Complex analysis and power series *

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1. Analytic functions

1.1 Notation *In this text, z refers to a complex number ($z \in \mathbb{C}$), while f and g represent functions $f : E \to \mathbb{C}$ and $g : F \to \mathbb{C}$, where $B(a;\bar{\delta}) \subseteq E \subseteq \mathbb{C}$, $B\left(a;\bar{\delta}\right) \subseteq F \subseteq \mathbb{C}$, $B\left(a;\bar{\delta}\right) = \left\{z \in \mathbb{C} : |z-a| < \bar{\delta}\right\}$, $0 < \bar{\delta} \leq \infty$ and $a \in \mathbb{C}$. In other words, f and g are functions with complex values whose domains are subsets E and F of the complex numbers containing an open ball centered at the point a.*

1.2 Definition LIMIT OF A COMPLEX FUNCTION. *Let $b \in \mathbb{C}$. We say that $f(z)$ converges to b when z tends to a, denoted*

$$\lim_{z\to a} f(z) = b\ ,$$

iff the following property holds: for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|z-a| < \delta \text{ and } z \neq a \Rightarrow |f(z)-b| < \varepsilon\ .$$

1.3 Definition RIGHT AND LEFT LIMITS. *Let $b \in \mathbb{C}$, $x \in \mathbb{R}$ and $f : E \to \mathbb{C}$, where $B\left(a;\bar{\delta}\right) \subseteq E \subseteq \mathbb{R}$ and $a \in \mathbb{R}$. We say that $f(x)$ converges to b when x tends to a from the left, denoted*

$$\lim_{x\to a^-} f(x) = b \text{ or } f(a-) = b\ ,$$

iff the following property holds: for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|x-a| < \delta \text{ and } x < a \Rightarrow |f(x)-b| < \varepsilon\ .$$

Similarly, we say that $f(x)$ converges to b when x tends to a from the right, denoted

$$\lim_{x\to a^+} f(x) = b \text{ or } f(a+) = b\ ,$$

iff the following property holds: for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|x-a| < \delta \text{ and } x > a \Rightarrow |f(x)-b| < \varepsilon\ .$$

1.4 Definition CONTINUOUS FUNCTION. *We say that the function f is continuous at point a iff*

$$\lim_{z\to a} f(z) = f(a)\ .$$

1.5 Definition DERIVATIVE OF A COMPLEX FUNCTION. *We say that the function f is differentiable at point a iff there exists a number $f'(a) \in \mathbb{C}$ such that*

$$\lim_{z\to a} \frac{f(z)-f(a)}{z-a} = f'(a)\ .$$

We call $f'(a)$ the derivative of $f(z)$ at a.

1.6 Remark We also denote $f'(z)$ by $\frac{d}{dz} f(z)$.

1.7 Proposition CONTINUITY OF DIFFERENTIABLE FUNCTIONS. *If the function f is differentiable at point a, then it is continuous at point a.*

1.8 Theorem PROPERTIES OF DIFFERENTIATION. *Let $z \in B\left(a;\bar{\delta}\right) \subseteq E \cap F$. If the functions f and g are differentiable at point z, then*

(1) $\frac{d}{dz}\left[c\, f(z)\right] = c\, f'(z)$,

(2) $\frac{d}{dz}\left[f(z) + g(z)\right] = f'(z) + g'(z)$,

(3) $\frac{d}{dz}\left[f(z)\, g(z)\right] = f'(z)\, g(z) + f(z)\, g'(z)$,

(4) $\frac{d}{dz}\left[\frac{f(z)}{g(z)}\right] = \frac{f'(z)g(z) - f(z)g'(z)}{g(z)^2}$, *provided* $g(z) \neq 0$.

1.9 Theorem CHAIN RULE. *Let $h : G \to \mathbb{C}$ where $B(f(a);\ \delta_0) \subseteq f(E) \subseteq G \subseteq \mathbb{C}$, $B(f(a);\ \delta_0) = \{z \in \mathbb{C} : |z - f(a)| < \delta_0\}$ and $0 < \delta_0 \leq \infty$. If the function f is differentiable at point a and if h is differentiable at point $f(a)$, then the composite function $H(z) = h[f(z)]$ is differentiable at point a and*

$$H'(a) = h'[f(a)]\, f'(a)\ .$$

1.10 Theorem DERIVATIVES OF IMPORTANT FUNCTIONS.

(1) *If c is a complex constant, then*

$$\frac{d}{dz}(c) = 0\ .$$

(2) *If n is a real constant,*

$$\frac{d}{dz}(z^n) = n\, z^{n-1},\ \text{provided } z \neq 0 \text{ when } n < 1\,.$$

(3) $\frac{d}{dz}(e^z) = e^z$.

