t.
\]  
(8.2)

8.1 Stationarity conditions

If the polynomial \( \varphi(z) = 1 - \varphi_1 z - \ldots - \varphi_p z^p \) has all its roots outside the unit circle, the equation (8.1) has one and only one weakly stationary solution, which can be written:

\[
X_t = \mu + \theta_{\infty} u_t = \mu + \sum_{j=0}^{\infty} \psi_j u_{t-j},
\]  
(8.3)

where

\[
\mu = \bar{\mu}/\varphi(B) = \bar{\mu}/(1 - \sum_{j=1}^{p} \varphi_j),
\]  
(8.4)

\[
\theta(B)/\varphi(B) \equiv \psi(B) = \sum_{j=0}^{\infty} \psi_j B^j.
\]  
(8.5)

The coefficients \( \psi_j \) are obtained by solving the equation

\[
\varphi(B)\psi(B) = \theta(B).
\]  
(8.6)

In this case, we also have:

\[
E(X_{t-j}u_t) = 0, \forall j \geq 1.
\]  
(8.7)

The \( \psi_j \) coefficients may be computed in the following way (setting \( \theta_0 = -1 \)):

\[
\left(1 - \sum_{k=1}^{p} \varphi_k B^k\right) \left(\sum_{j=0}^{\infty} \psi_j B^j\right) = 1 - \sum_{j=1}^{q} \theta_j B^j = -\sum_{j=1}^{q} \theta_j B^j
\]  
(8.8)

hence

\[
\varphi(B)\psi_j = -\theta_j, j = 0, 1, \ldots, q
\]

\[
= 0, j \geq q + 1,
\]  
(8.9)
where $\psi_j = 0$, for $j < 0$. Consequently,

\[
\psi_j = \sum_{k=1}^{p} \varphi_k \psi_{j-k} - \theta_j, \quad j = 0, 1, \ldots, q
\]

\[
= \sum_{k=1}^{p} \varphi_k \psi_{j-k}, \quad j \geq q + 1,
\]

and

\[
\psi_0 = 1, \\
\psi_1 = \varphi_1 \psi_0 - \theta_1 = \varphi_1 - \theta_1, \\
\psi_2 = \varphi_1 \psi_1 + \varphi_2 \psi_0 - \theta_2 = \varphi_1 \psi_1 + \varphi_2 - \theta_2 = \varphi_1^2 - \varphi_1 \theta_1 + \varphi_2 - \theta_2, \\
\vdots \\
\psi_j = \sum_{k=1}^{p} \varphi_k \psi_{j-k}, j \geq q + 1.
\]

(8.11)

The $\psi_j$ coefficients behave like the autocorrelations of an $AR(p)$ process, except for the initial coefficients $\psi_1, \ldots, \psi_q$.

8.2 Autocovariances and autocorrelations

Suppose:

a) the process $X_t$ is second-order stationary with $\sum_{j=1}^{p} \varphi_j \neq 1$;

b) $E(X_{t-j} u_t) = 0$, $\forall j \geq 1$.

(8.12)

By the stationarity assumption,

\[
E(X_t) = \mu, \forall t,
\]

(8.13)

hence

\[
\mu = \bar{\mu} + \sum_{j=1}^{p} \varphi_j \mu
\]

(8.14)

and

\[
E(X_t) = \mu = \bar{\mu}/ \left(1 - \sum_{j=1}^{p} \varphi_j\right).
\]

(8.15)
The mean is the same as in the case of a pure AR($p$) process. The MA($q$) part has no effect on the mean. Let us now rewrite the process in the form

$$\tilde{X}_t = \sum_{j=1}^{p} \varphi_j \tilde{X}_{t-j} + u_t - \sum_{j=1}^{q} \theta_j u_{t-j}$$  \hspace{1cm} (8.16)$$

where $\tilde{X}_t = X_t - \mu$. Consequently,

$$\tilde{X}_{t+k} = \sum_{j=1}^{p} \varphi_j \tilde{X}_{t+k-j} + u_{t+k} - \sum_{j=1}^{q} \theta_j u_{t+k-j},$$  \hspace{1cm} (8.17)

$$E(\tilde{X}_t \tilde{X}_{t+k}) = \sum_{j=1}^{p} \varphi_j E(\tilde{X}_t \tilde{X}_{t+k-j}) + E(\tilde{X}_t u_{t+k}) - \sum_{j=1}^{q} \theta_j E(\tilde{X}_t u_{t+k-j})$$  \hspace{1cm} (8.18)

$$\gamma(k) = \sum_{j=1}^{p} \varphi_j \gamma(k-j) + \gamma_{xu}(k) - \sum_{j=1}^{q} \theta_j \gamma_{xu}(k-j),$$  \hspace{1cm} (8.19)

where

$$\gamma_{xu}(k) = E(\tilde{X}_t u_{t+k}) = 0, \text{ if } k \geq 1,$$

$$\neq 0, \text{ if } k \leq 0,$$  \hspace{1cm} (8.20)

$$\gamma_{xu}(0) = E(\tilde{X}_t u_t) = \sigma^2.$$  

For $k \geq q + 1$,

$$\gamma(k) = \sum_{j=1}^{p} \varphi_j \gamma(k-j),$$  \hspace{1cm} (8.21)

$$\rho(k) = \sum_{j=1}^{p} \varphi_j \rho(k-j).$$  \hspace{1cm} (8.22)

