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SOME ROBUST EXACT RESULTS ON SAMPLE AUTOCORRELATIONS AND TESTS OF RANDOMNESS*

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Several exact results on the second moments of sample autocorrelations, for both Gaussian and non-Gaussian series, are presented. General formulae for the means, variances and covariances of sample autocorrelations are given for the case where the variables in a sequence are exchangeable. Bounds for the variances and covariances of sample autocorrelations from an arbitrary random sequence are derived. Exact and explicit formulae for the variances and covariances of sample autocorrelations from a Gaussian white noise are given. It is observed that the latter results hold for all spherically symmetric distributions. A simulation experiment, with Gaussian series, indicates that normalizing each sample autocorrelation with its exact mean and variance, instead of the usual approximate moments, can improve considerably the accuracy of the asymptotic N(0,1) distribution to obtain critical values for tests of randomness. The exact second moments of rank autocorrelations are also studied.

1. Introduction

Sample autocorrelations are one of the main instruments of time series analysis. They are especially useful to test the randomness of a time series and to assess dependence at various lags. Further, important economic hypotheses can be verified by testing the randomness of certain series: market efficiency [Fama (1970)], rational expectations [Kantor (1979)], the life cycle–permanent income hypothesis [Hall (1978)], etc. The efficiency of a speculative market, for example, may be assessed by testing whether first differences of relevant asset prices, like stock prices or exchange rates, are independent (the random walk hypothesis).

Several definitions of sample autocorrelations have been proposed. We consider here the most standard one, as it is used for example to identify time series models [Box and Jenkins (1976, p. 32)]: given n observations $X_1, \ldots, X_n$,

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the sample autocorrelation at lag k is

$$r_k = \sum_{i=1}^{n-k} (X_i - \bar{X})(X_{i+k} - \bar{X}) \Big/ \sum_{i=1}^{n} (X_i - \bar{X})^2, \qquad 1 \le k \le n-1, \quad (1.1)$$

where $\bar{X} = \sum_{i=1}^{n} X_i/n$ is the sample mean. We find especially important that the data be expressed in deviations from their sample mean because, in most practical situations, the true mean is unknown. This characteristic will play an important role below.

We will be concerned here by some exact distributional properties of sample autocorrelations, under the important null hypothesis of randomness. Both normal and non-normal distributions will be considered. Tests based on sample autocorrelations typically use critical values based on their asymptotic normal distribution [Bartlett (1946), Anderson (1971, ch. 8)]: both the moments of r_k (mean and variance) and the form of the distribution are usually approximate, especially when $k \ge 2$. Despite the fact that autocorrelation coefficients are widely applied in empirical research, few exact results have been published on their sampling properties, in particular for $k \ge 2$; see the reviews of Anderson (1971, ch. 6) and Kendall, Stuart and Ord (1983, ch. 48). Moran (1948) gave the exact mean of r_k, $k \ge 1$, for an arbitrary random series, and the exact variance of the first autocorrelation r_1 for a normal random series; later [Moran (1967)], he obtained an upper bound on the variance of r_1, valid for all random series. Pan Jie Jian (1968) gave an expression for the distribution of r_1 for the case of a normal white noise and Goldsmith (1977) tabulated it. Using the method of Sawa (1978), De Gooijer (1980) gave formulae that enable the numerical evaluation of the first four moments of each sample autocorrelation, when the data come from a general autoregressive moving-average Gaussian process: his formulae however are not explicit and require numerical integrations that may be expensive. Actually, no author has given exact and explicit formulae for the variances $\mathrm{var}(r_k)$, $k \ge 2$, or the covariances between the different autocorrelations, even when the series is a normal white noise. The vast majority of the results available either deal with alternative definitions of autocorrelations (coefficients with known mean, circular definition, etc.) or remain approximate.[1]

In this paper, we present several exact results on the first and second moments of sample autocorrelations, for both normal and non-normal series, and discuss their application in testing the randomness of a time series. We consider in turn four wide classes of series: (A) series of exchangeable random variables, (B) random series (or random samples), i.e., independent and identically distributed (i.i.d.) random variables with an arbitrary

[1] See, for example, T.W. Anderson (1971, ch. 6), O.D. Anderson (1982), Evans and Savin (1981), Kendall, Stuart and Ord (1983, ch. 48), Knoke (1977,1979), Phillips (1978), Tanaka (1983).

distribution, (C) series with a spherically symmetric distribution, (D) normal random series. Though we are most interested by the hypothesis of randomness (B or D), we will see that many results that hold for B or D actually hold under the more general assumptions A or C.