1.11 Theorem DERIVATIVE OF A REAL FUNCTION OF A COMPLEX VARIABLE. *Suppose the function f only takes real values at all points of the open ball $B\left(a;\bar{\delta}\right)$,* i.e. *$f(z) \in \mathbb{R}$ for any $z \in B\left(a;\ \bar{\delta}\right)$. If f is differentiable at point a, then $f'(a) = 0$.*

1.12 Definition ANALYTIC FUNCTION. *If there exists a positive constant $\varepsilon > 0$ such that the function f is differentiable at all points z such that $|z - a| < \varepsilon$, we say that the function is* analytic *at point a. If the function f is analytic at all points of a domain $D \subseteq \mathbb{C}$, we say that f is analytic on the domain D.*

1.13 Remark An *analytic* function is also called a *holomorphic function*.

1.14 Definition SINGULAR POINT. *If a function f is not analytic at point z_0, but for any $\varepsilon > 0$ there exists a point z_1 such that $|z_1 - z_0| < \varepsilon$ and f is analytic at z_1, we say that z_0 is a* singular point (*or a* singularity) *of the function f. If, furthermore, there exists a radius $R > 0$ such that f is analytic on the disk $0 < |z - z_0| < R$, we say that z_0 is an* isolated singular point *of the function f.*

1.15 Theorem OPERATIONS ON ANALYTIC FUNCTIONS. *If the functions f and g are analytic at point a, then*

(1) *the functions $f(z) + g(z)$ and $f(z)\,g(z)$ are analytic at point a;*

(2) *the function $f(z)/g(z)$ is analytic at point a provided $g(a) \neq 0$.*

1.16 Theorem COMPOSITION OF ANALYTIC FUNCTIONS. *Let $h : G \to \mathbb{C}$ where $B(f(a);\ \delta_0) \subseteq f(E) \subseteq G \subseteq \mathbb{C}$, $B(f(a);\ \delta_0) = \{z \in \mathbb{C} : |z - f(a)| < \delta_0\}$ and $0 < \delta_0 \leq \infty$. If the function f is analytic at point a and if h is analytic at point $f(a)$, then the composed function $(h \circ f)(z) = h[f(z)]$ is analytic at point a.*

1.17 Theorem INFINITE DIFFERENTIABILITY OF ANALYTIC FUNCTIONS. *If the function f is analytic at point $a \in \mathbb{C}$, then f has derivatives of all orders at a, and the derivative functions are also analytic at point a.*

1.18 Theorem IMPORTANT ANALYTIC FUNCTIONS.

(1) *Any polynomial of degree n,*

$$f(z) = a_0 + a_1 z + \cdots + a_n z^n \tag{1.1}$$

where $a_0, a_1, \ldots, a_n \in \mathbb{C}$, is analytic at all points $z \in \mathbb{C}$.

(2) *A rational function*

$$f(z) = P(z)/Q(z) \tag{1.2}$$

where $P(z)$ and $Q(z)$ are polynomials of degrees p and q, is analytic everywhere, except when $Q(z) = 0$.

(3) *The functions e^z, $\cos(z)$ and $\sin(z)$ are analytic everywhere.*

(4) *The function $\log(z)$ is analytic everywhere except at $z = 0$.*

2. Power series

2.1 Definition POWER SERIES. *Let $\{a_n\}_{n=0}^{\infty} \subseteq \mathbb{C}$, $z_0 \in \mathbb{C}$ and $z \in \mathbb{C}$. We call the series $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ a power series centered at z_0. The numbers a_n are the* coefficients *of the series.*

2.2 Remark In this definition and the sequel, we will use the convention $0^0 = 1$.

2.3 Theorem CONVERGENCE RADIUS OF A POWER SERIES (ABEL-HADAMARD). *Let $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ a power series and*

$$\alpha = \limsup_{n\to\infty} |a_n|^{1/n}, \quad R = 1/\alpha,$$

where $R = \infty$ when $\alpha = 0$, and $R = 0$ when $\alpha = \infty$. Then the series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ converges absolutely if $|z-z_0| < R$ and diverges if $|z-z_0| > R$. Further, if $0 \le \rho < R$, the convergence is uniform for $|z-z_0| \le \rho$.

2.4 Remark We call R the *convergence radius* of the series $\sum_{n=0}^{\infty} a_n (z-z_0)^n$. The expression $1/R = \limsup_{n\to\infty} |a_n|^{1/n}$ is the *Hadamard* formula for the convergence radius.

2.5 Corollary ABSOLUTE CONVERGENCE OF POWER SERIES. *If the power series $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ converges for $z = z_1$, where $z_1 \neq z_0$, then it converges absolutely for any z such that $|z-z_0| < |z_1 - z_0|$.*

2.6 Corollary BOUNDS ON COEFFICIENTS OF POWER SERIES. *Let $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ a power series whose convergence radius is R, and let $\varepsilon > 0$.*

(1) *If $0 < R \le \infty$, there exists an integer N, such that $|a_n| < \left(\frac{1}{R} + \varepsilon\right)^n$ for $n > N$.*

(2) *If $0 < R < \infty$, there is an infinity of values of n for which $|a_n| > \left(\frac{1}{R} - \varepsilon\right)^n$.*

(3) *If $R = 0$, there is an infinity of values of n for which $|a_n| > \varepsilon^n$.*

2.7 Theorem UNIFORM ABSOLUTE CONVERGENCE OF POWER SERIES. *If the power series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ converges absolutely for $z = z_1$, where $z_1 \neq z_0$, then it converges absolutely and uniformly on the closed disk $D = \{z \in \mathbb{C} : |z-z_0| \le |z_1 - z_0|\}$.*

2.8 Proposition CONVERGENCE RADIUS AND RATIO CRITERION. *Let $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ be a power series whose convergence radius is R. Then*

$$\liminf_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| \le R \le \limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| .$$

Further, if $\lim_{n\to\infty} |a_{n+1}/a_n|$ exists or $\lim_{n\to\infty} |a_{n+1}/a_n| = \infty$, then $R = \lim_{n\to\infty} |a_{n+1}/a_n|$.