The variance is given by

$$\gamma(0) = \sum_{j=1}^{p} \varphi_j \gamma(j) + \sigma^2 - \sum_{j=1}^{q} \theta_j \gamma_{xu}(-j)$$  \hspace{1cm} (8.23)

hence

$$\gamma(0) = \left[ \sigma^2 - \sum_{j=1}^{q} \theta_j \gamma_{xu}(-j) \right] / \left[ 1 - \sum_{j=1}^{p} \varphi_j \rho(j) \right].$$  \hspace{1cm} (8.24)
In operational notation, the autocovariances satisfy the equation
\[
\varphi(B)\gamma(k) = \theta(B)\gamma_{xu}(k), \, k \geq 0,
\]  
(8.25)
where \(\gamma(-k) = \gamma(k)\), \(B^j \gamma(k) \equiv \gamma(k-j)\) and \(B^j \gamma_{xu}(k) \equiv \gamma_{xu}(k-j)\). In particular,
\[
\varphi(B)\gamma(k) = 0, \, k \geq q + 1,
\]  
(8.26)
\[
\varphi(B)\rho(k) = 0, \, k \geq q + 1.
\]  
(8.27)
To compute the autocovariances, we can solve the equations (8.19) for \(k = 0, 1, \ldots, p\), and then apply (8.21). The autocovariances of an ARMA \((p, q)\) process behave like those of an \(AR(p)\) process, except that initial values are modified.

8.3 Example ARMA(1, 1) process

\[
X_t = \bar{\mu} + \varphi_1 X_{t-1} + u_t - \theta_1 u_{t-1}, \quad |\varphi_1| < 1
\]  
(8.28)
\[
\tilde{X}_t - \varphi_1 \tilde{X}_{t-1} = u_t - \theta_1 u_{t-1}
\]  
(8.29)
where \(\tilde{X}_t = X_t - \mu\). We have
\[
\gamma(0) = \varphi_1 \gamma(1) + \gamma_{xu}(0) - \theta_1 \gamma_{xu}(-1),
\]  
(8.30)
\[
\gamma(1) = \varphi_1 \gamma(0) + \gamma_{xu}(1) - \theta_1 \gamma_{xu}(0)
\]  
(8.31)
and
\[
\gamma_{xu}(1) = 0,
\]  
(8.32)
\[
\gamma_{xu}(0) = \sigma^2,
\]  
(8.33)
\[
\gamma_{xu}(-1) = E(\tilde{X}_t u_{t-1}) = \varphi_1 E(\tilde{X}_{t-1} u_{t-1}) + E(u_t u_{t-1}) - \theta_1 E(u_{t-1}^2)
\]  
(8.34)
\[
= \varphi_1 \gamma_{xu}(0) - \theta_1 \sigma^2 = (\varphi_1 - \theta_1)\sigma^2
\]
Thus,
\[
\gamma(0) = \varphi_1 \gamma(1) + \sigma^2 - \theta_1 (\varphi_1 - \theta_1)\sigma^2
\]  
(8.35)
\[
= \varphi_1 \gamma(1) + [1 - \theta_1 (\varphi_1 - \theta_1)]\sigma^2,
\]
\[
\gamma(1) = \varphi_1 \gamma(0) - \theta_1 \sigma^2
\]  
(8.36)
\[
= \varphi_1 \{\varphi_1 \gamma(1) + [1 - \theta_1 (\varphi_1 - \theta_1)]\sigma^2\} - \theta_1 \sigma^2,
\]  
(8.37)
hence
\[
\gamma(1) = \{\varphi[1 - \theta_{1}(\varphi_{1} - \theta_{1})] - \theta_{1}\} \sigma^{2}/(1 - \varphi_{1}^{2}) \\
= \{\varphi_{1} - \theta_{1}\varphi_{1}^{2} + \varphi_{1}\theta_{1}^{2} - \theta_{1}\} \sigma^{2}/(1 - \varphi_{1}^{2}) \\
= (1 - \theta_{1}\varphi_{1})(\varphi_{1} - \theta_{1}) \sigma^{2}/(1 - \varphi_{1}^{2}).
\] (8.37)

Similarly,
\[
\gamma(0) = \varphi_{1}\gamma(1) + [1 - \theta_{1}(\varphi_{1} - \theta_{1})]\sigma^{2} \\
= \frac{\sigma^{2}}{1 - \varphi_{1}^{2}} \{\varphi_{1}(1 - \theta_{1}\varphi_{1})(\varphi_{1} - \theta_{1}) + (1 - \varphi_{1}^{2})[1 - \theta_{1}(\varphi_{1} - \theta_{1})]\} \\
= \frac{\sigma^{2}}{1 - \varphi_{1}^{2}} \{\varphi_{1}^{2} - \theta_{1}\varphi_{1}^{3} + \varphi_{1}\theta_{1}^{2} - \varphi_{1}\theta_{1} + 1 - \varphi_{1}^{2} - \theta_{1}\varphi_{1} + \theta_{1}\varphi_{1}^{3} + \theta_{1}^{2} - \varphi_{1}^{2}\theta_{1}^{2}\} \\
= \frac{\sigma^{2}}{1 - \varphi_{1}^{2}} \{1 - 2\varphi_{1}\theta_{1} + \theta_{1}^{2}\}. 
\] (8.38)

Thus,
\[
\gamma(0) = (1 - 2\varphi_{1}\theta_{1} + \theta_{1}^{2}) \sigma^{2}/(1 - \varphi_{1}^{2}), 
\] (8.39)
\[
\gamma(1) = (1 - \theta_{1}\varphi_{1})(\varphi_{1} - \theta_{1}) \sigma^{2}/(1 - \varphi_{1}^{2}), 
\] (8.40)
\[
\gamma(k) = \varphi_{1}\gamma(k - 1), \text{ for } k \geq 2. 
\] (8.41)

9. Invertibility

9.1 Any second-order stationary AR(p) process can be written under an MA(\infty) form. Similarly, any second-order stationary ARMA(p, q) process can also be written under an MA(\infty) form. By analogy, it is natural to ask the question: can a MA(q) or ARMA(p, q) process be represented in a purely autoregressive form?

