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Journal of Econometrics 130 (2006) 123–142

JOURNAL OF Econometrics

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Distribution-free bounds for serial correlation coefficients in heteroskedastic symmetric time series

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Accepted 21 January 2005
Available online 9 April 2005

Abstract

We consider the problem of testing whether the observations $X_1, \ldots, X_n$ of a time series are independent with unspecified (possibly nonidentical) distributions symmetric about a common known median. Various bounds on the distributions of serial correlation coefficients are proposed: exponential bounds, Eaton-type bounds, Chebyshev bounds and Berry–Esséen–Zolotarev bounds. The bounds are exact in finite samples, distribution-free and easy to compute. The performance of the bounds is evaluated and compared with traditional serial

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doi:10.1016/j.jeconom.2005.01.034

dependence tests in a simulation experiment. The procedures proposed are applied to U.S. data on interest rates (commercial paper rate).

JEL classification: C14; C22; C12; C32; E4

Keywords: Autocorrelation; Serial dependence; Nonparametric test; Distribution-free test; Heterogeneity; Heteroskedasticity; Symmetric distribution; Robustness; Exact test; Bound; Exponential bound; Large deviations; Chebyshev inequality; Berry–Esséen; Interest rates

1. Introduction

Let $X_1, X_2, \ldots, X_n$ be a time series of length n. In many situations, it is of interest to test whether the X_t's are independent against an alternative of serial dependence, say, at lag k ($k \geqslant 1$). If under the null hypothesis the observations are assumed to be identically distributed with known mean μ, a natural test consists in rejecting the null hypothesis for large or small values of the autocorrelation coefficient

$$r_k = \sum_{t=1}^{n-k} (X_t - \mu)(X_{t+k} - \mu) \Big/ \sum_{t=1}^{n} (X - \mu)^2, \tag{1}$$

where $1 \leqslant k \leqslant n-1$. Under general regularity conditions, the distribution of r_k is approximately normal with mean zero and variance n^{-1}; see Anderson (1971, Chapter 8) or Brockwell and Davis (1991, Chapter 7).

When the observations are not identically distributed or their distributions are heavy-tailed, such a procedure can clearly be inappropriate. In this paper, we study the null hypothesis H_0 under which the observations $X_1, \ldots, X_n$ are independent but possibly nonidentically distributed, with distributions symmetric about known medians μ_t. No assumption about the existence of the moments of $X_1, \ldots, X_n$ is made, and the distribution of the observations can be discrete. Since X_t can be replaced by $X_t - \mu_t$, we can, without loss of generality, assume that $\mu_1 = \cdots = \mu_n = 0$. Consequently, we shall henceforth set $\mu_t = 0$, $t = 1, \ldots, n$.

The hypothesis H_0 is "nonparametric" in the sense that no finite-dimensional parameter vector can determine entirely the probability distribution of the observations $X_1, X_2, \ldots, X_n$. Following standard terminology [see Lehmann (1986, Sections 3.1 and 3.5)], a test of H_0 has *level* α if the probability of rejecting H_0 is *not greater than* α under any distribution of $X = (X_1, \ldots, X_n)'$ included in H_0 ($0 < \alpha < 1$). If moreover the supremum of the rejection probabilities over H_0 is equal to α, one says that the test has *size* α. Since H_0 covers a wide spectrum of probability distributions and because of the "parametric origin" of the coefficient r_k, the distribution of r_k under H_0 depends on the *form* of the distributions of the observations. Without additional assumptions, it is unknown. Consequently, no *similar* critical region of the type $|r_k| > c$ (where c is a nonstochastic critical point which depends on the level of the test) does exist: i.e., for $0 < c < 1$, the probability of the event $|r_k| > c$ is not constant over the set of data generating processes (DGP) in

H_0, and finding a valid critical value involves bounding the distribution of r_k over H_0 or considering data-dependent critical regions for r_k. In particular, there is strictly no guarantee that the actual sizes of tests based on the asymptotic (normal) distributions of r_k will be less than or equal to their nominal level (as tests of H_0) in finite samples. The same will hold a fortiori for critical values obtained under parametric assumptions, e.g., the assumption that $X_1, \ldots, X_n$ are independent and identically distributed (i.i.d.) random variables according to a $N(0, \sigma^2)$ distribution [in which case exact critical values may be computed using Imhof's algorithm]: such critical values—though they belong to daily practice—simply do not yield valid tests of the nonparametric hypothesis H_0.

The objective of this paper is to develop *finite-sample* (α-level) tests based on r_k for the *nonparametric null hypothesis* H_0. In other words, we need to ensure that the probability of rejecting H_0 is not greater than α under any DGP in H_0. This problem is quite distinct from the one where one tries to approximate the distribution of r_k under some *specific* distribution included in H_0 (like the i.i.d. Gaussian model). Following a classical nonparametric technique, we shall do this here by using an appropriate *conditioning*. When $X_1, X_2, \ldots, X_n$ are absolutely continuous, the vector of absolute values $|X| = (|X_1|, \ldots, |X_n|)'$ is a complete sufficient statistic for H_0. Further, classical arguments of similarity and Neyman structure lead one to consider tests that are conditional with respect to the complete sufficient statistic $|X|$; see Lehmann (1986, Chapter 4). Indeed, conditioning on $|X|$ is a *necessary* requirement to obtain a valid test under conditions of general heterogeneity (heteroskedasticity); see Lehmann and Stein (1949), Pratt and Gibbons (1981, Section 5.10), Dufour and Hallin (1991, Section 1), and Dufour (2003, Section 4.2). The conditional distribution of $X = (X_1, X_2, \ldots, X_n)'$ given $|X|$ is then determined by the distribution of the signs of $X_1, \ldots, X_n$. Since, under H_0, the signs are independent symmetric Bernoulli variables, the conditional distribution of r_k (given the vector of absolute values $|X|$) may in principle be computed, e.g., by enumeration. In practice, however, the conditional distribution of r_k depends on each specific sample, because it is a function of $|X|$, and so finding critical values may be difficult. This problem is also met in the well-known case of permutation t-tests; see Pratt and Gibbons (1981, Chapter 4).

For the problem of testing H_0 against location-shift alternatives, simple bounds for the conditional and unconditional distributions of the t-statistic were provided in Edelman (1986, 1990) and Dufour and Hallin (1991, 1993); similar bounds for general linear signed rank statistics have also been proposed in Dufour and Hallin (1992). Beyond the important advantage of exactness for any sample size, extensive comparisons in Dufour and Hallin (1991, 1992, 1993) indicate that the bounds studied (exponential, Chebyshev-type, Eaton-type, Berry–Esséen) can be surprisingly tight, especially if one takes the minimum of the various bounds.

In this paper, we give analogous results for tests of H_0 based on r_k against serial dependence alternatives. Four types of bounds are presented: (1) exponential bounds (Proposition 1); (2) improved Eaton bounds (Proposition 2); (3) Chebyshev-type bounds (Proposition 3); (4) Berry–Esséen–Zolotarev bounds (Proposition 4). The exponential bounds are based on the conditional moment generating function of r_k

(given $|X|$), the improved Eaton and Chebyshev-type bounds on conditional moments of r_k (a truncated third moment in the case of the Eaton bound), while the Berry–Esséen–Zolotarev bound is based on the normal distribution function. The exponential, Eaton, Chebyshev and Berry–Esséen bounds extend to the case of autocorrelation coefficients the bounds proposed in Dufour and Hallin (1991, 1992, 1993).