In section 2, we derive general formulae for the means, variances and covariances of sample autocorrelations, from an arbitrary series of exchangeable random variables, for all lags and sample sizes. Since random series belong to this class, these formulae hold for i.i.d. continuous random variables. An important case of variables that are exchangeable without being independent is the sequence of ranks from a sample of i.i.d. random variables. In the sequel, we apply and specialize these formulae. We obtain upper bounds on the variances as well as upper and lower bounds for the covariances of autocorrelation coefficients (at all lags) when the variables in the series are exchangeable. Consequently these hold for any sequence of i.i.d. variables, irrespective of the form of the distribution. The bounds are tight in the sense that they are very close to what one gets assuming the variables are i.i.d. normal. They can be used to obtain exact distribution-free conservative tests of randomness. In section 3, we specialize the general formulae to the case of rank autocorrelations obtained by replacing each observation in (1.1) by its rank. Previous studies of such coefficients gave only approximate expressions for $\mathrm{var}(r_k)$; see Wald and Wolfowitz (1943), Knoke (1977), Bartels (1982).

In section 4, we consider series of i.i.d. normal random variables and, more generally, series that obey a spherically symmetric (s.s.) distribution. We first remark that the distribution of sample autocorrelations is exactly the same under these two assumptions: accordingly, to study the latter case, we can assume normality. We then give exact and explicit formulae for the variances and covariances of sample autocorrelations, applicable to all lags and sample sizes. We observe that the exact variances in the normal case are remarkably close to the upper bounds given in section 2, except possibly when n is small ($n < 20$). Finally, in section 5, we consider the standard problem of testing the randomness of a normal time series using sample autocorrelations. We suggest that each coefficient r_k can and should be normalized with the exact mean and variance given above, as opposed to the often used approximate mean (zero) and variance: through a Monte Carlo simulation, we find that exactly normalized sample autocorrelations have distributions that are generally better approximated by the asymptotic $\mathrm{N}(0,1)$ distribution and thus yield more accurate critical values; in many cases, the difference is important.

2. Results for exchangeable variables

2.1. Definitions and notations

Let $X_1, \ldots, X_n$ be a sequence of exchangeable random variables: i.e., for any permutation $(d_1, \ldots, d_n)$ of the integers $(1, \ldots, n)$, the distribution of

$(X_{d_1},\ldots, X_{d_n})$ is the same as the distribution of $(X_1,\ldots, X_n)$. Clearly, independent and identically distributed random variables are exchangeable. On the other hand, exchangeable variables are not necessarily independent. For example, random variables having a joint symmetric normal distribution [see Rao (1973, p. 196)] are exchangeable even if the correlation ρ between any two of them is large (e.g., $\rho = 0.99$). The ranks of independent observations from a common continuous distribution have a uniform distribution and thus form a sequence of exchangeable variables; yet they are not independent. The same results on ranks actually hold if we only assume that the observations are exchangeable and have a continuous distribution, a common hypothesis in non-parametric statistics [see Hâjek and Šidák (1967, p. 37)]. We will use below the following property of exchangeable variables: if $M = M(X_1,\ldots, X_n)$ is a permutation-symmetric function of the observations, i.e.,

$$M(X_{d_1},\ldots, X_{d_n}) = M(X_1,\ldots, X_n),$$

for any permutation $(d_1,\ldots, d_n)$ of $(1,\ldots, n)$, then the variables $X_1 - M,\ldots, X_n - M$ are also exchangeable [see Fligner, Hogg and Killeen (1976)]. For further details on the notion of exchangeability, see Galambos (1982) and the references therein.

If we define

$$Z_i = X_i - \bar{X}, \qquad i = 1,\ldots, n,$$

where $\bar{X}$ is the mean of the X_i's, we can write

$$r_k = \sum_{i=1}^{n-k} Z_i Z_{i+k} \Big/ \sum_{i=1}^{n} Z_i^2, \qquad 1 \le k \le n-1. \tag{2.1}$$

If the X_i's are exchangeable, the Z_i's are also exchangeable since $\bar{X}$ is a permutation-symmetric function of $X_1,\ldots, X_n$.

Assuming $\mathrm{P}[X_1 = X_2 = \cdots = X_n] = 0$, we will now derive results on the variances and covariances of the sample autocorrelations that hold under the mere assumption of exchangeability of the variables $X_1,\ldots, X_n$. In particular, they hold whenever $X_1,\ldots, X_n$ are i.i.d. with an arbitrary continuous distribution.

2.2. *Variance of* r_k

Under the assumption that $X_1,\ldots, X_n$ are i.i.d. (with a continuous distribution), it is possible to show that

$$\mathrm{E}[r_k] = -\frac{(n-k)}{n(n-1)}, \qquad 1 \le k \le n-1; \tag{2.2}$$

see Moran (1948), Kendall, Stuart and Ord (1983, p. 551). However, one sees easily that the proof of this result depends only on the exchangeability of $Z_1, \dots, Z_n$ and thus the result holds whenever $X_1, \dots, X_n$ are exchangeable. We require $\mathrm{P}[X_1 = X_2 = \cdots = X_n] = 0$ to ensure that r_k exists with probability 1.