9.2 Consider the process MA(1):
\[
X_{t} = u_{t} - \theta_{1}u_{t-1}, \ t \in \mathbb{Z},
\] (9.1)
where \(\{u_{t} : t \in \mathbb{Z}\} \sim BB(0, \sigma^{2})\) and \(\sigma^{2} > 0\). We see easily that
\[
u_{t} = X_{t} + \theta_{1}u_{t-1}
\]
\begin{align*}
&= X_t + \theta_1 (X_{t-1} + \theta_1 u_{t-2}) \\
&= X_t + \theta_1 X_{t-1} + \theta_1^2 u_{t-2} \\
&= \sum_{j=0}^{n} \theta_1^j X_{t-j} + \theta_1^{n+1} u_{t-n-1} \quad (9.2)
\end{align*}

and
\begin{equation}
E \left[ \left( \sum_{j=0}^{n} \theta_1^j X_{t-j} - u_t \right)^2 \right] = E \left[ \left( \theta_1^{n+1} u_{t-n-1} \right)^2 \right] = \theta_1^{2(n+1)} \sigma^2 \to 0, \quad (9.3)
\end{equation}

provided \(|\theta_1| < 1\). Consequently, the series \(\sum_{j=0}^{n} \theta_1^j X_{t-j}\) converges in q.m. to \(u_t\) if \(|\theta_1| < 1\).

In other words, when \(|\theta_1| < 1\), we can write
\begin{equation}
\sum_{j=0}^{\infty} \theta_1^j X_{t-j} = u_t, \quad t \in \mathbb{Z}, \quad (9.4)
\end{equation}

or
\begin{equation}
(1 - \theta_1 B)^{-1} X_t = u_t, \quad t \in \mathbb{Z}, \quad (9.5)
\end{equation}

where \((1 - \theta_1 B)^{-1} = \sum_{j=0}^{\infty} \theta_1^j B^j\). The condition \(|\theta_1| < 1\) is equivalent to having the roots of the equation \(1 - \theta_1 z = 0\) outside the unit circle. If \(\theta_1 = 1\),
\begin{equation}
X_t = u_t - u_{t-1} \quad (9.6)
\end{equation}

and the series
\begin{equation}
(1 - \theta_1 B)^{-1} X_t = \sum_{j=0}^{\infty} \theta_1^j X_{t-j} = \sum_{j=0}^{\infty} X_{t-j} \quad (9.7)
\end{equation}

does not converge, for \(E(X_{t-j}^2)\) does not converge to 0 as \(j \to \infty\). Similarly, if \(\theta_1 = -1\),
\begin{equation}
X_t = u_t + u_{t-1} \quad (9.8)
\end{equation}

and the series
\begin{equation}
(1 - \theta_1 B)^{-1} X_t = \sum_{j=0}^{\infty} (-1)^j X_{t-j} \quad (9.9)
\end{equation}

does not converge either. These models are not invertible.

\textbf{9.3 Theorem} (Invertibility condition for a MA process) : Let \(\{X_t : t \in \mathbb{Z}\}\) be a second-
order stationary process such that

\[ X_t = \mu + \theta(B)u_t \]  \hspace{1cm} (9.10)

where \( \theta(B) = 1 - \theta_1 B - \ldots - \theta_q B^q \). Then the process \( X_t \) satisfies an equation of the form

\[ \sum_{j=0}^{\infty} \phi_j X_{t-j} = \bar{\mu} + u_t \]  \hspace{1cm} (9.11)

iff the roots of the polynomial \( \theta(z) \) are outside the unit circle. Further, when the representation (9.11) exists, we have:

\[ \bar{\phi}(B) = \theta(B)^{-1}, \quad \bar{\mu} = \theta(B)^{-1}\mu = \mu/ \left( 1 - \sum_{j=1}^{q} \theta_j \right). \]  \hspace{1cm} (9.12)

9.4 Corollary (Invertibility of an ARMA process) : Let \( \{X_t : t \in \mathbb{Z}\} \) be a second-order stationary ARMA process that satisfies the equation

\[ \varphi(B)X_t = \bar{\mu} + \theta(B)u_t \]  \hspace{1cm} (9.13)

where \( \varphi(B) = 1 - \varphi_1 B - \ldots - \varphi_p B^p \) and \( \theta(B) = 1 - \theta_1 B - \ldots - \theta_q B^q \). Then the process \( X_t \) satisfies an equation of the form

\[ \sum_{j=0}^{\infty} \phi_j X_{t-j} = \bar{\mu} + u_t \]  \hspace{1cm} (9.14)

iff the roots of the polynome \( \theta(z) \) are outside the unit circle. Further, when the representation (9.14) exists, we have:

\[ \bar{\phi}(B) = \theta(B)^{-1}\varphi(B), \quad \bar{\mu} = \theta(B)^{-1}\bar{\mu} = \mu/ \left( 1 - \sum_{j=1}^{q} \theta_j \right). \]  \hspace{1cm} (9.15)