All these bounds are exact in finite samples and simple to compute. They are applicable despite the presence of general forms of nonnormality and hetero-skedasticity (provided the symmetry hypothesis holds). In particular, no assumption on the existence of moments is required, and the variables considered may have continuous or discrete distributions. None of the bounds given uniformly dominates the others. While the three first classes of bounds are especially useful to obtain upper bounds for small tail areas, the Berry–Esséen bounds can be tighter for larger tail areas (i.e., tails associated with points that are closer to the center of the distribution) and yield lower bounds on tail areas as well. Conservative conditional (given $|X|$) as well as unconditional conservative p-values, or critical points, for tests based on r_k can be obtained from any one of these bounds. Since all the bounds are simple to compute, the obvious strategy here is to take the smallest p-value yielded by the different bounds (or, equivalently, the tightest critical point). Such p-values provide a useful nonparametric check on the significance of tests based on autocorrelation coefficients.

The exponential bounds are described in Section 2, the Eaton and Chebyshev bounds are given in Section 3, while the Berry–Esséen bounds are derived in Section 4. In Section 5, simulation results on the performance of the bounds are presented. In Section 6, we illustrate the use of the bounds by applying them to data on commercial paper interest rates in the U.S. We conclude in Section 7.

2. Exponential bounds

In the following proposition, we derive exponential bounds for the tail areas of the conditional distribution of r_k given $|X|$ under the null hypothesis that $X_1, \ldots, X_n$ are independent with distributions symmetric about zero. The notation a.s. means *almost surely*, while the symbol "$:=$" represents a definition. The proofs of the propositions appear in Appendix A.

Proposition 1 (*Exponential bounds*). *Let $X_1, \ldots, X_n$ be independent random variables with distributions symmetric about zero, $|X| = (|X_1|, \ldots, |X_n|)'$, and*

$$r_k := \sum_{t=1}^{n-k} X_t X_{t+k} \Big/ \sum_{t=1}^{n} X_t^2, \quad 1 \leqslant k \leqslant n-1, \tag{2}$$

$$w_{kt} := |X_t X_{t+k}| \Bigg/ \left(\sum_{\tau=1}^{n-k} X_\tau^2 X_{\tau+k}^2 \right)^{1/2}, \quad t = 1, \ldots, n-k, \tag{3}$$

where we use the convention $0/0 = 0$. *Then the conditional distribution of* r_k *given* $|X|$ *is symmetric about zero and*

$$\begin{aligned}\mathsf{P}[r_k \geqslant y \mid |X|] \leqslant B_k(y_k, |X|) \leqslant \exp(-y_k^2) \prod_{t=1}^{n-k} \cosh(w_{kt} y_k) \\ \leqslant \exp(-y_k^2)[\cosh(y_k/\sqrt{n_k^*})]^{n_k^*} \leqslant \exp(-y_k^2/2) \end{aligned} \tag{4}$$

a.s., for all $y > 0$ *and* $1 \leqslant k \leqslant n-1$, *where* $y_k := y/D_k(|X|)$, $\cosh(x) := (e^x + e^{-x})/2$, $n_k^* := card(\{t : |X_t X_{t+k}| \neq 0, 1 \leqslant t \leqslant n-k\})$ *is the number of products* $X_t X_{t+k}$ *different from zero,*

$$D_k(|X|) := \left(\sum_{t=1}^{n-k} X_t^2 X_{t+k}^2 \right)^{1/2} \Bigg/ \left(\sum_{t=1}^{n} X_t^2 \right), \tag{5}$$

$$B_k(y, |X|) := \inf_{z \geqslant 0} \left\{ \exp(-zy) \prod_{t=1}^{n-k} \cosh(w_{kt} z) \right\} \tag{6}$$

and the four bounds in (4) *are set equal to zero when* $D_k(|X|) = 0$.

From the symmetry of the conditional distribution of r_k, it is clear that $\mathsf{P}[|r_k| \geqslant y \mid |X|] = 2\mathsf{P}[r_k \geqslant y \mid |X|] = 2\mathsf{P}[r_k \leqslant -y \mid |X|]$ a.s., so that (4) can also be used to bound $\mathsf{P}[r_k \leqslant -y]$ and $\mathsf{P}[|r_k| \geqslant y \mid |X|]$ for any $y > 0$. In (4), four bounds on the tail areas $\mathsf{P}[r_k \geqslant y \mid |X|]$ are given. Denote them by $E_{1k} \leqslant E_{2k} \leqslant E_{3k} \leqslant E_{4k}$ in ascending order. These bounds are increasingly looser, but the larger ones are easier to compute. In particular, E_{2k}, E_{3k} and E_{4k} only require information about the second empirical moments of the sample (r_k and $\sum X_t^2$), which may be useful when the complete observation vector $X = (X_1, \ldots, X_n)'$ is not available to an investigator. The exponential bound $E_{4k} = \exp(-y_k^2/2)$ is similar to a bound given by Edelman (1986) and Efron (1969) for the case of t-statistics; for an earlier related result, see also Hoeffding (1963). In contrast with the case of t-statistics, however, this bound now explicitly depends on $|X|$ through $D_k(|X|)$. The second largest bound $E_{3k} = \exp(-y_k^2)[\cosh(y_k/\sqrt{n_k^*})]^{n_k^*}$ uniformly improves the latter by explicitly taking into account the sample size and the lag. It is based on a result given by Eaton (1970) for linear combinations of independent Bernoulli variables. For example, for $n-k = 10$ and $y_k = 3$, we have $E_{3k} = 0.0064$ while $E_{4k} = 0.0111$. Similarly, the bound $E_{2k} = \exp(-y_k^2)\prod_{t=1}^{n-k} \cosh(w_{kt} y_k)$ improves the two previous ones by explicitly taking into account the weights w_{kt}, $t = 1, \ldots, n-k$. When the weights are equal, i.e., $w_{kt} = 1/\sqrt{n_k^*}$, $t = 1, \ldots, n-k$, the bounds E_{2k} and E_{3k} coincide. In other cases, E_{2k} can yield substantial improvements over E_{3k}, especially when the data contain a large outlier. For example, if $w_{kt} \to 0, n_k^* = 10$ and $y_k = 3$, the ratio E_{2k}/E_{3k} converges to 0.1933. Finally, the smallest bound $E_{1k} \equiv B_k(y_k, |X|)$ is obtained by finding the infimum of the function $M_k(z) = \exp(-zy_k)\prod_{t=1}^{n-k} \cosh(w_{kt} z)$ for $z \geqslant 0$, and can yield substantial improvement over the previous bounds. The function $B_k(y, |X|)$ has the following more explicit expression:

$$\begin{aligned} B_k(y, |X|) &= 0 \quad \text{if } \sum_{t=1}^{n-k} w_{kt} < y, \\ &= (\tfrac{1}{2})^{n_k^*} \quad \text{if } \sum_{t=1}^{n-k} w_{kt} = y, \\ &= \exp(-z_k^* y) \prod_{t=1}^{n-k} \cosh(w_{kt} z_k^*) \quad \text{if } \sum_{t=1}^{n-k} w_{kt} > y, \end{aligned} \tag{7}$$

where z_k^* is the unique positive number that solves the equation

$$\sum_{t=1}^{n-k} w_{kt}[(1 - \mathrm{e}^{-2w_{kt}z_k^*})/(1 + \mathrm{e}^{-2w_{kt}z_k^*})] = y. \tag{8}$$

It is fairly easy to compute $B_k(y, |X|)$ by numerical methods; for further discussion, see Dufour and Hallin (1992, pp. 315–317).