To obtain the variance of r_k, we first observe that the numerator of r_k^2 can be written as

$$\left(\sum_{i=1}^{n-k} Z_i Z_{i+k}\right)^2 = \sum_{i=1}^{n-k} Z_i^2 Z_{i+k}^2 + 2 \sum_{i=1}^{n-2k} Z_i Z_{i+k}^2 Z_{i+2k} + \sum_{*} Z_i Z_{i+k} Z_j Z_{j+k},$$

where $\sum_*$ denotes summation over $i, j = 1, \dots, n-k$ such that i, $i+k$, j and $j+k$ are all distinct. From the exchangeability of $Z_1, \dots, Z_n$, we can write

$$\begin{aligned} \mathrm{E}[r_k^2] &= \mathrm{E}\left[\left(\sum_{i=1}^{n} Z_i^2\right)^{-2} \left\{(n-k) Z_1^2 Z_2^2 + 2(n-2k) Z_1^2 Z_2 Z_3 \right.\right. \\ &\qquad \left.\left. + \left((n-k)^2 - 2(n-2k) - (n-k)\right) Z_1 Z_2 Z_3 Z_4 \right\}\right] \\ &= \mathrm{E}\left[\left(\sum_{i=1}^{n} Z_i^2\right)^{-2} \left\{ \frac{(n-k)}{n(n-1)} \sum^{*} Z_i^2 Z_j^2 + \frac{2(n-2k)}{n(n-1)(n-2)} \sum^{*} Z_i^2 Z_j Z_l \right.\right. \\ &\qquad \left.\left. + \frac{\left((n-k)^2 - 2(n-2k) - (n-k)\right)}{n(n-1)(n-2)(n-3)} \sum^{*} Z_i Z_j Z_l Z_m \right\}\right], \end{aligned}$$

where $\sum^*$ denotes summation over all distinct suffixes varying from 1 to n. Denote the power sums by

$$S_r = \sum_{i=1}^{n} Z_i^r, \qquad r \geq 1.$$

Using the following identities [Kendall, Stuart and Ord (1983, p. 708)],

$$\sum^{*} Z_i^2 Z_j^2 = S_2^2 - S_4,$$

$$\sum^{*} Z_i^2 Z_j Z_l = 2S_4 - S_2^2,$$

$$\sum^{*} Z_i Z_j Z_l Z_m = 3S_2^2 - 6S_4,$$

we get that

$$\begin{aligned}\mathrm{E}[r_k^2] &= \frac{(n-k)}{n(n-1)}\left(1-\mathrm{E}[S_4/S_2^2]\right)\\ &\quad+\frac{\{2n(n-2k)-3(n-k)(n-k-1)\}}{n(n-1)(n-2)(n-3)}\left(2\mathrm{E}[S_4/S_2^2]-1\right)\\ &=\frac{1}{n(n-1)(n-2)(n-3)}\\ &\quad\times\left[\{-n^3+(k+3)n^2-k(n+6k)\}\mathrm{E}[S_4/S_2^2]\right.\\ &\quad\left.+\{n^2(n-k-4)+3(n-k)+3k(n+k)\}\right]. \qquad (2.3)\end{aligned}$$

The variance then follows from the familiar formula $\mathrm{var}(r_k)=\mathrm{E}[r_k^2]-(\mathrm{E}[r_k])^2$, where $\mathrm{E}[r_k]$ is given by (2.2). In order to obtain an explicit formula for $\mathrm{var}(r_k)$, all we need is $\mathrm{E}[S_4/S_2^2]$.

When $\mathrm{E}[S_4/S_2^2]$ cannot be evaluated analytically, the approximation discussed by Moran (1967, 1970) can be useful. Further, using Cauchy's inequality, it is easy to see that $S_4/S_2^2 \geq 1/n$ for any probability distribution on the Z_i's [Moran (1967, p. 397)].[2] Then, if we notice that the coefficient of $\mathrm{E}[S_4/S_2^2]$ in (2.3) is negative for all k (whenever $n>3$), we get an upper bound for $\mathrm{var}(r_k)$ by replacing $\mathrm{E}[S_4/S_2^2]$ by $1/n$:

$$\mathrm{var}(r_k) \leq \frac{n^4-(k+7)n^3+(7k+16)n^2+2(k^2-9k-6)n-4k(k-4)}{n(n-1)^2(n-2)(n-3)}, \qquad (2.4)$$

where $k \geq 1$ and $n>3$. For $k=1$, we retrieve the result of Moran (1967): $\mathrm{var}(r_1) \leq (n-2)/n(n-1)$. The bound (2.4) can be used to obtain exact upper limits on critical values for tests of randomness based on sample autocorrelations, without any assumption on the form of the distribution (except continuity). This can be done easily, for example, by using Chebyshev's inequality; for details, see Dufour and Roy (1984).

[2] When $Z_i=0$, $i=1,\dots,n$, we adopt the convention $S_4/S_2^2=1$.