10. Wold representation

10.1 We have seen that all second-order ARMA processes can be written in a causal \( MA(\infty) \) form. This property indeed holds for all second-order stationary processes.
10.2 Theorem (Wold) : Let \( \{X_t, t \in \mathbb{Z}\} \) be a second-order stationary process such that \( E(X_t) = \mu \). Then \( X_t \) can be written in the form

\[
X_t = \mu + \sum_{j=0}^\infty \psi_j u_{t-j} + v_t
\]

(10.1)

where \( \{u_t : t \in \mathbb{Z}\} \sim BB(0, \sigma^2) \), \( \sum_{j=0}^\infty \psi_j^2 < \infty \), \( E(u_t X_{t-j}) = 0 \), \( \forall j \geq 1 \), and \( \{v_t : t \in \mathbb{Z}\} \) is a process deterministic such that \( E(v_t) = 0 \) and \( E(u_s v_t) = 0 \), \( \forall s, t \). Further, if \( \sigma^2 > 0 \), the sequences \( \{\psi_j\} \) and \( \{u_t\} \) are unique, and

\[
u_t = \tilde{X}_t - P(\tilde{X}_t | \tilde{X}_{t-1}, \tilde{X}_{t-2}, ...)
\]

(10.2)

where \( \tilde{X}_t = X_t - \mu \).


10.3 If \( E(u_t^2) > 0 \) in Wold representation, we say the process \( X_t \) is regular. \( v_t \) is called the deterministic component of the process while \( \sum_{j=0}^\infty \psi_j u_{t-j} \) is its indeterministic component. When \( v_t = 0 \), \( \forall t \), the process \( X_t \) is said to be strictly indeterministic.

10.4 Corollary (Forward Wold representation) : Let \( \{X_t : t \in \mathbb{Z}\} \) be second-order a stationary process such that \( E(X_t) = \mu \). Then \( X_t \) can be written in the form

\[
X_t = \mu + \sum_{j=0}^\infty \bar{\psi}_j \bar{u}_{t+j} + \bar{v}_t
\]

(10.3)

where \( \{\bar{u}_t : t \in \mathbb{Z}\} \sim \bar{BB}(0, \bar{\sigma}^2) \), \( \sum_{j=0}^\infty \bar{\psi}_j^2 < \infty \), \( E(\bar{u}_t X_{t+j}) = 0 \), \( \forall j \geq 1 \), and \( \{\bar{v}_t : t \in \mathbb{Z}\} \) is a deterministic (with respect to \( \bar{v}_{t+1}, \bar{v}_{t+2}, ... \)) such that \( E(\bar{v}_t) = 0 \) and \( E(\bar{u}_s \bar{v}_t) = 0 \), \( \forall s, t \). Further, if \( \bar{\sigma}^2 > 0 \), the sequences \( \{\bar{\psi}_j\} \) and \( \{\bar{u}_t\} \) are uniquely defined, and

\[
\bar{u}_t = \tilde{X}_t - P(\tilde{X}_t | \tilde{X}_{t+1}, \tilde{X}_{t+2}, ...)
\]

(10.4)

where \( \tilde{X}_t = X_t - \mu \).
PROOF. The result follows on applying Wold theorem to the process $Y_t \equiv X_{t-1}$ qui is also second-order stationary. Q.E.D.

11. Generating functions and spectral density

11.1 Generating functions constitute a convenient technique representing or finding the autocovariance structure of a stationary process.

11.2 Definition (Generating function) : Let $(a_k : k = 0, 1, 2, \ldots)$ and $(b_k : k = \ldots, -1, 0, 1, \ldots)$ two sequences of complex numbers. Let $D(a) \subseteq \mathbb{C}$ the set of points $z \in \mathbb{C}$ for which the series $\sum_{k=0}^{\infty} a_k z^k$ converges, and let $D(b) \subseteq \mathbb{C}$ the set of points $z$ for which the series $\sum_{k=-\infty}^{\infty} b_k z^k$ converges. Then the functions

$$a(z) = \sum_{k=0}^{\infty} a_k z^k, \quad z \in D(a) \quad (11.1)$$

and

$$b(z) = \sum_{k=-\infty}^{\infty} b_k z^k, \quad z \in D(b) \quad (11.2)$$

are called the generating functions of the sequences $a_k$ and $b_k$ respectively.

11.3 Proposition (Convergence annulus of a generating function) : Let $(a_k : k \in \mathbb{Z})$ be a sequence of complex numbers. Then the generating function

$$a(z) = \sum_{k=-\infty}^{\infty} a_k z^k \quad (11.3)$$

converges for $R_1 < |z| < R_2$ where

$$R_1 = \limsup_{k \to \infty} |a_{-k}|^{1/k}, \quad (11.4)$$

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\[ R_2 = \frac{1}{\limsup_{k \to \infty} |a_k|^{1/k}}, \] (11.5)

and diverges for \(|z| < R_1\) or \(|z| > R_2\). If \(R_2 < R_1\), \(a(z)\) converges nowhere and, if \(R_1 = R_2\), \(a(z)\) diverges everywhere except possibly, for \(|z| = R_1 = R_2\). Further, when \(R_1 < R_2\), the coefficients \(a_k\) are uniquely defined, and

\[ a_k = \frac{1}{2\pi i} \int_C \frac{a(z) \, dz}{(z - z_0)^{k+1}}, \quad k = 0, \pm 1, \pm 2, \ldots \] (11.6)

where \(C = \{z \in \mathbb{C} : |z - z_0| = R\}\) and \(R_1 < R < R_2\).