2.9 Theorem CONVERGENCE CONDITION ON THE UNIT CIRCLE. *Let $\sum_{n=0}^{\infty} a_n z^n$ be a power series whose convergence radius is 1. If $\{a_n\}_{n=0}^{\infty}$ is a sequence of real numbers such that*

(a) *$a_{n+1} \le a_n, \forall n$, and*

(b) *$\lim_{n\to\infty} a_n = 0$,*

then the series $\sum_{n=0}^{\infty} a_n z^n$ converges at any point of the circle $|z| = 1$, except possibly at $z = 1$.

2.10 Theorem CONTINUITY OF POWER SERIES ON THE UNIT CIRCLE (ABEL). *If the series* $\sum_{n=0}^{\infty} a_n$ *converges, then the function* $\sum_{n=0}^{\infty} a_n z^n$*, where* $|z| < 1$*, tends to* $\sum_{n=0}^{\infty} a_n$ *when* $z \to 1$ *so that* $|1-z| / (1-|z|)$ *remains bounded.*

2.11 Corollary CONTINUITY OF REAL POWER SERIES ON THE UNIT CIRCLE. *If* $\{a_n\}_{n=0}^{\infty}$ *is a sequence of real numbers such that* $\sum_{n=0}^{\infty} a_n$ *converges, and if the power series* $f(x) = \sum_{n=0}^{\infty} a_n x^n$ *converges for* $|x| < 1$*, where* $x \in \mathbb{R}$*, then* $\lim\limits_{x \to 1^-} f(x)$ *exists and*

$$\lim_{x \to 1^-} f(x) = \sum_{n=0}^{\infty} a_n .$$

2.12 Remark If the series $\sum_{n=0}^{\infty} a_n$ does not converge, the limit $\lim\limits_{x \to 1^-} f(x)$ may or may not exist. In general, the existence of the limit $\lim\limits_{x \to 1^-} f(x)$ does not guarantee the convergence of the series $\sum_{n=0}^{\infty} a_n$. There are however cases where the existence of the limit $\lim\limits_{x \to 1^-} f(x)$ implies the convergence of $\sum_{n=0}^{\infty} a_n$ (*Tauberian theorems*). The following theorem provides an example.

2.13 Theorem CRITERION FOR CONVERGENCE AND CONTINUITY OF REAL POWER SERIES ON THE UNIT CIRLE (TAUBER). *If* $\{a_n\}_{n=0}^{\infty}$ *is a sequence of real numbers such that* $f(x) = \sum_{n=0}^{\infty} a_n x^n$ *converges for* $|x| < 1$*, where* $x \in \mathbb{R}$*, if* $\lim\limits_{n \to \infty} (n a_n) = 0$ *and if* $\lim\limits_{x \to 1^-} f(x)$ *exists, then the series* $\sum_{n=0}^{\infty} a_n$ *converges and* $\lim\limits_{x \to 1^-} f(x) = \sum_{n=0}^{\infty} a_n$*.*

2.14 Theorem UNICITY OF POWER SERIES COEFFICIENTS. *If* $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ *and* $\sum_{n=0}^{\infty} b_n (z-z_0)^n$ *are two power series which converge for* $|z-z_0| < R$*, where* $R > 0$*, and if the limits of these series coincide on a sequence of points* $\{z_k\}_{k=1}^{\infty}$ *such that* $0 < |z_k| < R$*,* $\forall k$*, and* $\lim\limits_{k \to \infty} z_k = z_0$*, then*

$$a_n = b_n , \quad \forall n .$$

2.15 Corollary UNICITY OF POWER SERIES COEFFICIENTS IN A CIRCLE. *If* $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ *and* $\sum_{n=0}^{\infty} b_n (z-z_0)^n$ *are two power series which converge for* $|z-z_0| < R$*, where* $R > 0$*, and if the limits of these series coincide for any* z *in the circle* $|z-z_0| < R$*, then*

$$a_n = b_n , \quad \forall n .$$

2.16 Theorem DIFFERENTIABILITY OF POWER SERIES. *Let* $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$ *for* $|z-z_0| < R$*, where* $R > 0$ *and* $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ *is a power series whose convergence radius is* R*. Then the function* $f(z)$ *is analytic (and thus differentiable) on the disk* $|z-z_0| < R$*, and*

$$f'(z) = \sum_{n=1}^{\infty} n\, a_n (z-z_0)^{n-1}$$

where the power series $\sum_{n=1}^{\infty} n\ a_n (z-z_0)^{n-1}$ *has convergence radius* R*. If, furthermore,* $0 < R < \infty$ *and* $f(z)$ *is a function such that* $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$ *at every point where the series*

$\sum_{n=0}^{\infty} a_n (z-z_0)^n$ converges, then there is at least one point on the circle $|z-z_0| = R$ where the function $f(z)$ is not analytic.

2.17 Remark In other words, we can obtain the derivative of the function $f(z) = \sum_{n=1}^{\infty} a_n (z-z_0)^n$ by differentiating the series term by term, and the derivative series has the same convergence radius as the original series.

2.18 Corollary DIFFERENTIABILITY AT ALL ORDERS OF POWER SERIES. *Let $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$ for $|z-z_0| < R$, where $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ is a power series whose convergence radius is R. Then the function $f(z)$ has derivatives of all orders, and these derivatives can be obtained by differentiating the series term by term. The derivative series all have the same convergence radius R, and*

$$a_n = \frac{f^{(n)}(z_0)}{n!},\ n = 0,1,2,...$$

where $f^{(n)}(z)$ is the derivative of order n of $f(z)$.