2.3. *Covariance between r_k and r_h*

Let $k < h$. The numerator of $r_k r_h$ can be written as

$$\sum_{i=1}^{n-k}\sum_{j=1}^{n-h} Z_i Z_{i+k} Z_j Z_{j+h} = \sum_{i=1}^{n-h} Z_i^2 Z_{i+k} Z_{i+h} + \sum_{j=1}^{n-h-k} Z_{j+h}^2 Z_{j+h+k} Z_j$$
$$+ \sum_{i=1}^{n-h-k} Z_i Z_{i+k}^2 Z_{i+h+k}$$
$$+ \sum_{j=1}^{n-h} Z_{j+(h-k)} Z_{j+h}^2 Z_j + \sum_{*} Z_i Z_{i+k} Z_j Z_{j+h},$$

where $\sum_*$ denotes summation over $i = 1, \ldots, n-k$ and $j = 1, \ldots, n-h$ such that i, $i+k$, j and $j+h$ are all distinct. By a development similar to the one used to obtain $\mathrm{E}[r_k^2]$, we find (for $k < h$)

$$\mathrm{E}[r_k r_h] = \mathrm{E}\big[S_2^{-2}\{[2(n-h)+2(n-h-k)]\, Z_1^2 Z_2 Z_3$$
$$+ [(n-k)(n-h) - 4(n-h) + 2k]\, Z_1 Z_2 Z_3 Z_4\}\big]$$
$$= \mathrm{E}\left[S_2^{-2}\left\{\frac{[4(n-h)-2k]}{n(n-1)(n-2)}(2S_4 - S_2^2)\right.\right.$$
$$\left.\left. + \frac{[(n-h)(n-k-4)+2k]}{n(n-1)(n-2)(n-3)}(3S_2^2 - 6S_4)\right\}\right]$$
$$= \frac{\{(n-h)(n+k)-2kh\}}{n(n-1)(n-2)(n-3)}\left(2\mathrm{E}[S_4/S_2^2] - 1\right). \tag{2.5}$$

The covariance follows from the familiar formula

$$\mathrm{cov}(r_k, r_h) = \mathrm{E}[r_k r_h] - \mathrm{E}[r_k]\mathrm{E}[r_h].$$

It is possible to find bounds on the covariances by using the following inequality on S_4/S_2^2: for any sequence of real numbers $Z_1, \ldots, Z_n$,

$$1/n \le S_4/S_2^2 \le 1. \tag{2.6}$$

The lower bound was given above. To get the upper bound, set

$$W_i = Z_i \Big/ \left(\sum_{j=1}^{n} Z_j^2 \right)^{1/2}, \qquad i = 1, \ldots, n.$$

It is then immediate that

$$S_4/S_2^2 = \sum_{i=1}^{n} W_i^4 \leq \sum_{i=1}^{n} W_i^2 = 1.$$

We obtain bounds for $\mathrm{E}[r_k r_h]$ and $\mathrm{cov}(r_k, r_h)$ from (2.5) and (2.6). If $(n-h)(n+k) - 2kh \geq 0$ (this inequality holds if $k, h \leq n/2$), we have (for $k < h$)

$$-\frac{\{(n-h)(n+k)-2kh\}}{n^2(n-1)(n-3)} \leq \mathrm{E}[r_k r_h] \leq \frac{\{(n-h)(n+k)-2kh\}}{n(n-1)(n-2)(n-3)}. \tag{2.7}$$

Bounds for $\mathrm{cov}(r_k, r_h)$ follow by subtracting $\mathrm{E}[r_k]\mathrm{E}[r_h]$ from each member of (2.7). Up to order n^{-3}, the bounds are (for $k < h$)

$$-\frac{2(n-h+3)}{n^3} + \mathrm{O}(n^{-4}) \leq \mathrm{cov}(r_k, r_h) \leq \frac{2(k+2)}{n^3} + \mathrm{O}(n^{-4}). \tag{2.8}$$

For $(n-h)(n+k) - 2kh < 0$, upper and lower bounds in (2.7) are interchanged.

3. Rank serial correlations

Let $X_1, \ldots, X_n$ be exchangeable continuous random variables and let $(R_1, \ldots, R_n)$ be the corresponding vector of ranks. Then

$$\mathrm{P}[(R_1, \ldots, R_n) = (d_1, \ldots, d_n)] = 1/n!,$$

for any permutation $(d_1, \ldots, d_n)$ of $(1, \ldots, n)$, and thus the ranks are also exchangeable variables. The rank serial correlation at lag k is defined by

$$\imath_k = \sum_{i=1}^{n-k} (R_i - \bar{R})(R_{i+k} - \bar{R}) \Big/ \sum_{i=1}^{n} (R_i - \bar{R})^2, \qquad 1 \leq k \leq n-1, \tag{3.1}$$

where

$$\bar{R} = \frac{1}{n}\sum_{i=1}^{n} R_i = \frac{n+1}{2}.$$

In this case, the denominator of r_k is constant so that it is equivalent to study the rank serial covariances

$$C_k = \sum_{i=1}^{n-k} (R_i - \bar{R})(R_{i+k} - \bar{R}), \qquad 1 \le k \le n-1.$$

Wald and Wolfowitz (1943) proposed to use a circular version of r_k to test randomness and proved its asymptotic normality. Rank serial correlations, in circular and non-circular form, were studied further or compared with other tests by various authors; e.g., Stuart (1956), Knoke (1977), Dufour (1981), Bartels (1982).