11.4 Proposition (Sums and products of generating functions) : Let \((a_k : k \in \mathbb{Z})\) and \((b_k \in \mathbb{Z})\) two sequences of complex numbers such that the generating functions \(a(z)\) and \(b(z)\) converge for \(R_1 < |z| < R_2\), where \(0 \leq R_1 < R_2 \leq \infty\). Then,

1. the generating function of the sum \(c_k = a_k + b_k\) is \(c(z) = a(z) + b(z)\);
2. if the product sequence

\[ d_k = \sum_{j=-\infty}^{\infty} a_j b_{k-j} \] (11.7)

converges for any \(k\), the generating function of the sequence \(d_k\) is

\[ d(z) = a(z)b(z). \] (11.8)

Further, the series \(c(z)\) and \(d(z)\) converge for \(R_1 < |z| < R_2\).

11.5 We will be especially interested by generating functions of autocovariances \(\gamma_k\) and autocorrelations \(\rho_k\) of a second-order stationary process \(X_t\):

\[ \gamma_x(z) = \sum_{k=-\infty}^{\infty} \gamma_k z^k, \] (11.9)

\[ \rho_x(z) = \sum_{k=-\infty}^{\infty} \rho_k z^k = \frac{\gamma_x(z)}{\gamma_0}. \] (11.10)
We see immediately that the generating function with a white noise \( \{u_t : t \in \mathbb{Z}\} \sim BB(0, \sigma^2) \) is constant:

\[
\gamma_u(z) = \sigma^2, \quad \rho_u(z) = 1.
\]  \(11.11\)

**11.6 Proposition** (Convergence of the generating function of the autocovariances): Let \( \gamma_k, k \in \mathbb{Z} \), the autocovariances of a second-order stationary process \( X_t \), and \( \rho_k, k \in \mathbb{Z} \), the corresponding autocorrelations.

1. If \( R \equiv \limsup_{k \to \infty} |\rho_k|^{1/k} < 1 \), the generating functions \( \gamma_x(z) \) and \( \rho_x(z) \) converge for \( R < |z| < 1/R \).

2. If \( R = 1 \), the functions \( \gamma_x(z) \) and \( \rho_x(z) \) diverge everywhere, except possibly on the circle \( |z| = 1 \).

3. If \( \sum_{k=0}^{\infty} |\rho_k| < \infty \), the functions \( \gamma_x(z) \) and \( \rho_x(z) \) converge absolutely and uniformly on the circle \( |z| = 1 \).

**11.7 Proposition** (Unicity): Let \( \gamma_k \) and \( \rho_k, k \in \mathbb{Z} \), autocovariance and autocorrelation sequences such that

\[
\gamma(z) = \sum_{k=-\infty}^{\infty} \gamma_k z^k = \sum_{k=-\infty}^{\infty} \gamma'_k z^k, \quad (11.12)
\]

\[
\rho(z) = \sum_{k=-\infty}^{\infty} \rho_k z^k = \sum_{k=-\infty}^{\infty} \rho'_k z^k \quad (11.13)
\]

where the series considered converge for \( R < |z| < 1/R \), where \( R \geq 0 \). Then \( \gamma_k = \gamma'_k \) and \( \rho_k = \rho'_k \) for any \( k \in \mathbb{Z} \).

**11.8 Proposition** (Generating function of the autocovariances of a MA(\( \infty \)) process): Let \( \{X_t : t \in \mathbb{Z}\} \) a second-order stationary process such that

\[
X_t = \sum_{j=-\infty}^{\infty} \psi_j u_{t-j} \quad (11.14)
\]
where \( \{u_t : t \in \mathbb{Z}\} \sim BB(0, \sigma^2) \). If the series
\[
\psi(z) = \sum_{j=-\infty}^{\infty} \psi_j z^j
\]
and \( \psi(z^{-1}) \) converge absolutely, then
\[
\gamma_x(z) = \sigma^2 \psi(z) \psi(z^{-1}).
\]  

11.9 Corollary (Generating function of the autocovariances of an ARMA process) : Let \( \{X_t : t \in \mathbb{Z}\} \) a second-order stationary and causal ARMA\((p, q)\) process, such that
\[
\varphi(B)X_t = \bar{\mu} + \theta(B)u_t
\]
where \( \{u_t : t \in \mathbb{Z}\} \sim BB(0, \sigma^2), \varphi(z) = 1 - \varphi_1 z - \ldots - \varphi_p z^p \) and \( \theta(z) = 1 - \theta_1 z - \ldots - \theta_q z^q \). Then the generating function of the autocovariances of \( X_t \) is
\[
\gamma_x(z) = \sigma^2 \frac{\theta(z)}{\varphi(z)} \frac{\theta(z^{-1})}{\varphi(z^{-1})}
\]
for \( R < |z| < 1/R \), where
\[
0 < R = \max\{|G_1|, |G_2|, \ldots, |G_p|\} < 1
\]
and \( G_1^{-1}, G_2^{-1}, \ldots, G_p^{-1} \) are the roots of the polynome \( \varphi(z) \).