Since they depend on $|X|$ only through $D_k(|X|)$, the two largest bounds E_{3k} and E_{4k} in (4) also yield simple unconditional bounds: for all $y > 0$,

$$\mathsf{P}[r_k \geqslant y D_k(|X|)] \leqslant \exp(-y^2)[\cosh(y/\sqrt{n-k})]^{n-k} \leqslant \exp(-y^2/2). \tag{9}$$

However, in most practical cases, the weights w_{kt} are known so that the better bounds E_{1k} and E_{2k} are available: conditional critical values based on the latter always yield less conservative tests (both conditionally and unconditionally).

3. Bounds based on moments

The exponential bounds described in Proposition 1 are based on the conditional moment generating function of r_k given $|X|$. In this section, we give two sets of bounds based on considering appropriate conditional moments of r_k. The first one applies results from Eaton (1970), Pinelis (1994) and Dufour and Hallin (1993), and is based on minimizing a truncated third order moment. We denote by $\varphi(y) = (2\pi)^{-1/2}\exp(-y^2/2)$ and $\Phi(y)$ the N(0, 1) density and distribution functions, and by $(y)_+$ the positive part of any real number y, i.e., $(y)_+ = \max(0, y)$.

Proposition 2 (*Improved Eaton–Pinelis bounds*). *Under the assumptions and notations of Proposition* 1, *we have*

$$\begin{aligned} \mathsf{P}[r_k \geqslant y \mid |X|] &\leqslant \min\{B_{\mathrm{E}}(y_k; n_k^*), 0.5y_k^{-2}, 0.5\} := B_{\mathrm{EP}}^*(y_k; n_k^*) \\ &\leqslant \min\{B_{\mathrm{E}}(y_k), 0.5y_k^{-2}, 0.5\} := B_{\mathrm{EP}}(y_k), \end{aligned} \tag{10}$$

a.s., for all $y > 0$, *where*

$$B_{\mathrm{E}}(y;m) := (0.5) \inf_{0 \leqslant c < y} \left\{ (0.5)^m \sum_{j=0}^{m} \binom{m}{j} f_c[(j-(m/2))/(m/4)^{1/2}]/(y-c)^3 \right\}, \tag{11}$$

$f_c := [(|x|-c)_+]^3$, $\binom{m}{j} := m!/[j!(m-j)!]$, *and*

$$\begin{aligned} B_{\mathrm{E}}(y) &:= \inf_{0 \leqslant c < y} \int_c^{\infty} \left(\frac{z-c}{y-c} \right)^3 \varphi(z)\,\mathrm{d}z \\ &= \inf_{0 \leqslant c < y} \{ [\varphi(c)(2+c^2) - (1-\Phi(c))(c^3+3c)]/(y-c)^3 \}. \end{aligned} \tag{12}$$

Calculation of the bounds, $B^*_{\mathrm{EP}}(y;m)$ and $B_{\mathrm{EP}}(y)$ is discussed in Dufour and Hallin (1993), where the associated (conservative) critical values for standard significance levels are also reported. It is of interest to note that the bound B_{EP} enjoys an optimality property in the sense that it is tightest among all bounds based on expectations of convex functions of a standard normal variable; see Pinelis (1994) and Dufour and Hallin (1993). Note also that the function $B_{\mathrm{E}}(y;m)$ is monotonic increasing in m, i.e., $B_{\mathrm{E}}(y;m) \leqslant B_{\mathrm{E}}(y;m+1)$ for $y > 0$.

Another related method consists in bounding the tail areas of r_k with Chebyshev-type inequalities. As observed in Dufour and Hallin (1992), such bounds can be quite tight, especially if they are based on higher-order moments (i.e., moments of order greater than 2). We summarize these in the following proposition.

Proposition 3 (*Generalized Chebyshev bounds*). *Let the assumptions and notations of Proposition* 1 *hold. Then, for any positive even integer p and for any* $y > 0$,

$$\begin{aligned} \mathsf{P}[r_k \geqslant y \mid |X|] &\leqslant \frac{\mathsf{E}(r_k^p \mid |X|)}{2y^p} \leqslant \frac{D_k(|X|)^p \mathsf{E}[Y(n_k^*)^p]}{2y^p} \\ &\leqslant \left[\frac{(p-1)(p-3)\cdots 3\cdot 1}{2y^p} \right] D_k(|X|)^p \end{aligned} \tag{13}$$

and

$$\mathsf{P}[r_k \geqslant y \mid |X|] \leqslant \left[\frac{(p_k^*-1)(p_k^*-3)\cdots 3\cdot 1}{2y^{p_k^*}} \right] D_k(|X|)^{p_k^*} \tag{14}$$

a.s., where $Y(m)$ *refers to a* Bin $(m, 0.5)$ *random variable,* $p_k^* = \max\{2, \bar{p}_k\}$ *and* $\bar{p}_k$ *is the largest even integer such that* $\bar{p}_k < 1 + y_k^2$.

To implement the first bound in (13), we need the conditional moments of r_k given $|X|$. These can be established easily from (24), (25) and (26) in the proof of Proposition 1 and equations (3.2)–(3.6) in Dufour and Hallin (1992); the appropriate expressions are given in Appendix B. Even moments $\mathsf{E}[r_k^p \mid |X|]$ of order greater than 12 can be established by analogous methods, but the algebra is correspondingly more involved. These moments as well as higher order ones can also be established by using symbolic manipulation programs. The standardized binomial moments can be computed up to any desired order from formulae (3.8) and (3.9) in Dufour and

Hallin (1992), and so the two larger bounds in (13) above can be obtained easily for any value of p. Clearly, the bounds in (13) can be computed for several values of p and the minimum of these bounds again provides a valid bound. The bound (14) is the explicit solution of this minimization process (over all even values of $p \geqslant 2$) based on the third bound in (13), which is based on the moments of a N(0, 1) distribution.

4. Berry–Esséen–Zolotarev bounds

The results of the two previous sections yield *upper* bounds on the tail areas of autocorrelation coefficients under the null hypothesis of independence, and they can therefore be used to check whether we can safely reject the null hypothesis at a given level under relatively weak nonparametric assumptions. Further, these bounds are reasonably tight only when y is not too small (say, $y > 1.5$). In many cases, it would also be helpful to have a *lower* bound which could be used to decide whether an autocorrelation coefficient unambiguously lies in the acceptance region of the (conditional) test based on r_k.

Unfortunately, it appears much more difficult to obtain lower bounds similar to the upper bounds previously given. In order to obtain such lower bounds as well as upper bounds whose behavior may be more satisfactory for lower values of y, we will consider bounds of the Berry–Esséen type. More precisely, in the following proposition, we combine results of van Beek (1972) and Zolotarev (1965) to bound the difference between the conditional distribution of r_k and the standard normal one.