In order to obtain the exact variance–covariance structure of the rank (non-circular) autocorrelations, we need to evaluate $\mathrm{E}[S_4/S_2^2]$. In this situation, we see easily that

$$S_2 = \frac{n(n+1)(n-1)}{12}, \qquad S_4 = \frac{n(n^2-1)(3n^2-7)}{240}. \tag{3.2}$$

Consequently,

$$\frac{S_4}{S_2^2} = \frac{3}{5}\,\frac{(3n^2-7)}{n(n^2-1)}. \tag{3.3}$$

$\mathrm{Var}(r_k)$ and $\mathrm{cov}(r_k, r_h)$ can be obtained directly by substituting (3.3) in (2.3) and (2.5). For example, the variance of r_1 is

$$\mathrm{var}(r_1) = (5n^3 - 19n^2 + 10n + 16)/\left[5n^2(n-1)^2\right]. \tag{3.4}$$

4. Results for normal and spherically symmetric distributions

We will now specialize the above results to the case of a normal random sample. Since results obtained under the normality assumption remain exactly valid for the more general class of spherically symmetric distributions, we will cast them in this framework.

4.1. Spherically symmetric distributions

Let X and μ be $n \times 1$ vectors with X random and μ fixed. The vector X has a spherically symmetric (s.s) distribution about μ if and only if $G(X-\mu)$ has the same distribution as $X-\mu$ for all orthogonal $n \times n$ matrices G. Chmielewski (1981) provides an extensive bibliography on this class of distributions. Statistical applications are discussed by Kariya and Eaton (1977) and King (1979, 1980).

The density of a vector X with a s.s. distribution, if it exists, is a function of the norm of $X-\mu$ only and its characteristic function $\phi(t)$ is of the form $\phi(t)=\psi(t't)\exp(it'\mu)$, where $t=(t_1,\dots,t_n)' \in \mathbb{R}^n$. The class of s.s. distributions includes such distributions as the multivariate normal and the multivariate Student-t with covariance matrix $\sigma^2 I_n$, a multivariate Cauchy, a multivariate exponential, etc.

Let $X=(X_1,\dots,X_n)'$ and $\mu=\mu 1$, where $1=(1,\dots,1)'$ is $n \times 1$. Denote $Z_i=X_i-\bar{X}$, $i=1,\dots,n$, and $Z=(Z_1,\dots,Z_n)'$. We can write

$$Z=MX, \tag{4.1}$$

where $M=I_n-(1/n)11'$ is a $n \times n$ symmetric idempotent matrix of rank $n-1$. Further we can find a $n \times n$ orthogonal matrix P such that

$$P'MP=\begin{pmatrix} I_{n-1} & 0 \\ 0 & 0 \end{pmatrix}.$$

Let $P=(P_1,P_2)$ where P_1 is $n \times (n-1)$ and P_2 is $n \times 1$. Then, if X has a s.s. distribution about μ, the vector $W=Z/\|Z\|$ has a distribution identical to the one of the vector $P_1(U/\|U\|)$, where U has a multinormal distribution $\mathrm{N}(0, I_{n-1})$; $\|\cdot\|$ denotes the Euclidean norm. We can see this as follows. Let $v=P'X=(v_1', v_2')'$, where $v_1=P_1'X$ and $v_2=P_2'X$. It is then simple to check that

$$Z=P_1 v_1, \qquad Z'Z=v_1'v_1, \qquad W=P_1(v_1/\|v_1\|), \tag{4.2}$$

where $P_1'P_1=I_{n-1}$ and $P_1'1=0$. Further, by considering the characteristic function of v_1, we can see easily that v_1 has a s.s. distribution about zero. The result then follows by applying Theorem 2.1 of Kariya and Eaton (1977).

A useful consequence of this property is the following: any statistic of the form $T(W)$ has a distribution which is independent of the functional form of the s.s. distribution of X, provided $\mu=\mu 1$. We can thus study its distribution assuming X is $\mathrm{N}(\mu 1, I_n)$. In particular, from the definition of sample autocorrelations, we have

$$r_k=\sum_{i=1}^{n-k} W_i W_{i+k}, \qquad 1 \le k \le n-1,$$

where $\boldsymbol{W} = (W_1, \ldots, W_n)'$. Therefore, the vector of sample autocorrelations has the same distribution whenever $\boldsymbol{X}$ has a s.s. distribution with $\boldsymbol{\mu} = \mu \boldsymbol{1}$: we can study its distribution by assuming $\boldsymbol{X}$ is $\mathrm{N}(\mu\boldsymbol{1}, I_n)$.[3]