2.19 Theorem INTEGRABILITY OF POWER SERIES. *Let $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ be a power series whose convergence radius is R, let $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$ for $|z-z_0| < R$, C a contour (continuous curve) in the interior of the convergence circle $|z-z_0| < R$, and $g(z)$ a continuous function on C. Then*

$$\int_C f(z)\, g(z)\, dz = \sum_{n=0}^{\infty} a_n \int_C g(z)(z-z_0)^n\, dz\,.$$

2.20 Definition TWO-SIDED POWER SERIES. *Let $\{a_n\}_{n=-\infty}^{\infty}$, $z_0 \in \mathbb{C}$ and $z \in \mathbb{C}$. We call* two-sided power series *a series of the form $\sum_{n=-\infty}^{\infty} a_n (z-z_0)^n$. This series converges when the two series $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ and $\sum_{n=-\infty}^{-1} a_n (z-z_0)^n$ converge. Otherwise, we say it diverges.*

2.21 Proposition CONVERGENCE ANNULUS OF TWO-SIDED POWER SERIES. *Let $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ and $\sum_{n=1}^{\infty} a_{-n} (z-z_0)^n$ be power series whose convergence radii are R_1 and R_2 respectively, where $R_1 > 0$ and $R_2 > 0$.*

(1) *If $1/R_2 < R_1$, the series $\sum_{n=-\infty}^{\infty} a_n (z-z_0)^n$ converges for $1/R_2 < |z-z_0| < R_1$ and diverges when $|z-z_0| > R_1$ or $|z-z_0| < 1/R_2$.*

(2) *If $R_1 < 1/R_2$, the series $\sum_{n=-\infty}^{\infty} a_n (z-z_0)^n$ diverges everywhere.*

(3) *If $R_1 = 1/R_2$, the series $\sum_{n=-\infty}^{\infty} a_n (z-z_0)^n$ diverges everywhere except possibly on the circle $|z-z_0| = R_1$.*

3. Taylor and Laurent series

3.1 Theorem TAYLOR SERIES. *Let f be an analytic function at any point of the open disk*

$$D = \{y \in \mathbb{C} : |z-z_0| < R\},\ \text{where } z_0 \in \mathbb{C} \text{ and } 0 < R \le \infty.$$

Then there exists a unique sequence $\{a_n\}_{n=0}^{\infty}$ *in* $\mathbb{C}$ *such that*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \ , \ \forall z \in D \, .$$

Further,

$$a_n = f^{(n)}(z_0)/n! = \frac{1}{2\pi i} \int\limits_C \frac{f(z)\, dz}{(z - z_0)^{n+1}} \ , \ n = 0, 1, 2, \ldots$$

where $C = \{z \in \mathbb{C} : |z - z_0| = \rho\}$ *and* ρ *is any radius such that* $0 < \rho < R$.

3.2 Remark In other words, an analytic function on the interior of a circle centered at z_0 can be written in the interior of this circle as a power series of $(z - z_0)$. Further, this series is unique. The integral $\int\limits_C$ is evaluated counterclockwise.

3.3 Corollary CAUCHY INEQUALITIES. *Under the conditions of Theorem* **3.1**, *suppose that* $|f(z)| \leq M$ *for* $z \in C(\rho)$, *where* $C(\rho) = \{z \in \mathbb{C} : |z - z_0| = \rho\}$ *and* $0 < \rho < R$. *Then*

$$|a_n| = |f^n(z_0)|/n! \leq M/\rho^n, n = 0, 1, 2, \ldots \, .$$

3.4 Remark The Cauchy inequalities entail: for $\rho < 1$, the coefficients of the Taylor series must decline at an exponential rate which depends on the convergence radius.

3.5 Corollary EQUIVALENCE BETWEEN ANALYTICITY AND THE EXISTENCE OF A TAYLOR SERIES. *Let* $D = \{z \in \mathbb{C} : |z - z_0| < R\}$ *where* $z_0 \in \mathbb{C}$ *and* $0 < R \leq \infty$. *Then a function* f *is analytic on the domain* D *iff there exists a unique sequence* $\{a_n\}_{n=0}^{\infty}$ *in* $\mathbb{C}$ *such that*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \forall z \in D \, .$$

3.6 Theorem ZEROS OF ANALYTIC FUNCTIONS. *Let* f *be an analytic function at point* z_0, *such that* $f(z_0) = 0$. *If* $f^{(n)}(z_0) = 0$, $n = 1, 2, \ldots, m-1$, *but* $f^{(m)}(z_0) \neq 0$, *where* $m \geq 1$, *then there exists a radius* $R > 0$ *such that the function* f *can be written*

$$f(z) = (z - z_0)^m g(z)$$

for $|z - z_0| < R$, *where the function* $g(z)$ *is analytic at* z_0, *and* $g(z) \neq 0$ *for* $|z - z_0| < R$. *If* $f^{(n)}(z_0) = 0$, $n = 1, 2, \ldots,$ *then there exists a radius* $R > 0$ *such that* $f(z) = 0$ *for* $|z - z_0| < R$.

3.7 Remark The latter theorem implies that the zeros of a non-zero analytic function are *isolated*: unless all the derivatives of f are zero, we can find a radius $R > 0$ such that z_0 is the only point where the function cancels in the disk $|z - z_0| < R$. We call z_0 a *root* of the function f, and m its *multiplicity*.