Proposition 4 (*Berry–Esséen–Zolotarev bounds*). *Under the assumptions and notations of Proposition* 1 *and provided $X_t X_{t+k} \neq 0$ for at least one t $(1 \leqslant t \leqslant n-k)$, we have*

$$\begin{aligned} &|\mathsf{P}[r_k \geqslant y \mid |X|] - \Phi[y/D_k(|X|)]| \\ &\quad \leqslant \Delta := \min\left\{ 0.7975 \sum_{t=1}^{n} |w_{kt}|^3, 0.366145 \left(\sum_{t=1}^{n} |w_{kt}|^3 \right)^{1/4} \right\} \\ &\quad \leqslant 0.366145 \end{aligned} \tag{15}$$

for all y, where $\Phi(y)$ denotes the N(0,1) *distribution function.*

It is clear that inequality (15) can provide both upper and lower bounds on the tail areas of r_k:

$$\begin{aligned} \mathrm{BE}_L := 1 - \Phi[y/D_k(|X|)] - \Delta \leqslant \mathsf{P}[r_k \geqslant y \mid |X|] \\ \leqslant 1 - \Phi[y/D_k(|X|)] + \Delta := \mathrm{BE}_U \quad \text{a.s.} \end{aligned} \tag{16}$$

This implies that the normal approximation is good when $\sum |w_{kt}|^3$ is small. It also follows from (15) that the conditional distribution of r_k given $|X|$—hence also its unconditional distribution—converges to a normal distribution when $\sum |w_{kt}|^3$ goes to zero. But, of course, the main interest of (15) lies in the fact that it is an operational finite-sample approximation result, not a convergence theorem.

5. Simulation experiment

In order to provide some evidence on the size and power of the proposed bounds, we considered an AR(1) process of the form

$$X_t = \varphi X_{t-1} + u_t, \quad t = 1, \ldots n, \tag{17}$$

$$u_t = d_t v_t, \quad t = 1, \ldots, n, \tag{18}$$

where the variables v_t, $t = 1, \ldots, n$, are i.i.d., the d_t's are scale parameters which determine the form of the heteroskedasticity, and $X_t = 0$ (fixed). Two types of distributions for v_t were considered:

$$\text{(G)} \quad v_t \overset{\text{i.i.d.}}{\sim} \mathrm{N}(0, 1), \quad t = 1, \ldots, n, \tag{19}$$

$$\text{(C)} \quad v_t \overset{\text{i.i.d.}}{\sim} \text{Cauchy}, \quad t = 1, \ldots, n. \tag{20}$$

For the error heterogeneity, the patterns described in Table 1 were studied.

Results of our simulation are reported in Tables 2–4. In these tables, the statistics $|t(r_1)|, |\tilde{t}(r_1)|$, $|t(\hat{\rho}_k)|$ and $|\bar{t}(\hat{\rho}_k)|$ represent four alternative ways of standardizing traditional (parametric) autocorrelation coefficients, while E_{11} is the best exponential bound. The autocorrelation statistics are:

$$|t(r_k)| = |\sqrt{n} r_k|, \quad |\bar{t}(r_k)| = |r_k/\sigma_k|, \tag{21}$$

$$|t(\hat{\rho}_k)| = |\sqrt{n}\hat{\rho}_k|, \quad |\bar{t}(\hat{\rho}_k)| = |(\hat{\rho}_k - \mu_k)/\sigma_k|, \tag{22}$$

where r_k is defined in (2),

$$\hat{\rho}_k = \sum_{t=1}^{n-k} (X_t - \bar{X})(X_{t+k} - \bar{X}) \Bigg/ \sum_{t=1}^{n} (X_t - \bar{X})^2 \tag{23}$$

Table 1
Heteroskedasticity patterns studied

Model	Type		
M1	Homoskedasticity	$d_t = 1$	$t = 1, \ldots, n$
M2	One outlier I	$d_t = 10$	if $t = n/2$
		$= 1$	otherwise
M3	One outlier II	$d_t = 100$	if $t = n/2$
		$= 1$	otherwise
M4	Exponential I	$d_t = e^{t/10}$	$t = 1, \ldots, n$
M5	Exponential II	$d_t = e^{t/2}$	$t = 1, \ldots, n$
M6	Two outliers I	$d_t = 10$	if $t = \frac{n}{2}$ or $\frac{n}{2} + 1$
		$= 1$	otherwise
M7	Two outliers II	$d_t = 100$	if $t = \frac{n}{2}$ or $\frac{n}{2} + 1$
		$= 1$	otherwise
M8	Two outliers III	$d_t = 10^6$	if $t = \frac{n}{2}$ or $\frac{n}{2} + 1$
		$= 1$	otherwise

Table 2
Empirical levels of serial dependence tests at nominal level $\alpha = 0.05$

Error distribution (v_t)	N(0, 1)					Cauchy				
Heteroskedasticity type	M1	M2	M5	M7	M8	M1	M2	M5	M7	M8
Sample size: $n = 30$	*Asymptotic tests and bounds*									
$\|t(r_1)\|$	3.90	1.67	33.60	49.46	51.32	2.47	2.43	18.68	23.44	34.36
$\|\tilde{t}(r_1)\|$	4.96	2.20	36.17	52.26	54.42	2.95	2.88	20.34	25.86	36.92
$\|t(\hat{\rho}_k)\|$	4.22	1.91	31.78	47.29	49.11	2.43	2.32	17.58	21.91	32.69
$\|\tilde{t}(\hat{\rho}_k)\|$	4.65	2.17	34.45	49.90	51.95	2.88	2.62	19.15	23.93	34.90
E_{11}	0.95	0.87	2.28	0.86	0.00	1.10	1.33	2.84	1.68	0.01
Best bound	1.11	1.00	2.28	0.87	0.00	1.16	1.36	2.84	1.68	0.01
Tests based on Imhof critical values										
$\|t(r_1)\|$	5.09	2.30	36.59	52.75	54.88	3.04	2.96	20.77	26.34	37.30
$\|\tilde{t}(r_1)\|$	5.45	2.49	37.39	53.71	55.65	3.21	3.17	21.41	27.13	37.97
$\|t(\hat{\rho}_k)\|$	5.57	2.53	35.01	50.98	52.37	2.95	3.10	19.91	24.78	35.58
$\|\tilde{t}(\hat{\rho}_k)\|$	5.20	2.47	35.74	51.34	53.54	3.08	2.82	20.05	25.48	36.12
Test based on global size correction										
$\|t(r_1)\|$	0.02	0.00	5.07	0.00	0.00	0.03	0.00	1.73	0.12	0.00
$\|\tilde{t}(r_1)\|$	0.02	0.00	5.07	0.00	0.00	0.03	0.00	1.73	0.12	0.00
$\|t(\hat{\rho}_k)\|$	0.02	0.00	4.86	0.00	0.00	0.02	0.00	1.60	0.12	0.00
$\|\tilde{t}(\hat{\rho}_k)\|$	0.03	0.00	5.10	0.00	0.00	0.02	0.01	1.74	0.18	0.00
Tests based on model-specific size correction										
$\|t(r_1)\|$	4.86	5.01	5.07	5.10	5.30	5.17	5.15	5.24	4.94	5.01
$\|\tilde{t}(r_1)\|$	4.86	5.01	5.07	5.10	5.30	5.17	5.15	5.24	4.94	5.01
$\|t(\hat{\rho}_k)\|$	4.80	4.73	4.86	4.90	5.17	5.15	5.23	5.35	5.22	4.87
$\|\tilde{t}(\hat{\rho}_k)\|$	4.60	4.93	5.10	5.29	4.77	5.11	5.04	5.22	5.10	5.28
Sample size: $n = 60$	*Asymptotic tests and bounds*									
$\|t(r_1)\|$	4.20	2.83	50.90	65.69	66.21	2.88	2.92	30.90	30.08	48.90
$\|\tilde{t}(r_1)\|$	4.89	3.22	51.92	66.58	67.12	3.07	3.09	31.61	30.75	49.77
$\|t(\hat{\rho}_k)\|$	4.24	2.81	49.98	65.08	65.57	2.84	2.88	30.30	29.55	48.09
$\|\tilde{t}(\hat{\rho}_k)\|$	4.77	3.05	51.27	65.82	66.49	2.95	3.13	31.19	30.39	49.12
E_{11}	0.72	0.91	2.31	0.62	0.00	1.15	1.08	2.95	1.45	0.00
Best bound	1.01	1.21	2.31	0.62	0.00	1.24	1.10	2.95	1.45	0.00
Tests based on Imhof critical values										
$\|t(r_1)\|$	4.99	3.29	52.05	66.69	67.23	3.12	3.12	31.75	30.90	49.87
$\|\tilde{t}(r_1)\|$	5.20	3.48	52.33	66.96	67.59	3.21	3.19	31.93	31.12	50.17
$\|t(\hat{\rho}_k)\|$	5.12	3.34	51.29	66.19	66.70	3.09	3.09	31.25	30.51	49.13
$\|\tilde{t}(\hat{\rho}_k)\|$	4.95	3.16	51.85	66.29	66.91	3.07	3.20	31.57	30.81	49.55
Tests based on global size correction										
$\|t(r_1)\|$	0.00	0.00	5.09	0.00	0.00	0.00	0.00	1.86	0.08	0.00
$\|\tilde{t}(r_1)\|$	0.00	0.00	5.09	0.00	0.00	0.00	0.00	1.86	0.08	0.00
$\|t(\hat{\rho}_k)\|$	0.00	0.00	5.20	0.00	0.00	0.00	0.00	1.88	0.08	0.00
$\|\tilde{t}(\hat{\rho}_k)\|$	0.00	0.00	5.03	0.00	0.00	0.00	0.00	1.88	0.07	0.00
Tests based on model-specific size correction										
$\|t(r_1)\|$	4.72	5.26	5.07	5.19	4.83	5.12	4.99	5.42	4.83	4.65
$\|\tilde{t}(r_1)\|$	4.72	5.26	5.07	5.19	4.83	5.12	4.99	5.42	4.83	4.65
$\|t(\hat{\rho}_k)\|$	4.59	5.09	5.11	5.34	4.96	5.03	4.85	5.35	4.85	5.03
$\|\tilde{t}(\hat{\rho}_k)\|$	4.67	5.28	5.07	5.16	4.95	5.07	4.98	5.35	4.61	4.93