4.2. Exact variances and covariances

To obtain explicit formulae for $\mathrm{var}(r_k)$ and $\mathrm{cov}(r_k, r_h)$, we need $\mathrm{E}[S_4/S_2^2]$. Since

$$S_4/S_2^2 = \sum_{i=1}^{n} W_i^4,$$

where $\boldsymbol{W} = \boldsymbol{Z}/\|\boldsymbol{Z}\|$, we know from the previous section that the distribution of S_4/S_2^2 is the same for all s.s. distributions. Assuming normality, Moran (1948) found that

$$\mathrm{E}\left[S_4/S_2^2\right] = \frac{3(n-1)}{n(n+1)}. \tag{4.3}$$

If we substitute (4.3) into (2.3), we find after some algebra:

$$\mathrm{var}(r_k) = \frac{n^4 - (k+3)n^3 + 3kn^2 + 2k(k+1)n - 4k^2}{(n+1)n^2(n-1)^2}, \tag{4.4}$$

where $1 \leq k \leq n-1$. With $k = 1$, we retrieve the result of Moran (1948),

$$\mathrm{var}(r_1) = (n-2)^2/\left[n^2(n-1)\right].$$

For large n, the exact variance for a normal sample, say σ_{kN}^2, is almost identical to the upper bound σ_{kU}^2 obtained for exchangeable random variables. Since

$$\sigma_{kN}^2 = \frac{n-(k+2)}{n^2} + \mathrm{O}(n^{-3}), \qquad \sigma_{kU}^2 = \frac{n-k}{n^2} + \mathrm{O}(n^{-3}),$$

it is immediate that

$$\lim_{n\to\infty} \sigma_{kU}^2/\sigma_{kN}^2 = 1.$$

We computed the exact ratio $\sigma_{kU}^2/\sigma_{kN}^2$ for various values of k and n. We

[3] This result can also be derived from an unpublished theorem given by King (1979, ch. 5) in the context of linear regression models.

found that the upper bound is nearly attained in the normal case even for samples as small as 20. With $n \geq 25$, the ratio is smaller than or equal to 1.10 for $k \leq 20$ and, with $n \geq 40$, the ratio is smaller than or equal to 1.05 for $k \leq 25$.

Similarly, we derive the covariance between r_k and r_h and get

$$\operatorname{cov}(r_k, r_h) = \frac{2\{kh(n-1)-(n-h)(n^2-k)\}}{(n+1)n^2(n-1)^2}. \tag{4.5}$$

where $1 \leq k < h \leq n-1$. Developing up to order n^{-2}, we have

$$\operatorname{cov}(r_k, r_h) = -2/n^2 + O(n^{-3}),$$

which is in agreement with a result of Fuller (1976, p. 242).

Another statistic considered by Knoke (1977) is $T = \sum_{j=1}^{n-1} r_j/j$; critical values were determined from a normal approximation with the exact mean obtained from (2.2) and an empirical variance. If we use the formula

$$\operatorname{var}(T) = \sum_{j=1}^{n-1} \sum_{k=1}^{n-1} \operatorname{cov}(r_j, r_k)/jk,$$

and substitute the expressions (4.4) and (4.5) of this paper, we get the exact variance of T. For example, for $n = 10$, 16, 32, and 64, the exact variances are 0.0455, 0.0394, 0.0278 and 0.0174, respectively, while the empirical variances obtained by Knoke were 0.041, 0.036, 0.027 and 0.018.

5. Monte Carlo results

Tests of randomness that use sample autocorrelations r_k are usually based on an asymptotic normal distribution. Further, even though the exact mean of r_k and the variance of r_1 (in the normal case) have been available for some time [Moran (1948)], many authors still use or recommend using the approximate mean zero and the approximate standard errors $n^{-1/2}$ [Box and Pierce, (1970), Box and Jenkins (1976, ch. 6)] or $\{(n-k)/n(n+2)\}^{1/2}$ [Ljung and Box (1978)]. The latter standard error is correct when the sample mean is not subtracted from the observations and the true mean is zero, but is not exact when the observations are centered. It is worthwhile to see what is the gain realized by replacing the approximate mean and variance by the exact mean in (2.2) and the exact variance in (4.4).

To investigate this issue, we conducted the following Monte Carlo experiment. For each of five different series lengths ($n = 10$, 20, 30, 50, 100), 10,000 independent realizations of a normal white noise were generated using the