3.8 Theorem FACTORIZATION OF AN ANALYTIC FUNCTION. *Let f be an analytic function on an open convex domain $U \subseteq \mathbb{C}$. If the function f has only a finite number p of distinct roots $z_1, \ldots, z_p$, then the function f can be written*

$$f(z) = (z - z_1)^{m_1} \ldots (z - z_p)^{m_p} g(z), z \in U$$

where $m_1, \ldots, m_p$ are the multiplicities of the roots $z_1, \ldots, z_p$ and $g(z)$ is an analytic function on U such that $g(z) \neq 0$ for any $z \in U$.

3.9 Remark In other words, an analytic function with a finite number of roots is finite can be written as the product of a polynomial with the same roots and an analytic function which is different from zero everywhere. An open disk $C = \{z \in \mathbb{C} : 0 \leq (z - z_0) < R\}$ where $R > 0$ is a convex set. The latter theorem remains valid when U is a convex and connected set.

3.10 Theorem SIMPLIFICATION RULE. *Let $U \subseteq \mathbb{C}$ an open and connected set. If f and g are two analytic functions on U such that*

$$f(z) g(z) = 0, \quad \forall z \in U ,$$

then $f(z) = 0$, $\forall z \in U$, or $g(z) = 0$, $\forall z \in U$.

3.11 Remark If f, g and h are three analytic functions on U such that $f(z) h(z) = g(z) h(z)$, $\forall z \in U$, and if $h(z) \neq 0$ for at least one value of $z \in U$, then

$$[f(z) - g(z)] h(z) = 0$$

and we can conclude that $f(z) = g(z)$, $\forall z \in U$.

3.12 Theorem LOCAL SEPARABILITY OF ANALYTIC FUNCTIONS. *Let f be an analytic function which is not constant on an open connected set U. Then, for $w \in \mathbb{C}$ and $z_0 \in U$, there exists a radius $R > 0$ such that $f(z) \neq w$ for $0 < (z - z_0) < R$.*

3.13 Remark In other words, if the function is not constant, we can find a radius $R > 0$ such that $f(z)$ takes the value w at least one time in the disk $0 \leq |z - z_0| < R$.

3.14 Theorem LAURENT SERIES. *Let C_0 and C_1 be two circles centered at z_0 such that C_0 is contained in C_1*, i.e.

$$C_0 = \{z \in \mathbb{C} : |z - z_0| = R_0\}, C_1 = \{z \in \mathbb{C} : |z - z_0| = R_1\} \text{ where } 0 \leq R_0 < R_1 \leq \infty .$$

Let f be an analytic function on C_0 and C_1 as well as on the domain between these two circles. Then there exists a unique two-sided sequence $\{a_n\}_{n=-\infty}^{\infty}$ in $\mathbb{C}$ such that

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

for any z such that $R_0 < |z - z_0| < R_1$, where

$$\begin{aligned} a_n &= \frac{1}{2\pi i}\int_{C_1} \frac{f(z)\,dz}{(z-z_0)^{n+1}}\ , \text{ for } n = 0, 1, 2, \ldots \\ &= \frac{1}{2\pi i}\int_{C_0} \frac{f(z)\,dz}{(z-z_0)^{n+1}}\ , \text{ for } n = -1, -2, \ldots . \end{aligned}$$

Further, for any circle $C = \{z \in \mathbb{C} : |z - z_0| = R\}$ where $R_0 < R < R_1$,

$$a_n = \frac{1}{2\pi i}\int_{C} \frac{f(z)\,dz}{(z-z_0)^{n+1}}\ , \ n = 0, \pm 1, \pm 2, \ldots .$$

3.15 Remark The line integrals $\int_{C_0}$, $\int_{C_1}$ and $\int_{C}$ are evaluated counterclockwise.

3.16 Corollary LAURENT SERIES NEAR AN ISOLATED SINGULARITY. *If f is an analytic function at any point of the disk $|z - z_0| < R$, where $R > 0$, except possibly at z_0, then there exists a unique two-sided sequence $\{a_n\}_{n=-\infty}^{\infty}$ in $\mathbb{C}$ such that*

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

for any z such that $0 < |z - z_0| < R$.

3.17 Remark In other words, if z_0 is a singular point of the function f, the function f can be represented by a Laurent series on the disk $0 < |z - z_0| < R$. If, furthermore, $a_n = 0$ for $n < 0$, the Laurent series reduces to a Taylor series, and we can redefine the function f at z_0 so that the latter is analytic at z_0 and thus everywhere on the disk $0 \le |z - z_0| < R$. In such a case, we say that the singular point z_0 is removable. When a function f is analytic at any point of the disk $|z - z_0| < R$, it is clear we must have $a_n = 0$ for $n < 0$.