Table 3
Empirical powers of serial dependence tests at level $\alpha = 0.05$ under $X_t = 0.2X_{t-1} + u_t$

Error distribution (v_t)	N(0,1)					Cauchy				
Test\Model	M1	M2	M5	M7	M8	M1	M2	M5	M7	M8
Sample size: $n = 30$	*Asymptotic tests and bounds*									
E_{11}	4.81	7.15	3.51	19.51	48.06	13.37	14.59	7.29	19.78	44.49
Best bound	5.66	7.54	3.51	19.52	48.06	13.59	14.72	7.29	19.78	44.49
Tests based on global size correction										
$\lvert t(r_1)\rvert$	0.22	0.02	7.77	0.19	0.00	0.08	0.11	2.98	0.66	0.00
$\lvert\tilde{t}(r_1)\rvert$	0.22	0.02	7.77	0.19	0.00	0.08	0.11	2.98	0.66	0.00
$\lvert t(\hat{\rho}_k)\rvert$	0.12	0.02	6.61	0.03	0.00	0.03	0.08	2.36	0.25	0.00
$\lvert\bar{t}(\hat{\rho}_k)\rvert$	0.37	0.05	8.57	0.60	0.00	0.10	0.13	3.48	0.83	0.00
Tests based on model-specific size correction										
$\lvert t(r_1)\rvert$	18.71	23.05	7.77	24.77	25.27	16.03	16.16	8.59	14.08	16.75
$\lvert\tilde{t}(r_1)\rvert$	18.71	23.05	7.77	24.77	25.27	16.03	16.16	8.59	14.08	16.75
$\lvert t(\hat{\rho}_k)\rvert$	12.36	13.78	6.61	20.38	20.32	10.95	10.72	7.37	11.67	13.63
$\lvert\bar{t}(\hat{\rho}_k)\rvert$	17.73	21.54	8.57	24.93	25.26	15.71	15.91	8.99	14.90	17.42
Sample size: $n = 60$	*Asymptotic tests and bounds*									
E_{11}	11.54	13.92	3.54	21.85	49.02	26.04	25.75	7.40	25.78	44.41
Best bound	13.71	15.26	3.55	21.85	49.02	26.32	26.09	7.40	25.79	44.41
Tests based on global size correction										
$\lvert t(r_1)\rvert$	0.00	0.00	8.00	0.11	0.00	0.01	0.00	3.16	0.33	0.00
$\lvert\tilde{t}(r_1)\rvert$	0.00	0.00	8.00	0.11	0.00	0.01	0.00	3.16	0.33	0.00
$\lvert t(\hat{\rho}_k)\rvert$	0.00	0.00	7.54	0.00	0.00	0.00	0.00	2.88	0.25	0.00
$\lvert\bar{t}(\hat{\rho}_k)\rvert$	0.01	0.00	8.33	0.20	0.00	0.02	0.00	3.37	0.36	0.00
Tests based on model-specific size correction										
$\lvert t(r_1)\rvert$	33.22	41.38	8.00	24.79	25.42	43.25	42.27	8.66	12.34	17.18
$\lvert\tilde{t}(r_1)\rvert$	33.22	41.38	8.00	24.79	25.42	43.25	42.27	8.66	12.34	17.18
$\lvert t(\hat{\rho}_k)\rvert$	27.21	33.31	7.54	22.97	23.38	29.29	27.99	8.14	11.11	16.55
$\lvert\bar{t}(\hat{\rho}_k)\rvert$	32.26	39.92	8.33	25.20	25.75	41.73	41.69	9.00	12.50	18.09

is the usual "centered" autocorrelation coefficient, while $\mu_k = -(n-k)/[n(n+1)]$ and $\sigma_k^2 = (n-k)/[n(n+2)]$ are the adjusted mean and variance suggested in Dufour and Roy (1985) for the case of a sequence of i.i.d. observations.