Table 1

Empirical levels of tests based on sample autocorrelations for different normalizations (in percentage).[a]

			n = 10			n = 20				n = 30				
			k			k				k				
Test	% level	Side	1	3	5	1	3	5	10	1	3	5	10	15
S1	5	R	0.8	0.5	0.0	1.9	1.6	1.4	0.1	2.5	2.5	2.0	1.2	0.4
		L	5.5	2.8	0.0	5.8	4.7	3.6	0.8	6.5	5.7	4.5	2.8	0.8
		B	2.2	0.7	0.0	3.0	2.7	1.6	0.0	4.3	3.5	2.6	1.4	0.2
	10	R	3.2	2.6	0.9	5.1	4.6	3.8	1.9	5.9	5.7	5.2	3.9	2.4
		L	13.4	8.8	3.3	12.7	11.1	9.1	4.4	13.0	12.0	10.2	7.8	4.3
		B	6.3	3.3	0.0	7.7	6.3	4.9	0.9	9.0	8.1	6.5	4.1	1.2
	20	R	8.9	8.1	7.9	13.0	11.5	10.8	8.9	14.1	13.5	12.9	11.7	9.5
		L	28.2	23.0	18.0	25.6	23.9	22.0	15.5	25.2	23.9	22.2	18.7	14.7
		B	16.7	11.3	4.1	17.8	15.7	12.9	6.3	18.9	17.7	15.5	11.6	6.7
S2	5	R	1.9	2.7	3.3	2.6	2.9	3.1	3.8	3.1	3.4	3.5	3.8	4.4
		L	9.8	9.4	9.4	7.7	7.8	7.6	7.5	7.6	7.7	7.1	7.4	7.2
		B	5.3	5.8	4.7	4.6	5.1	5.2	5.2	5.4	5.5	5.2	5.6	5.1
	10	R	5.2	6.0	8.3	6.3	6.4	6.6	8.4	6.7	7.2	7.4	7.9	8.7
		L	18.6	17.7	18.6	14.9	15.0	14.7	14.7	14.6	14.5	13.7	13.7	13.7
		B	11.7	12.1	12.7	10.3	10.7	10.7	11.3	10.7	11.1	10.6	11.2	11.6
	20	R	11.4	12.5	16.9	14.5	14.2	14.6	17.2	14.9	15.3	15.5	16.5	17.8
		L	33.2	31.7	32.2	27.6	27.6	27.5	27.2	26.4	25.9	25.5	25.8	25.8
		B	23.8	23.7	26.9	21.1	21.4	21.3	23.1	21.3	21.8	21.0	21.6	22.4
S3	5	R	5.6	5.6	5.9	4.9	5.1	5.0	5.3	4.8	5.0	5.0	5.1	5.5
		L	4.6	4.8	3.3	4.6	4.8	5.0	5.0	5.2	5.2	4.8	5.3	4.9
		B	4.6	4.9	2.6	4.2	4.7	4.5	4.3	4.9	5.1	4.7	5.1	4.5
	10	R	10.6	10.5	12.6	10.4	9.9	10.0	10.9	9.8	10.2	10.0	10.4	10.6
		L	10.6	10.2	10.3	10.1	10.2	10.2	10.2	10.6	10.7	10.1	10.6	10.7
		B	10.2	10.5	9.3	9.5	9.9	10.0	10.3	9.9	10.2	9.9	10.3	10.4
	20	R	20.7	20.3	22.8	20.9	20.2	20.4	21.5	19.9	20.0	20.2	20.4	20.8
		L	21.5	21.4	22.3	20.4	20.6	20.9	21.1	21.0	20.7	20.5	20.5	21.0
		B	21.2	20.7	23.0	20.5	20.1	20.3	21.1	20.4	20.8	20.1	21.1	21.3

Table 1 (continued)