3.18 Corollary GENERALIZED CAUCHY INEQUALITIES. *Under the conditions of Theorem* **3.14**, *suppose that $|f(z)| \le M$ for $z \in C(R)$, where $C(R) = \{z \in \mathbb{C} : |z - z_0| = R\}$ and $R_0 < R < R_1$. Then*

$$|a_n| \le M/R^n\ , \ n = 0,\ \pm 1,\ \pm 2, \ldots .$$

3.19 Definition PRINCIPAL AND REGULAR PARTS OF A LAURENT SERIES. *In a Laurent series $\sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$, we call the series $\sum_{n=-\infty}^{-1} a_n (z - z_0)^n$ the* principal part *of the series, while the series $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ is called the* regular part *of the series.*

4. Sums, products and ratios of power series

4.1 Theorem POINTWISE CONVERGENCE. *Let $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ and $\sum_{n=0}^{\infty} b_n (z - z_0)^n$ be two convergent power series whose limits are $f(z)$ and $g(z)$ respectively at a given point z. Then the* *following properties hold:*

(1) $cf(z) = \sum_{n=0}^{\infty} c a_n (z-z_0)^n, \forall c \in \mathbb{C}$;

(2) $f(z) + g(z) = \sum_{n=0}^{\infty} (a_n + b_n)(z-z_0)^n$;

(3) *if* $f(z)$ *or* $g(z)$ *converges absolutely, then*

$$f(z)\,g(z) = \sum_{n=0}^{\infty} c_n (z-z_0)^n \tag{4.1}$$

where $c_n = \sum_{k=0}^{n} a_k b_{n-k}$; *furthermore, if the two series* $f(z)$ *and* $g(z)$ *converge absolutely, the series* $\sum_{n=0}^{\infty} c_n (z-z_0)^n$ *converges absolutely;*

(4) *if*

(a) $b_0 \neq 0$,

(b) *the series* $h(z) = \sum_{n=0}^{\infty} d_n (z-z_0)^n$ *where the coefficients* d_n *are obtained by solving the equations* $\sum_{k=0}^{n} a_k b_{n-k} = a_n$, $n = 0,\ 1,\ \ldots$, *converges,*

(c) $g(z)$ *or* $h(z)$ *converges absolutely,*

(d) $g(z) \neq 0$,

then

$$\frac{f(z)}{g(z)} = \sum_{n=0}^{\infty} d_n (z-z_0)^n\ . \tag{4.2}$$

4.2 Theorem CONVERGENCE IN A CIRCLE. *Let* $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$ *and* $g(z) = \sum_{n=0}^{\infty} b_n (z-z_0)^n$ *be two power series whose convergence radii are* R_1 *and* R_2 *respectively, where* $R_1 > 0$ *and* $R_2 > 0$. *Then*

(1) *for any* $c \in \mathbb{C}$, *the series* $\sum_{n=0}^{\infty} c a_n (z-z_0)^n$ *converges absolutely for* $|z-z_0| < R_1$ *and*

$$\sum_{n=0}^{\infty} c\, a_n (z-z_0)^n = c\, f(z) \ \text{ for } |z-z_0| < R_1\,; \tag{4.3}$$

(2) *the series* $\sum_{n=0}^{\infty} (a_n + b_n)(z-z_0)^n$ *converges absolutely for* $|z-z_0| < \min\{R_1, R_2\}$ *and*

$$\sum_{n=0}^{\infty} (a_n + b_n)(z-z_0)^n = f(z) + g(z) \ \text{ for } |z-z_0| < \min\{R_1, R_2\}\,; \tag{4.4}$$

(3) *the series* $\sum_{n=0}^{\infty} c_n (z-z_0)^n$, *where* $c_n = \sum_{k=0}^{n} a_k b_{n-k}$, *converges absolutely for* $|z-z_0| < \min\{R_1, R_2\}$, *and*

$$\sum_{n=0}^{\infty} c_n (z-z_0)^n = f(z)\, g(z)\,, \ \text{ for } |z-z_0| < \min\{R_1,\ R_2\}\,; \tag{4.5}$$

(4) *if* $g(z) \neq 0$ *for* $|z - z_0| < R$, *where* $0 < R \leq \min\{R_1, R_2\}$, *and* $\{d_n\}_{n=0}^{\infty}$ *is the sequence of coefficients obtained by solving the equations*

$$\sum_{k=0}^{n} d_k b_{n-k} = a_n, n = 0,\ 1, \dots, \tag{4.6}$$

then the series $\sum_{n=0}^{\infty} d_n (z - z_0)^n$ *converges absolutely for* $|z - z_0| < R$, *and*

$$\sum_{n=0}^{\infty} d_n (z - z_0)^n = f(z)/g(z)\ ,\ \text{for } |z - z_0| < R; \tag{4.7}$$

when $g(z_0) = b_0 \neq 0$, *the coefficients* d_n *are unique and there exists a radius* $R > 0$ *such that* $g(z) \neq 0$ *for* $|z - z_0| < R$.

4.3 Theorem MACLAURIN SERIES FOR A RATIONAL FUNCTION. *If*

$$f(z) = P(z)/Q(z) \tag{4.8}$$

where $P(z) = \sum_{n=0}^{p} a_n z^n$ *and* $Q(z) = \sum_{n=0}^{q} a_n z^n$ *are polynomials of degree* p *and* q *respectively, and* $Q(0) \neq 0$, *then*

$$f(z) = \sum_{n=0}^{\infty} d_n z^n,\ \text{for } |z| < R\ ,$$

where $R = \min\{|z_1^*|, \dots, |z_q^*|\} > 0$, $z_1^*, \dots, z_q^*$ *are the roots (possibly non distinct) of polynomial* $Q(z)$ *and the coefficients* d_n *are obtained by solving the equations*

$$\sum_{k=0}^{n} d_k b_{n-k} = a_n,\ n = 0,\ 1, \dots, \tag{4.9}$$

with $a_n \equiv 0$ *for* $n > p$ *and* $b_n \equiv 0$ *for* $n > q$. *Further, the series* $\sum_{n=0}^{\infty} d_n z^n$ *converges absolutely for* $|z| < R$.