For each of the above parametric statistics, we also report the results of tests based on four ways of computing critical values: (1) standard asymptotic normal critical values; (2) critical points based on the Imhof algorithm assuming the observations are i.i.d. Gaussian (the Imhof critical values were also cross-checked by simulation); (3) critical values obtained by simulation under each specific distribution and heteroskedasticity pattern considered; (4) critical values based on the largest critical point that we found over the set of distributions and heteroskedasticity patterns considered—i.e. models Ml–M8 with v_t following a N(0, 1) or a Cauchy distribution—which are all included in the null hypothesis of independence. Of course, the second method is the best choice under the assumptions made by the Imhof algorithm, but will not control the level (in the sense of ensuring that the

Table 4
Empirical powers of serial dependence tests at level $\alpha = 0.05$ under $X_t = 0.9X_{t-1} + u_t$

Error distribution (v_t)	N(0,1)					Cauchy				
Test\Model	M1	M2	M5	M7	M8	M1	M2	M5	M7	M8
Sample size: $n = 30$	*Asymptotic tests and bounds*									
E_{11}	97.94	97.95	19.45	83.83	84.70	94.39	94.67	35.89	90.42	89.65
Best bound	98.20	98.18	19.45	84.21	85.06	94.55	94.92	35.95	90.81	89.97
Tests based on global size correction										
$\lvert t(r_1)\rvert$	92.69	97.09	40.57	79.63	79.43	93.46	94.40	56.06	88.42	86.01
$\lvert \tilde{t}(r_1)\rvert$	92.69	97.09	40.57	79.63	79.43	93.46	94.40	56.06	88.42	86.01
$\lvert t(\hat{\rho}_k)\rvert$	80.15	90.72	40.76	71.37	71.49	86.36	87.83	54.99	81.97	80.07
$\lvert \tilde{t}(\hat{\rho}_k)\rvert$	85.21	93.59	47.30	73.70	73.76	89.35	90.57	61.06	84.20	81.78
Tests based on model-specific size correction										
$\lvert t(r_1)\rvert$	99.65	99.95	40.57	84.29	83.85	98.99	99.09	71.44	92.65	88.98
$\lvert \tilde{t}(r_1)\rvert$	99.65	99.95	40.57	84.29	83.85	98.99	99.09	71.44	92.65	88.98
$\lvert t(\hat{\rho}_k)\rvert$	98.52	99.71	40.76	77.68	77.46	98.28	98.41	71.21	88.70	84.87
$\lvert \tilde{t}(\hat{\rho}_k)\rvert$	99.06	99.84	47.30	79.27	79.06	98.59	98.64	77.24	89.86	85.98
Sample size: $n = 60$	*Asymptotic tests and bounds*									
E_{11}	100	100	18.45	86.29	85.85	99.09	99.15	35.00	96.03	90.32
Best bound	100	100	18.45	86.55	86.23	99.11	99.18	35.10	96.28	90.52
Tests based on global size correction										
$\lvert t(r_1)\rvert$	99.11	99.55	39.31	80.56	80.27	97.93	98.15	55.43	92.95	86.61
$\lvert \tilde{t}(r_1)\rvert$	99.11	99.55	39.31	80.56	80.27	97.93	98.15	55.43	92.95	86.61
$\lvert t(\hat{\rho}_k)\rvert$	97.54	98.57	40.00	76.78	75.75	97.02	97.44	55.51	90.50	83.31
$\lvert \tilde{t}(\hat{\rho}_k)\rvert$	98.11	98.92	42.82	77.46	76.66	97.43	97.68	58.09	91.07	83.98
Tests based on model-specific size correction										
$\lvert t(r_1)\rvert$	100	100	39.31	84.95	84.35	99.68	99.62	70.70	96.31	89.96
$\lvert \tilde{t}(r_1)\rvert$	100	100	39.31	84.95	84.35	99.68	99.62	70.70	96.31	89.96
$\lvert t(\hat{\rho}_k)\rvert$	100	100	40.00	81.52	81.11	99.58	99.57	70.87	94.98	87.73
$\lvert \tilde{t}(\hat{\rho}_k)\rvert$	100	100	42.82	82.35	81.69	99.60	99.59	74.26	95.35	88.13

probability of rejecting the null hypothesis of independence is not larger than the level) in other cases covered by the null hypothesis of independence (e.g., with heteroskedasticity); for further discussion of the Imhof algorithm, see Imhof (1961), Koerts and Abrahamse (1969) and Dufour and King (1991). The third method provides a theoretical benchmark that cannot be achieved in practice, because the heteroskedasticity pattern is not specified by the null hypothesis of independence. The fourth method is the one closest to what one would like to do for a distribution-free test that is robust to non-normality and heteroskedasticity of unknown form, based on these statistics. It is not clear, however, that the (marginal) distributions of these statistics can be bounded in a useful way under the (very wide) null hypothesis considered by the conditional bounds we propose [for further of discussion of this point, see Pratt and Gibbons (1981, Chapter 4), Dufour and Hallin (1991, Section 1), Dufour (2003, Section 4.2)]. We do not have a way of producing provably valid critical values for these tests. So the "size-corrected" critical values used for the unconditional tests remain too "small" and the powers presented overestimate the true power of these procedures for the nonparametric null hypothesis studied.

All tests are performed at the 0.05 nominal level. Sample sizes $n = 30, 60$ were considered. Results on empirical frequencies of type I errors (empirical level) appear in Table 2, while powers for $\varphi = 0.2, 0.9$ appear in Tables 3–4.[1] Size and power frequencies were evaluated using 10,000 replications. Critical values for the "size-corrected" tests were obtained out of a preliminary simulation involving 100,000 replications. Most calculations were performed with Fortran 90 programs (Sun Workshop Compiler Fortran 90 4.2) on a Unix server. Critical values based on the Imhof algorithm were obtained using the SHAZAM computer program (version 9, Whistler et al., 2001).

We see from these results that the bounds constitute the only method that allows one to control the level of the test for all the patterns considered, in the sense that the probability of type I error is never larger than the nominal level 0.05 of the test. By contrast, the probability of type I error can get as large as 0.54 for $n = 30$ and 0.65 for $n = 60$ in the limited number cases considered in this experiment, so the size of the tests considered is at least as large as these numbers, even though the nominal size is 0.05.[2] In particular, tests based on exact critical values designed for i.i.d. Gaussian observations behave very poorly in such circumstances.

Once standard tests are corrected for size, the bounds can lead to substantial power gains. This holds despite the fact that our "size corrections" are incomplete, so the powers of the tests that are not based on bounds are overestimated. The adjustments required to correct the size of these procedures are simply too "large" to yield useful tests of the nonparametric hypothesis considered. This shows clearly that the distribution-free bounds presented in this paper can at least provide a useful check on the reliability of serial dependence tests that are not provably distribution-free.

We also observed that the tightest exponential bound $E_{11} = B_1(y_1, |X|)$ yields the best results in terms of power (for a level of 0.05), with a performance that is very close to the one provided by the minimum value over all the bounds (which may be supplied by a different bound, depending on the sample).

[1]More complete results are available from the discussion paper version of this article (Dufour et al., 2004). In particular, these include results for three sample sizes ($n = 30, 60, 100$) and a larger set of values of the autoregressive coefficient ($\varphi = 0.2, 0.3, 0.4, 0.5, 0.6, 0.8, 0.9$).

[2]Under a sufficiently important heteroskedasticity, it is not clear that traditional test statistics have the usual asymptotic normal distribution, so there is no presumption that standard asymptotic theory will work well here or exhibit convergence. This can be contrasted with the fact that the conditional distribution-free tests proposed here are provably exact under the same circumstances.

6. Application to commercial paper rate

In this section, we illustrate how the bounds derived above can be used by applying them to U.S. data on interest rates. We will study the autocorrelation structure of the first and second differences of the logarithm of the commercial paper rate [denoted by $\ln(r_t)$] from 1951 to 1983 (quarterly, 132 observations). The source of the data is Balke and Gordon (1986, pp. 789–808).