11.10 Proposition (Generating function of the autocovariances of a filtered process) : Let \( \{X_t : t \in \mathbb{Z}\} \) a second-order stationary process and
\[
Y_t = \sum_{j=-\infty}^{\infty} c_j X_{t-j}, t \in \mathbb{Z},
\]
where \( \{c_j : j \in \mathbb{Z}\} \) is a sequence of real constants such that \( \sum_{j=-\infty}^{\infty} |c_j| < \infty \). If the series \( \gamma_x(z) \) and \( c(z) = \sum_{j=-\infty}^{\infty} c_j z^j \) converge absolutely, then
\[
\gamma_y(z) = c(z) c(z^{-1}) \gamma_x(z).
\]
11.11 Definition (Spectral density) : Let $X_t$ a second-order stationary process such that the generating function of the autocovariances $\gamma_x(z)$ converge for $|z|=1$. The spectral density of the process $X_t$ is the function

$$f_x(\omega) = \frac{1}{2\pi} \left[ \gamma_0 + 2 \sum_{k=1}^{\infty} \gamma_k \cos(\omega k) \right]$$

where the coefficients $\gamma_k$ are the autocovariances of the process $X_t$. The function $f_x(\omega)$ is defined for all the values of $\omega$ such that the series $\sum_{k=1}^{\infty} \gamma_k \cos(\omega k)$ converges.

11.12 Remark If the series $\sum_{k=1}^{\infty} \gamma_k \cos(\omega k)$ converges, it is immediate that $\gamma_x(e^{-i\omega})$ converge and

$$f_x(\omega) = \frac{1}{2\pi} \gamma_x(e^{-i\omega}) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \gamma_k e^{-i\omega k}$$

where $i = \sqrt{-1}$.

11.13 Proposition (Convergence and properties of the spectral density) : Let $\gamma_k, k \in \mathbb{Z}$, be an autocovariance function such that $\sum_{k=0}^{\infty} |\gamma_k| < \infty$. Then

(1) the series

$$f_x(\omega) = \frac{\gamma_0}{2\pi} + \frac{1}{\pi} \sum_{k=1}^{\infty} \gamma_k \cos(\omega k)$$

converges absolutely and uniformly in $\omega$ ;

(2) the function $f_x(\omega)$ is continuous ;

(3) $f_x(\omega + 2\pi) = f_x(\omega)$ and $f_x(-\omega) = f_x(\omega), \forall \omega$ ;

(4) $\gamma_k = \int_{-\pi}^{\pi} f_x(\omega) \cos(\omega k) d\omega, \forall k$ ;

(5) $f_x(\omega) \geq 0$ ;
(6) \( \gamma_0 = \int_{-\pi}^{\pi} f_x(\omega) d\omega \).

11.14 Proposition (Spectral densities of special processes): Let \( \{X_t : t \in \mathbb{Z}\} \) be a second-order stationary process with autocovariances \( \gamma_k, k \in \mathbb{Z} \).

1. If \( X_t = \mu + \sum_{j=-\infty}^{\infty} \psi_j u_{t-j} \) where \( \{u_t : t \in \mathbb{Z}\} \sim BB(0, \sigma^2) \) and \( \sum_{j=-\infty}^{\infty} |\psi_j| < \infty \), then

\[
 f_x(\omega) = \frac{\sigma^2}{2\pi} \psi(e^{i\omega})\psi(e^{-i\omega}) = \frac{\sigma^2}{2\pi} |\psi(e^{i\omega})|^2. \tag{11.25}
\]

2. If \( \phi(B)X_t = \bar{\mu} + \theta(B)u_t \), where \( \phi(B) = 1 - \varphi_1 B - \ldots - \varphi_p B^p \), \( \theta(B) = 1 - \theta_1 B - \ldots - \theta_q B^q \) and \( \{u_t : t \in \mathbb{Z}\} \sim BB(0, \sigma^2) \), then

\[
 f_x(\omega) = \frac{\sigma^2}{2\pi} \left| \frac{\theta(e^{i\omega})}{\phi(e^{i\omega})} \right|^2. \tag{11.26}
\]

3. If \( Y_t = \sum_{j=-\infty}^{\infty} c_j X_{t-j} \) where \( \{c_j : j \in \mathbb{Z}\} \) is a sequence of real constants such that \( \sum_{j=-\infty}^{\infty} |c_j| < \infty \), and if \( \sum_{k=0}^{\infty} |\gamma_k| < \infty \), then

\[
 f_y(\omega) = \left| c(e^{i\omega}) \right|^2 f_x(\omega). \tag{11.27}
\]

12. Inverse autocorrelations

12.1 Definition (Autocorrelations inverses): Let \( f_x(\omega) \) the spectral density of a second-order stationary process \( \{X_t : t \in \mathbb{Z}\} \). If the function \( 1/ f_x(\omega) \) is also a spectral density, the autocovariances \( \gamma_x^{(I)}(k), k \in \mathbb{Z} \), associated with the inverse spectrum inverse \( 1/ f_x(\omega) \) are called the inverse autocovariances of the process \( X_t \), i.e.

\[
 \gamma_x^{(I)}(k) = \int_{-\pi}^{\pi} \frac{1}{f_x(\omega)} \cos(\omega k) d\omega, k \in \mathbb{Z}. \tag{12.1}
\]
The inverse autocovariances satisfy the equation
\[
\frac{1}{f_x(\omega)} = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \gamma_x^{(I)}(k) \cos(\omega k) = \frac{1}{2\pi} \gamma_x^{(I)}(0) + \frac{1}{\pi} \sum_{k=1}^{\infty} \gamma_x^{(I)} \cos(\omega k). \tag{12.2}
\]
The inverse autocorrelations are
\[
\rho_x^{(I)}(k) = \frac{\gamma_x^{(I)}(k)}{\gamma_x^{(I)}(0)}, k \in \mathbb{Z}. \tag{12.3}
\]
A sufficient condition for the function $1/f_x(\omega)$ to be a spectral density is that the function $1/f_x(\omega)$ be continuous on the interval $-\pi \leq \omega \leq \pi$, which entails that $f_x(\omega) > 0$, \forall \omega.