5. Singularities

5.1 Definition POLE AND ESSENTIAL SINGULARITY. *Let* f *be an analytic function on the disk* $0 < |z - z_0| < R$. *We say that* f *has a pole at point* z_0 *if* $\lim_{z \to z_0} |f(z)| = \infty$. *If the point* z_0 *is a singular point which is neither removable nor a pole, we say that it is an essential singular point.*

5.2 Theorem CHARACTERIZATION OF ISOLATED SINGULARITIES. *Let* f *be an analytic function with an isolated singular point at* z_0. *Then*

(1) z_0 *is a removable singular point*

$$\Leftrightarrow \lim_{z \to z_0} (z - z_0) f(z) = 0$$

$\Leftrightarrow$ $\lim_{z \to z_0} f(z) = c$, *for some* $c \in \mathbb{C}$.

(2) z_0 *is a pole*

$\Leftrightarrow$ *the function* $g(z) = 1/f(z)$ *has a removable singular point at* z_0

$\Leftrightarrow$ *there is a positive integer* m $(m > 0)$ *and an analytic function* $h(z)$ *on a disk* $|z - z_0| < R$, *where* $R > 0$, *such that* $h(z_0) \neq 0$ *and* $f(z) = h(z)/(z - z_0)^m$

$\Leftrightarrow$ *there is a positive integer* m *such that* $\lim_{z \to z_0} (z - z_0)^m f(z) = c$, *where* $c \in \mathbb{C}$

$\Leftrightarrow$ *there is a positive integer* m *such that the function* $g(z) = (z - z_0)^m f(z)$ *has a removable singular point at* z_0.

5.3 Definition ORDER OF A POLE. *If* z_0 *is a pole of the function* f *such that*

$$\lim_{z \to z_0} (z - z_0)^m f(z) = c \neq 0, \text{ for some } c \in \mathbb{C},$$

we say that z_0 *is a pole of order* m.

5.4 Theorem SINGULARITIES AND LAURENT SERIES. *Let* f *be an analytic function with an isolated singular point at* z_0 *with Laurent series is*

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n \text{ for } 0 < |z - z_0| < R. \tag{5.1}$$

Then

(1) z_0 *is a removable singular point* $\Leftrightarrow$ $a_n = 0, \forall n < 0$;

(2) z_0 *is a pole of order* m $\Leftrightarrow$ $a_{-m} \neq 0$ *and* $a_n = 0$ *for* $n < -m$;

(3) z_0 *is an essential singular point*

$\Leftrightarrow$ $a_n \neq 0$ *for an infinite number of negative values of* n.

5.5 Theorem BEHAVIOR OF AN ANALYTIC FUNCTION NEAR AN ESSENTIAL SINGULARITY (PICARD). *Let* f *be an analytic function on the disk* $0 < |z - z_0| < R$. *If* z_0 *is an essential singular point, then for any complex number* $c \in \mathbb{C}$, *except possibly one, there exists a sequence* $\{z_n\}_{n=1}^{\infty}$ *converging to* z_0 *such that* $f(z_n) = c$, $\forall n$.

5.6 Remark Picard's theorem means that in any neighborhood of z_0 and for any complex number c (except possibly one), the function f takes the value c an infinite number of times.

6. Partial fractions

6.1 Theorem PARTIAL FRACTION EXPANSION OF RATIONAL FUNCTIONS. *Consider the rational function $f(z) = P(z)/Q(z)$ where $P(z) = \sum_{n=0}^{p} a_n z^n$ is a polynomial of degree p $(a_p \neq c)$ and $Q(z) = (z-z_1)^{m_1}(z-z_2)^{m_2}\cdots(z-z_q)^{m_q}$ is a polynomial of degree $q_* = \sum_{j=1}^{q} m_j$ with q distinct roots $z_1, \dots, z_q$ of multiplicities $m_1, \dots, m_q$ respectively $(q \geq 1,\ m_j \geq 1$ for $j = 1, \dots, q)$. Then the function $f(z)$ can be uniquely written in the form*

$$f(z) = G(z) + \sum_{j=1}^{q} G_j\left[1/(z-z_j)\right]$$

for any $z \in \mathbb{C}$ such that $z \neq z_j$, $j = 1, \dots, q$, where

$$G_j\left[1/(z-z_j)\right] = \sum_{k=1}^{m_j} A_{jk}/(z-z_j)^k,$$

$A_{jk} \in \mathbb{C}$, and $G(z)$ is a polynomial. Further,

(1) *if $p < q_*$, $G(z) \equiv 0$,*

(2) *if $p \geq q_*$ and the polynomials $P(z)$ and $Q(z)$ have no common root, the degree of $G(z)$ is $p - q_*$.*

6.2 Theorem FACTORIZATION OF AN ANALYTIC FUNCTION WITH FINITE NUMBER OF POLES. *Let f be an analytic function everywhere on an open domain $U \subseteq \mathbb{C}$ except at a finite number of singular points $z_1, \dots, z_q$ which are poles of orders $m_1, \dots, m_q$ respectively $(q \geq 1,\ m_j \geq 1$ for $j = 1, \dots, q)$. Then there exists a function $g(z)$ analytic everywhere on U such that $g(z_j) \neq 0$, $j = 1, \dots, p$, and*

$$f(z) = g(z)/\left[(z-z_1)^{m_1}(z-z_2)^{m_2}\cdots(z-z_q)^{m_q}\right]$$

for $z \in U$ and $z \neq z_j$, $j = 1, \dots, q$. If, furthermore, the function f has a finite number of zeros, the function f can be written

$$f(z) = \frac{P(z)}{Q(z)} h(z)$$

for $z \in U$ and $z \neq z_j$, $j = 1, \dots, q$, where $P(z)$ and $Q(z)$ are polynomials with no common root, $Q(z) = (z-z_1)^{m_1}(z-z_2)^{m_2}\cdots(z-z_q)^{m_q}$ and $h(z) \neq 0$ for $z \in U$.

