Finite-sample inference and bounds methods in econometrics and statistics *

Jean-Marie Dufour †

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† Canada Research Chair Holder (Econometrics). Centre de recherche et développement en économique (C.R.D.E.), Centre interuniversitaire de recherche en analyse des organisations (CIRANO), and Département de sciences économiques, Université de Montréal. Mailing address: Département de sciences économiques, Université de Montréal, C.P. 6128 succursale Centre-ville, Montréal, Québec, Canada H3C 3J7. TEL: 1 514 343 2400; FAX: 1 514 343 5831; e-mail: jean.marie.dufour@umontreal.ca. Web page: http://www.fas.umontreal.ca/SCECO/Dufour .

Contents

List of Definitions, Propositions and Theorems

1. Hypothesis testing and nuisance parameters

Testing an hypothesis H_0 usually involves finding a test statistic $T(H_0)$ with 2 characteristics:

1. the stochastic behavior (distribution) of $T(H_0)$ under H_0 must be known;
2. the general way in which the distribution of $T(H_0)$ is affected under the alternative must also be known (e.g. $T(H_0)$ may tend to take **large** or **small** values with greater possibilities under the alternative)
   $\longrightarrow$ Fundamental that the quantiles of the distribution function of $T(H_0)$ be either **uniquely defined** or (at least) **bounded**

Otherwise, the behavior of $T(H_0)$ under H_0 is **not interpretable** and $T(H_0)$ **cannot** be the basis of a **valid** test of H_0

Common difficulty: nuisance parameters

$$\theta = \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} \tag{1.1}$$

$$H_0 : \theta_1 = \theta_1^0 \tag{1.2}$$

$$\text{Test: } T\left(\theta_1^0\right) > c(\alpha) \tag{1.3}$$

If the distribution of $T(\theta_1^0)$ does not depend on θ_2, it is uniquely determined and can be found either by analytical methods or by simulation.
If the distribution of $T(\theta_1^0)$ depends on θ_2, it is not uniquely determined:
θ_2 is a nuisance parameter.

In many econometric and statistical problems, it is difficult to find the exact distribution of test statistics and confidence sets.
Two basic reasons:

1. deriving the relevant distributions may require complex calculations;

   and / or

2. distribution may involve nuisance parameters.

Most common approach to such distributional problems: use a **large-sample approximation.**

Important characteristic of such approximations in many situations, the asymptotic distribution does not involve nuisance parameters [e.g. N(0,1), chi-square]
$\longrightarrow$ great flexibility.

Main interest of asymptotic approximations: generate approximations useful in finite-samples

Shortcomings:

1. Finite-sample distribution may involve nuisance parameters
2. Accuracy of the approximation is typically unknown and may be **arbitrarily bad** especially with nuisance parameters (non-uniform convergence)

Approximation arbitrarily bad $\Longrightarrow$ Tests statistic not interpretable

2. Basic techniques to deal with nuisance parameters

1. **Transforming**:
   Find a transformation that reduces the data for a statistic $T(\theta_1^0)$ whose distribution does not depend on θ_2 [e.g. reduction to a maximal invariant statistic]

   - t and F-statistics in classical linear regression
   - reduction of observations in cash or signs

2. **Conditioning**
   on a statistic S such that the conditional distribution of $T(\theta_1^0)$ given S does not depend on θ_2:

   - tests with Neyman structure;
   - permutation tests;
   - conditioning on explanatory variables.

3. **Bounding**:
   find a bound on the distribution of $T(\theta_1^0)$ which is valid irrespective of the unknown value of θ_2:

$$\begin{aligned} \sup_{\theta_2} P_{(\theta_1^0,\theta_2)}\left[T(\theta_1^0) > x\right] &\leq B_{\theta_1^0}(x) \\ \inf_{\theta_2} P_{(\theta_1^0,\theta_2)}\left[T(\theta_1^0) > x\right] &\geq C_{\theta_1^0}(x) \end{aligned}$$

3. Approaches for building bounds procedures

Four approaches:

1. Bounding the statistics of interest by other random variables with more tractable distributions
   $\longrightarrow$ Bounds on distribution functions
2. Bounding directly the distribution function of interest (or its tail areas) by some function (not necessarily obtained as the distribution function of random variable)
3. Sequential confidence procedures
4. Projection techniques