For these two series, we report in Tables 5 and 6 the usual centered version of traditional autocorrelations $\hat{\rho}_k$ [defined in (23)] and the uncentered autocorrelations r_k [in (2)], for $k = 1, \ldots, 20$. Since both series have means very close to zero, there is very little difference between the two sets of autocorrelation coefficients (see also the t-statistics reported in the tables). Even though we are mostly interested by the minimal upper bound on the p-value for testing independence, we also report the individual bounds for the sake of comparison (but one would not normally report all this information). The bounds reported are for two-sided tests, i.e., we compute bounds on $\mathsf{P}[|r_k| \geqslant y \mid |X|] = 2\mathsf{P}[r_k \geqslant |y| \mid |X|]$ at $y = \hat{r}_k$ (observed value of r_k). The upper bounds on $\mathsf{P}[r_k \geqslant |y| \mid |X|]$ computed are based on the four exponential bounds $E_{1k} \leqslant E_{2k} \leqslant E_{3k} \leqslant E_{4k}$ from (4), the improved Eaton–Pinelis-type bounds B^*_{EP} and B_{EP} from (10), Chebyshev bounds based on the exact conditional even moments of r_k, binomial moments and normal moments as given in (13)–(14), and the Berry–Esséen–Zolotarev type bound $\mathrm{BE_U}$ given by (16). The Chebyshev bound (C) based on the exact moments of r_k is the minimal value yielded by the six first even moments ($p = 2, 4, \ldots, 12$), the one based on the binomial moments is the best over the first 15 even moments ($p = 2, 4, \ldots, 30$), while the normal moment bound is based on (14). We also report the Berry–Esséen lower bound obtained from (16). All the upper bounds we consider (except B^*_{EP} and B_{EP}) can take values larger than 1.0: since a probability cannot be greater than 1.0, any one of these bounds can be improved by taking the minimum given by the bound the 1.0. Consequently, when an upper bound exceeds one, we report 1.0 in the table. Similarly, when the lower bound is less than zero, we report 0.0 in the table.

From the results in Table 5, we see that the series $X_t = (1 - B)\ln(r_t) = \ln(r_t) - \ln(r_{t-1})$ exhibits four autocorrelations r_k (at lags $k = 1, 2, 6, 7$) whose absolute values exceed two asymptotic standard errors ($|r_k| \geqslant 2/\sqrt{n} = 2/\sqrt{131} = 0.175$). Among these, three ($k = 1, 6, 7$) are clearly significant at level 0.05 under the assumption that $X_1, \ldots, X_n$ have distributions symmetric about zero. It is also of interest to note that the autocorrelations at lags $k = 3, 5, 8, 9, 12, 13, 14, 18$ are clearly not significant at level 0.05. Depending on cases, the best upper bound is obtained by using either a Chebyshev (C), Eaton-type (B^*_{EP}) or Berry–Esséen bound.

The autocorrelations for the second differences $X_t = (1 - B)^2 \ln(r_t)$ in Table 6 exhibit only one autocorrelation (at $k = 2$) whose absolute value is greater than two asymptotic standard errors ($|r_k| \geqslant 2/\sqrt{n} = 2/\sqrt{130} = 0.175$). The nonparametric upper bound on the p-value for $|r_2|$ indicates that this is significant even for a level as low as 0.00002; the best upper bound is given here by the exponential bound E_1. In this case, all the upper bounds (except the Berry–Esséen one) indicate that this is significant at level 0.005. The Berry–Esséen lower bound indicates that the

Table 5
U.S. Commercial paper interest rate (logarithm, first differences): $X_t = (1-B)\ln(r_t)$. Autocorrelations and bounds on p-values for two-sided tests. Sample: 1951:2–1983:4 (quarterly, $n = 131$ observations)[a]

k	$\hat{\rho}_k$	r_k	Exponential bounds				Eaton-type bounds		Chebyshev-type bounds					Berry–Esséen	Best upper bound	Type	Lower bound
			E_1	E_2	E_3	E_4	B^*_{EP}	B_{EP}	C (p^*)		CB (p^*)		CN	BE_U			BE_U
1	0.3339	0.3385	0.0124	0.0200	0.0332	0.0347	0.0169	0.0175	0.0100	(12)	0.0235	(8)	0.0243	0.5457	0.0100*	C	0.0000
2	−0.2105	−0.2023	0.4277	0.4516	0.5053	0.5079	0.3494	0.3502	0.3353	(4)	0.3648	(2)	0.3648	0.6884	0.3353	C	0.0000
3	−0.0393	−0.0326	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	(2)	1.0000	(2)	1.0000	1.0000	1.0000		0.2144
4	0.0911	0.0969	1.0000	1.0000	1.0000	1.0000	0.9179	0.9179	0.9179	(2)	0.9179	(2)	0.9179	0.6903	0.6903	BE_U	0.0000
5	−0.0499	−0.0415	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	(2)	1.0000	(2)	1.0000	0.9371	0.9371	BE_U	0.2565
6	−0.2490	−0.2383	0.0314	0.0357	0.0460	0.0478	0.0240	0.0248	0.0246	(10)	0.0327	(8)	0.0338	0.4110	0.0240*	B^*_{EP}	0.0000
7	−0.2132	−0.2027	0.0467	0.0499	0.0598	0.0618	0.0320	0.0328	0.0350	(8)	0.0435	(8)	0.0446	0.3674	0.0320*	B^*_{EP}	0.0000
8	−0.0174	−0.0086	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	(2)	1.0000	(2)	1.0000	1.0000	1.0000		0.4774
9	−0.0230	−0.0136	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	(2)	1.0000	(2)	1.0000	1.0000	1.0000		0.4522
10	−0.1787	−0.1686	0.3146	0.3208	0.3458	0.3489	0.2255	0.2267	0.2307	(4)	0.2446	(4)	0.2460	0.5000	0.2255	B^*_{EP}, C	0.0000
11	−0.1190	−0.1088	0.7822	0.7837	0.7973	0.7993	0.5452	0.5452	0.5452	(2)	0.5452	(2)	0.5452	0.5684	0.5452	B^*_{EP}, C	0.0000
12	−0.0179	−0.0077	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	(2)	1.0000	(2)	1.0000	1.0000	1.0000		0.4990
13	0.0118	0.0210	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	(2)	1.0000	(2)	1.0000	1.0000	1.0000		0.2967
14	0.0296	0.0400	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	(2)	1.0000	(2)	1.0000	0.9983	0.9983	BE_U	0.1485
15	0.0788	0.0905	1.0000	1.0000	1.0000	1.0000	0.9427	0.9427	0.9427	(2)	0.9427	(2)	0.9427	0.7598	0.7598	BE_U	0.0000
16	0.1127	0.1228	0.7597	0.7636	0.7867	0.7888	0.5374	0.5374	0.5374	(2)	0.5374	(2)	0.5374	0.6671	0.5374	B^*_{EP}, C	0.0000
17	0.1005	0.1088	0.7276	0.7296	0.7459	0.7481	0.5085	0.5085	0.5085	(2)	0.5085	(2)	0.5085	0.5762	0.5085	B^*_{EP}, C	0.0000
18	−0.0055	0.0025	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	(2)	1.0000	(2)	1.0000	1.0000	1.0000		0.4941
19	−0.1109	−0.1029	0.6813	0.6830	0.6973	0.6997	0.4761	0.4761	0.4761	(2)	0.4761	(2)	0.4761	0.5297	0.4761	B^*_{EP}, C	0.0000
20	−0.0640	−0.0567	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	(2)	1.0000	(2)	1.0000	0.7679	0.7679	BE_U,	0.0312

t-statistic for zero mean = 0.9875.