			n = 50						n = 100						
			k						k						
Test	% level	Side	1	3	5	10	15	25	1	3	5	10	15	25	50
S1	5	R	3.1	2.8	2.9	2.1	1.7	0.5	3.8	3.6	3.5	3.4	3.0	2.1	1.0
		L	6.1	5.4	5.0	3.8	2.8	1.0	5.6	5.6	5.3	5.1	4.2	3.3	1.2
		B	4.2	3.9	3.3	2.6	1.7	0.3	4.1	4.6	3.9	3.9	3.2	2.3	0.6
	10	R	7.2	6.5	6.4	5.5	5.1	2.3	7.6	7.7	7.8	7.3	6.8	5.6	3.1
		L	12.1	11.3	10.4	9.3	7.2	4.2	11.6	11.0	10.8	10.6	9.6	8.0	3.9
		B	9.1	8.1	7.9	5.8	4.5	1.4	9.4	9.2	8.8	8.4	7.1	5.3	2.1
	20	R	16.1	14.8	15.1	13.2	13.8	9.8	16.7	16.8	16.5	16.3	15.5	14.2	11.2
		L	23.1	23.3	22.0	20.8	18.2	14.3	23.2	22.2	22.5	22.0	20.7	19.0	13.1
		B	19.3	17.8	16.8	14.8	12.3	6.6	19.3	18.7	18.6	17.9	16.4	13.6	7.0
S2	5	R	3.4	3.3	3.8	3.8	4.4	4.1	3.9	3.9	3.9	4.3	4.2	3.9	4.9
		L	6.7	6.4	6.4	6.5	6.1	6.5	6.0	6.0	6.2	6.3	5.9	6.1	5.7
		B	4.9	4.8	4.9	4.8	5.0	5.0	4.5	5.1	4.8	5.2	5.1	4.7	5.3
	10	R	7.8	7.5	7.6	7.7	9.0	8.6	8.0	8.3	8.5	8.5	8.5	8.3	9.8
		L	13.0	12.6	12.4	12.7	11.8	12.8	12.0	11.6	11.7	12.3	12.0	11.7	11.6
		B	10.1	9.7	10.2	10.3	10.4	10.6	9.9	9.9	10.1	10.6	10.1	10.0	10.6
	20	R	16.8	15.8	16.6	15.7	17.8	17.8	17.0	17.4	17.4	17.8	17.6	17.5	19.0
		L	23.8	24.7	23.9	24.0	23.7	23.6	23.7	22.8	23.5	23.5	23.2	22.9	21.9
		B	20.7	20.1	20.0	20.4	20.8	21.5	20.0	19.8	20.2	20.8	20.5	20.0	21.3
S3	5	R	5.0	4.6	4.9	5.0	5.6	5.0	4.8	4.8	4.9	5.3	5.1	4.8	5.5
		L	4.8	4.8	4.8	4.9	4.6	5.1	4.7	5.1	4.9	5.2	4.8	4.9	4.8
		B	4.6	4.7	4.7	4.7	4.9	4.5	4.6	5.0	4.8	5.1	5.0	4.6	5.2
	10	R	10.1	9.7	10.0	9.9	10.7	10.2	9.9	9.9	10.1	10.2	10.0	9.9	10.8
		L	9.9	9.8	9.7	10.0	9.6	10.5	10.2	9.9	9.9	10.3	10.2	10.1	10.0
		B	9.9	9.5	9.8	9.9	10.2	10.0	9.5	9.9	9.8	10.4	9.9	9.7	10.3
	20	R	20.5	19.8	20.3	19.5	21.1	20.4	19.7	20.1	20.4	20.3	20.5	19.8	21.0
		L	19.7	20.2	19.7	20.4	19.9	20.5	20.4	20.2	20.2	20.8	20.3	20.2	19.8
		B	20.0	19.4	19.7	19.9	20.3	20.7	20.1	19.7	20.1	20.5	20.2	20.0	20.9

[a] Tests are based on asymptotic N(0,1) approximation of $R_k = (r_k - \mu_k)/\sigma_k$, where $\mu_k = 0$ for S1 and S2, $\mu_k = -(n-k)/\{n(n-1)\}$ for S3, $\sigma_k = n^{-1/2}$ for S1, $\sigma_k = \{(n-k)/n(n+2)\}^{1/2}$ for S2, and $\sigma_k = \{\mathrm{var}(r_k)\}^{1/2}$ from formula (4.4) for S3. R and L refer to one-sided tests against positive and negative dependence, respectively. B refers to a two-sided test. The standard error of the empirical levels is 0.2% for the nominal level 5%, 0.3% for 10% and 0.4% for 20%.

subroutine GGUBS of IMSL (1980), and for each realization, sample autocorrelations r_k at several lags were computed. We then examined the quality of the asymptotic N(0, 1) approximation for three different versions of the normalized statistics $R_k = (r_k - \mu_k)/\sigma_k$. The three normalizations S1, S2 and S3 were defined as follows: for S1, $\mu_k = 0$ and $\sigma_k = n^{-1/2}$; for S2, $\mu_k = 0$ and $\sigma_k = \{(n-k)/n(n+2)\}^{1/2}$; for S3, μ_k is the exact mean in (2.2) and σ_k the exact standard error from (4.4). To appreciate the accuracy of the N(0, 1) approximation, we examined the empirical frequencies of rejection of the null hypothesis of randomness by tests with three different nominal levels (5, 10 and 20 percent). Further, for each value of n and k, we considered three types of tests: one-sided tests against positive serial dependence (R), one-sided tests against negative serial dependence (L) and two-sided tests (B).

The results of the experiment are presented in table 1. We make the following observations. First, for S1, the N(0, 1) distribution provides a relatively poor approximation, even for series of 100 observations. Second, the approximation is better for S2, but the empirical significance levels of the one-sided tests remain appreciably different from the theoretical levels (at least for short series of 50 observations or less). Third, the best results are obtained with the normalization S3: the agreement between the empirical and the theoretical levels is very good both for one-sided and two-sided tests and the approximation is satisfactory even for series of 10 observations. These results clearly suggest that the normalization based on the exact mean and variance of r_k is preferable to the approximate normalizations often used. Further, it is easy to implement the exact formulae in computer programs. We thus strongly recommend to use the exact means and variances when testing randomness with sample autocorrelations.

Note finally that tail probabilities for sample autocorrelations (in the normal case) can in principle be obtained by using the methods of Imhof (1961) or Pan Jie Jian (1968); see Goldsmith (1977), Sneek (1983), Ali (1984). This remains, however, relatively costly and no table of exact critical values for sample autocorrelations is yet available (for $k \geq 2$). Clearly, simple improvements in the quality of the asymptotic normal approximation, as described above, remain an attractive practical alternative.

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