3.1. Bounding the statistic of interest by other statistics

Given a statistic T used in building a test on confidence set with a complicated distribution (possibly involving nuisance parameters), one tries to find other statistics T_1 and T_2 with more tractable distributions and such that

$$\begin{aligned} T_1 &\leq T \leq T_2\,, \\ P\left[T_1 \geq x\right] &\leq P\left[T \geq x\right] \leq P\left[T_2 \geq x\right]\,. \end{aligned}$$

Approach applied in:

DUFOUR, J.-M. (1989): “Nonlinear Hypotheses, Inequality Restrictions, and Non-Nested Hypotheses: Exact Simultaneous Tests in Linear Regressions,” *Econometrica*, 57, 335–355.

3.1. Bounding tail areas by some function

$$P_\theta\left[T \geq x\right] \leq G(x)$$

where $G(x)$ is not necessarily obtained from the distribution of a random variables.

1. Exponential inequalities;
2. Chebyshev inequalities (based on second and higher-order moments);
3. Berry-Esseen bounds.

There are cases (e.g. in nonparametric statistics) where such bounds can be used and combined to get fairly tight bounds on tail areas.

Approach used in:

DUFOUR, J.-M. (1991): "Kimball's Inequality and Bounds Tests for Comparing Several Regressions under Heteroskedasticity," in *Economic Structural Change. Analysis and Forecasting*, ed. by P. Hackl, and A. Westlund, pp. 49–57. Springer-Verlag, Berlin.

DUFOUR, J.-M., AND S. MAHSEREDJIAN (1993): "Tabulation of Farebrother's Test of Linear Restrictions: A Solution," *Econometric Theory*, 9, 697–702.

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3.1. Projection techniques

3.2. Sequential confidence procedure

Useful with nuisance parameters

$$\theta = \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} \tag{3.1}$$

where

$$\begin{array}{l} \theta_1\text{: vector of nuisance parameter} \\ \theta_2\text{: vector of parameters of interest} \end{array} \tag{3.2}$$

Problem: inference about θ_2 (confidence set on test)

Suppose 2 conditions are satisfied:

1. it is possible to build exact confidence set C_1 for θ_1

$$P\left[\theta_1 \in C_1\right] = 1 - \alpha_1\,; \tag{3.3}$$

2. if θ_1 is known, it is possible to build a confidence set $C_2(\theta_1)$ for θ_2 such that

$$P[\theta_2 \in C_2(\theta_1)] = 1 - \alpha_2\,. \tag{3.4}$$

Procedure:

1. Build on exact confidence set C_1 for θ_1:

$$P\left[\theta_1 \in C_1\right] \underset{(\geq)}{=} 1 - \alpha_1$$

2. Build a simultaneous confidence set C for θ_1 and θ_2:

$$C = \{(\theta_1, \theta_2) : \theta_1 \in C_1\,,\ \theta_2 \in C_2(\theta_1)\}$$
$$P\left[(\theta_1, \theta_2) \in C\right] \geq 1 - (\alpha_1 + \alpha_2)$$

3. Use a projection (or an intersection) method to deduce conservative (or a liberal) confidence set for θ_2:

$$U = \{\theta_2 : (\theta_1, \theta_2) \in C \text{ for some } \theta_1 \in C_1\}$$
$$P\left[\theta_2 \in U\right] \geq 1 - (\alpha_1 + \alpha_2)$$

$$L = \{\theta_2 : (\theta_1, \theta_2) \in C \text{ for all } \ \theta_1 \in C_1\} \tag{3.5}$$
$$P\left[\theta_2 \in L\right] \leq (1 - \alpha_2) + \alpha_1 \tag{3.6}$$

4. Conservative and liberal critical regions can be deduced from there confidence sets:
   $\theta_2^0 \notin U$ is a conservative critical region for H_0:$\theta_2 = \theta_2^0$ with level $\alpha \equiv \alpha_1 + \alpha_2$;
   $\theta_2 \notin L$ is a liberal continual region for $H_0 : \theta_2 = \theta_2^0$ with level $\alpha = \alpha_1 - \alpha_2$.

5. By combining a conservative and a liberal confidence region with the same level one gets a generalized bounds tests

Approach applied to linear regression with AR(1) errors in:

DUFOUR, J.-M. (1990): "Exact Tests and Confidence Sets in Linear Regressions with Autocorrelated Errors," *Econometrica*, 58, 475–494.

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