If the process $X_t$ is a second-order stationary $ARMA(p, q)$ process such that
\[
\varphi_p(B)X_t = \bar{\mu} + \theta_q(B)u_t \tag{12.4}
\]
where $\varphi_p(B) = 1 - \varphi_1B - \ldots - \varphi_pB^p$ and $\theta_q(B) = 1 - \theta_1B - \ldots - \theta_qB^q$ are des polynomes which have all their roots outside the unit circle and $\{u_t : t \in \mathbb{Z}\} \sim BB(0, \sigma^2)$, then
\[
f_x(\omega) = \frac{\sigma^2}{2\pi} \left| \frac{\theta_q(e^{i\omega})}{\varphi_p(e^{i\omega})} \right|^2 \tag{12.5}
\]
and
\[
\frac{1}{f_x(\omega)} = \frac{2\pi}{\sigma^2} \left| \frac{\varphi_p(e^{i\omega})}{\theta_q(e^{i\omega})} \right|^2. \tag{12.6}
\]
The inverse autocovariances $\gamma_x^{(I)}(k)$ are the autocovariances associated with the model
\[
\theta_q(B)X_t = \bar{\mu} + \varphi_p(B)v_t \tag{12.7}
\]
where $\{v_t : t \in \mathbb{Z}\} \sim BB(0, 1/\sigma^2)$ and $\bar{\mu}$ is some constant. Consequently, the inverse autocorrelations of an $ARMA(p, q)$ process behave like the autocorrelations of an $ARMA(q, p)$. For an process $AR(p)$ process,
\[
\rho_x^{(I)}(k) = 0, \text{ for } k > p. \tag{12.8}
\]
For a $MA(q)$ process, the inverse partial autocorrelations (i.e. the partial autocorrelations
associated with the inverse autocorrelations) are equal to zero for \( k > q \). These properties can be used for identifying the order of a process.

13. Multiplicity of representations

13.1. Backward representation ARMA models

By the backward Wold theorem, we know that any strictly indeterministic second-order stationary process \( X_t : t \in \mathbb{Z} \) can be written in the form

\[
X_t = \mu + \sum_{j=0}^{\infty} \bar{\psi}_j \bar{u}_{t+j}
\]  

(13.1)

where \( \bar{u}_t \) is a white noise such that \( E(X_{t-j} \bar{u}_t) = 0 \), \( \forall j \geq 1 \). In particular, if

\[
\varphi_p(B)(X_t - \mu) = \theta_q(B)u_t
\]

(13.2)

where the polynomials \( \varphi_p(B) = 1 - \varphi_1 B - ... - \varphi_p B^p \) and \( \theta_q(B) = 1 - \theta_1 B - ... - \theta_q B^q \) have all their roots outside the unit circle and \( \{u_t : t \in \mathbb{Z}\} \sim BB(0, \sigma^2) \), the spectral density of \( X_t \) is

\[
f_x(\omega) = \frac{\sigma^2}{2\pi} \left| \frac{\theta_q(e^{i\omega})}{\varphi_p(e^{i\omega})} \right|^2.
\]

(13.3)

Consider the process

\[
Y_t = \frac{\varphi_p(B^{-1})}{\theta_q(B^{-1})} (X_t - \mu) = \sum_{j=0}^{\infty} c_j (X_{t+j} - \mu).
\]

(13.4)

Pour the Proposition 11.14, the spectral density of \( Y_t \) is

\[
f_y(\omega) = \left| \frac{\varphi_p(e^{i\omega})}{\theta_q(e^{i\omega})} \right|^2 f_x(\omega) = \frac{\sigma^2}{2\pi}
\]

(13.5)

and thus \( \{Y_t : t \in \mathbb{Z}\} \sim BB(0, \sigma^2) \). If we define \( \bar{u}_t = Y_t \), we see that

\[
\frac{\varphi_p(B^{-1})}{\theta_q(B^{-1})} (X_t - \mu) = \bar{u}_t
\]

(13.6)
or
\[
\varphi_p(B^{-1}) X_t = \bar{\mu} + \theta_q(B^{-1}) \bar{u}_t, \tag{13.7}
\]
and
\[
(10.1.7) X_t - \varphi_1 X_{t+1} - ... - \varphi_p X_{t+p} = \bar{\mu} + \bar{u}_t - \theta_1 \bar{u}_{t+1} - ... - \theta_q \bar{u}_{t+q} \tag{13.8}
\]
where \((1 - \varphi_1 - ... - \varphi_p) \mu = \bar{\mu}\). We call (13.6) or (13.8) the backward representation of the \(X_t\) process.