6.3 Theorem PARTIAL FRACTION EXPANSION OF AN ANALYTIC FUNCTION WITH FINITE NUMBER OF POLES. *Let f be an analytic function everywhere on an open domain $U \subseteq \mathbb{C}$ except at a finite finite number of singular points $z_1, \dots, z_q$ which are poles of orders $m_1, \dots, m_q$ $(q \geq 1, m_j \geq 1$ for $j = 1, \dots, q)$. Then the function f can be written in a unique way in the form*

$$f(z) = g(z) + \sum_{j=1}^{q} G_j\left[1/(z-z_j)\right]$$

for any $z \in U$ *such that* $z \neq z_j,\ j = 1, \ldots, q,$ *where*

$$G_j[1/(z-z_j)] = \sum_{k=1}^{m_j} A_{jk}/(z-z_j)^k,$$

$A_{jk} \in \mathbb{C}$, *and* $G(z)$ *is analytic everywhere on* U.

7. Proofs and references

1. Churchill and Brown (1984, chapters 2 and 3) and Ahlfors (1979, chapter 2).

1.2. Ahlfors (1979, section 1.1, p. 22).

1.4. Ahlfors (1979, section 1.1, p. 23).

1.5. Ahlfors (1979, section 1.1, p. 23).

1.7. Ahlfors (1979, section 1.2, p. 24).

1.8 - 1.9. Churchill and Brown (1984, section 15, p. 41, and section 21, pp. 57-58).

1.10. Churchill and Brown (1984, section 15, p. 41 et section 21, pp. 57-58).

1.11. Ahlfors (1979, section 1.1, p. 23).

1.12. Churchill and Brown (1984, section 19, p. 50).

1.14. Churchill and Brown (1984, section 54, p. 156).

1.15 - **1.16**. Churchill and Brown (1984, section 19, p. 51).

1.17. Churchill and Brown (1984, section 39, pp. 111-114).

1.18. Churchill and Brown (1984, section 9, p. 27, and Chapter 3).

2. Ahlfors (1979, Chapter 2), Churchill and Brown (1984, Chapter 5), and Rudin (1976, Chapter 3).

2.3. Ahlfors (1979, section 2.4, p. 38) and Rudin (1976, section 3.39, p. 69).

2.6. Wilf (1990, section 2.4, Theorem 2.4.2, p. 44).

2.7 - **2.8**. Deshpande (1986, section 6.1, pp. 62-64).

2.9. Rudin (1976, section 3.44, p. 71).

2.10. Ahlfors (1979, section 2.5, pp. 41-42).

2.11 - **2.13**. Devinatz (1968, section 4.5, pp. 170-171).

2.14. Gillert, Küstner, Kellwich and Kästner (1986, section 21.2, p. 527).

2.16. Ahlfors (1979, section 2.4, p. 38), Churchill and Brown (1984, section 49, p. 144) and Wilf (1990, section 2.4, Theorem 2.4.2, pp. 44-45).

2.18. Churchill and Brown (1984, section 50, pp. 146-147).

2.19. Churchill and Brown (1984, section 44, pp. 126-128, and section 39,).

3.1. Churchill and Brown (1984, section 44, pp. 126-128, and section 39, pp. 111-114.).

3.3. Silverman (1974, section 10.1, p. 139).

3.5. This is a direct consequence of Theorem **3.1**.

3.6. Churchill and Brown (1984, section 53, pp. 152-153).

3.8. Deshpande (1986, section 10.2, p. 139).

3.10 - 3.12. Deshpande (1986, section 10.1, Propositions 10.3 and 10.4, p. 135).

3.14. Churchill and Brown (1984, sections 46 and 50, pp. 132-136 and 146-148).

3.16. Silverman (1974, section 11.1, p. 158).

4. Churchill and Brown (1984, Section 51, pp. 148-153) and Deshpande (1986, Section 6.1).

4.1. Statement (3) is a consequence of the Cauchy-Mertens theorem; see Devinatz (1968, Section 4.5, pp. 168-169). Statement (4) is a consequence of (3).

5. Deshpande (1986, Chapter 12).

5.1. Deshpande (1986, Section 12.2, Propositions 12.2 - 12.3, pp. 154-155).

5.2 - 5.3. Deshpande (1986, section 10.2, p. 139).

5.4. Deshpande (1986, Section 12.3, Proposition 12.9, p. 163).

5.5. Silverman (1974, Section 57, p. 241) and Churchill and Brown (1984, Section 56, p. 161).

6.1. Ahlfors (1979, Section 1.4, pp. 31-32) and Lentin and Rivaud (1964, chapitre II, section 19, pp. 234-238).

6.2. Deshpande (1986, Section 13.1, Proposition 13.1, pp. 169-170).

6.3. Deshpande (1986, Section 13.1, p. 171).

Other useful references include: Cartan (1961), Gillert et al. (1986), Gradshteyn and Ryzhik (1980), Knopp (1956), Knopp (1990), Rudin (1987), Silverman (1972), Spiegel (1964).

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