### 13.2. Multiple moving-average representations

Let \(\{X_t\} \sim ARIMA(p, d, q)\). Then
\[
W_t = (1 - B)^d X_t \sim ARMA(p, q). \tag{13.9}
\]
If we suppose that \(E(W_t) = 0\), \(W_t\) satisfies an equation of the form
\[
\varphi_p(B) W_t = \theta_q(B) u_t \tag{13.10}
\]
or
\[
W_t = \frac{\theta_q(B)}{\varphi_p(B)} u_t = \psi(B) u_t. \tag{13.11}
\]
To determine an appropriate \(ARMA\) model, one typically estimates the autocorrelations \(\rho_k\). The latter are uniquely determined by the generating function of the autocovariances:
\[
\gamma_x(z) = \sigma^2 \psi(z) \psi(z^{-1}) = \sigma^2 \frac{\theta_q(z)}{\varphi_p(z)} \frac{\theta_q(z^{-1})}{\varphi_p(z^{-1})}. \tag{13.12}
\]
If
\[
\theta_q(z) = 1 - \theta_1 z - ... - \theta_q z^q = (1 - H_1 z) ...(1 - H_q z) = \prod_{j=1}^q (1 - H_j z), \tag{13.13}
\]
then
\[
\gamma_x(z) = \frac{\sigma^2}{\varphi_p(z) \varphi_p(z^{-1})} \prod_{j=1}^q (1 - H_j z)(1 - H_j z^{-1}). \tag{13.14}
\]
However
\[
(1 - H_j z)(1 - H_j z^{-1}) = 1 - H_j z - H_j z^{-1} + H_j^2 = H_j^2(1 - H_j^{-1} z - H_j^{-1} z^{-1} + H_j^{-2})
\]
\[
= H_j^2(1 - H_j^{-1} z)(1 - H_j^{-1} z^{-1}) \tag{13.15}
\]
\[
gamma_x(z) = \left[ \sigma^2 \prod_{j=1}^{q} H_j^2 \right] \frac{\theta_q'(z) \theta_q'(z^{-1})}{\varphi_p(z) \varphi_p(z^{-1})} \prod_{j=1}^{q} (1 - H_j^{-1}z) \prod_{j=1}^{q} (1 - H_j^{-1}z^{-1})
\]

(13.16)

where

\[
\bar{\sigma}^2 = \sigma^2 \prod_{j=1}^{q} H_j^2,
\]

(13.17)

\[
\theta_q'(z) = \prod_{j=1}^{q} (1 - H_j^{-1}z).
\]

(13.18)

\[\gamma_x(z)\] in (13.16) can be viewed as the generating function of a process of the form

\[\varphi_p(B)W_t = \theta_q'(B)\bar{u}_t = \left[ \prod_{j=1}^{q} (1 - H_j^{-1}B) \right] \bar{u}_t\]

(13.19)

while \[\gamma_x(z)\] in (13.14) is the generating function of

\[\varphi_p(B)W_t = \theta_q(B)u_t = \left[ \prod_{j=1}^{q} (1 - H_jB) \right] u_t.
\]

(13.20)

The processes (13.19) and (13.20) have the same autocovariance function and thus cannot be distinguished by looking at their seconds moments.

13.1 Example

\[\bar{u}_t\]

(13.21)

have the same autocorrelation function.

In general, the models

\[\varphi_p(B)W_t = \left[ \prod_{j=1}^{q} (1 - H_j^{-1}B) \right] \bar{u}_t
\]

(13.23)

all have the same autocovariance function (and are thus indistinguishable). Since it is easier
with an invertible model, we select
\[ H^*_{j} = \begin{cases} H_j, & \text{if } |H_j| < 1 \\ H^{-1}_j, & \text{if } |H_j| > 1 \end{cases}, \tag{13.24} \]
where $|H_j| \leq 1$, in order to have an invertible model.

### 13.3. Redundant parameters

Suppose $\varphi_p(B)$ and $\theta_q(B)$ have a common factor, say $G(B)$:
\[ \varphi_p(B) = G(B)\varphi_{p_1}(B), \theta_q(B) = G(B)\theta_{q_1}(B). \tag{13.25} \]

Consider the models
\[ \varphi_p(B)W_t = \theta_q(B)u_t \tag{13.26} \]
\[ \varphi_{p_1}(B)W_t = \theta_{q_1}(B)u_t. \tag{13.27} \]

The $MA(\infty)$ representations of these two models are
\[ W_t = \psi(B)u_t, \tag{13.28} \]
where
\[ \psi(B) = \frac{\theta_q(B)}{\varphi_p(B)} = \frac{\theta_{q_1}(B)G(B)}{\varphi_{p_1}(B)G(B)} = \frac{\theta_{q_1}(B)}{\varphi_{p_1}(B)} \equiv \psi_1(B) \tag{13.29} \]
and
\[ W_t = \psi_1(B)u_t. \tag{13.30} \]

(13.26) and (13.27) have the same $MA(\infty)$ representation, hence also the same autocovariance generating functions:
\[ \gamma_x(z) = \sigma^2\psi(z)\psi(z^{-1}) = \sigma^2\psi_1(z)\psi_1(z^{-1}). \tag{13.31} \]

It is not possible to distinguish a series generated by (13.26) form one produced with (13.27). Among these two models, we will select the simpler one, i.e. (13.27). Further, if we tried to estimate (13.26) rather than (13.27), we would meet singularity problems (in the covariance matrix of the estimators).
References