[a]The p-values reported one for two-sided tests. Each one is obtained by bounding $\mathsf{P}[|r_k| \geqslant y \mid |X|] = 2\mathsf{P}[r_k \geqslant y \mid |X|]$ where we take $y = |\hat{r}_k|$ (the observed value of $|r_k|$). The exponential bounds $E_1 \leqslant E_2 \leqslant E_3 \leqslant E_4$ are based on Proposition 1, the Eaton-type bounds on $B^*_{EP} \leqslant B_{EP}$ on Proposition 2, the Chebyshev bounds on Proposition 3, and the Berry–Esséen bounds on Proposition 4. C is the best Chebyshev bound based on the exact even moments of order $p = 2, 4, \ldots, 12$, CB the best Chebyshev bound based on the centered binomial moments of order $p = 2, 4, \ldots, 30$, and CN is based on (14); the moment yielding the best bound (p^*) is given in parentheses. Best upper bounds lower than 0.05 have been starred (*).

Table 6
U.S. Commercial paper interest rate (logarithm, first differences): $X_t = (1 - B)^2 \ln(r_t)$. Autocorrelations and bounds on p-values for two-sided tests. Sample: 1951:3–1983:4 (quarterly, $n = 130$ observations)

k	$\hat{\rho}_k$	r_k	Exponential bounds				Eaton-type bounds		Chebyshev-type bounds					Berry–Esséen	Best upper bound	Type	Lower bound
			E_1	E_2	E_3	E_4	B^*_{EP}	B_{EP}	C (p^*)		CB (p^*)		CN	$\mathrm{BE_U}$			$\mathrm{BE_U}$
1	−0.0874	−0.0874	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	(2)	1.0000	(2)	1.0000	0.9820	0.9820	$\mathrm{BE_U}$	0.0000
2	−0.5392	−0.5392	0.00002	0.0013	0.0042	0.0046	0.0019	0.0020	0.0006	(12)	0.0030	(12)	0.0033	0.5790	0.00002*	E1	0.0000
3	0.0282	0.0282	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	(2)	1.0000	(2)	1.0000	1.0000	1.0000		0.2340
4	0.2043	0.2044	0.3403	0.3474	0.3749	0.3776	0.2473	0.2483	0.2518	(4)	0.2685	(4)	0.2699	0.5397	0.2473	B^*_{EP}	0.0000
5	0.0444	0.0445	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	(2)	1.0000	(2)	1.0000	1.0000	1.0000		0.0592
6	−0.1754	−0.1754	0.2373	0.2449	0.2710	0.2739	0.1708	0.1719	0.1774	(4)	0.1887	(4)	0.1897	0.5024	0.1708	B^*_{EP}	0.0000
7	−0.1201	−0.1201	0.5735	0.5773	0.5997	0.6020	0.4165	0.4165	0.4165	(2)	0.4165	(2)	0.4165	0.5697	0.4165	B^*_{EP},C	0.0000
8	0.1448	0.1449	0.3858	0.3899	0.4111	0.4139	0.2750	0.2760	0.2863	(4)	0.3005	(4)	0.3022	0.4902	0.2750	B^*_{EP}	0.0000
9	0.1187	0.1187	0.4409	0.4484	0.4790	0.4817	0.3283	0.3292	0.3399	(4)	0.3512	(2)	0.3512	0.5774	0.3283	B^*_{EP}	0.0000
10	−0.1560	−0.1560	0.3192	0.3337	0.3729	0.3757	0.2458	0.2469	0.2410	(4)	0.2668	(4)	0.2683	0.6288	0.2410	C	0.0000
11	−0.0318	−0.0318	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	(2)	1.0000	(2)	1.0000	0.9564	0.9564	$\mathrm{BE_U}$	0.2093
12	0.0612	0.0610	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	(2)	1.0000	(2)	1.0000	0.9182	0.9182	$\mathrm{BE_U}$	0.0000
13	0.0017	0.0017	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	(2)	1.0000	(2)	1.0000	1.0000	1.0000		0.6052
14	−0.0308	−0.0307	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	(2)	1.0000	(2)	1.0000	1.0000	1.0000		0.1379
15	0.0082	0.0082	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	(2)	1.0000	(2)	1.0000	1.0000	1.0000		0.4532
16	0.0364	0.0364	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	(2)	1.0000	(2)	1.0000	1.0000	1.0000		0.1480
17	0.0699	0.0700	1.0000	1.0000	1.0000	1.0000	0.8318	0.8318	0.8318	(2)	0.8318	(2)	0.8318	0.6949	0.6949	$\mathrm{BE_U}$	0.0000
18	−0.0022	−0.0022	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	(2)	1.0000	(2)	1.0000	1.0000	1.0000		0.4939
19	−0.1100	−0.1100	0.6430	0.6499	0.6806	0.6830	0.4654	0.4654	0.4654	(2)	0.4654	(2)	0.4654	0.6795	0.4654	B^*_{EP}, C	0.0000
20	−0.0295	−0.0295	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	(2)	1.0000	(2)	1.0000	1.0000	1.0000		0.2008

t-statistic for zero mean = −0.0790.

autocorrelations at lags $k = 5, 12, 13$ are clearly not significant at level 0.05. Overall the second differences of $\ln(r_t)$ seem to have a simpler autocorrelation structure than the first differences $(1 - B)\ln(r_t)$.

7. Conclusion

In this paper, we suggested several ways of bounding the distribution of serial correlation coefficients, under a nonparametric null hypothesis of serial independence, allowing for both *discrete* and *continuous* distributions as well as general heterogeneity of unknown form. As required in the case of a sufficiently general heteroskedasticity, the proposed technique is based on the conditional distribution of the autocorrelations given the absolute values of the observations, which is then bounded by considering the distribution of the signs. The bounds proposed are valid for any sample size and do not rely on asymptotic approximations. In order to do that we assumed that the observations have symmetric (nonidentical) distributions with respect to known medians.

These are, of course, real restrictions. But minimal distributional assumptions are needed to get testable hypotheses. The symmetry assumption is quite common in econometrics and statistics and holds for many important distributional families (e.g., Gaussian distributions, Cauchy distributions, a wide class a stable laws, etc.) Relaxing it will require the introduction of alternative assumptions, such as i.i.d. observations (which precludes heteroskedasticity); see Dufour and Roy (1985) and Hallin and Puri (1992). Of course, which set of restrictions is most appropriate will depend on the context.

The assumption that the observations have known medians can be relaxed more easily. For example, if we assume that the observations have the same median, it is possible to obtain an exact confidence interval for this unknown median (which plays the role of a nuisance parameter), for example by inverting a sign test or the nonparametric t test described in Dufour and Hallin (1991). One can then test serial independence by using a two-stage confidence procedure similar to the ones proposed in Dufour (1990), Campbell and Dufour (1997) and Dufour and Kiviet (1998) in other contexts. Designing such a procedure, or alternative ones that would deal such nuisance parameters, goes beyond the scope of the present article and will be considered in a subsequent paper.

Acknowledgements

The authors thank two anonymous referees and the Editor, Ronald Gallant, for several useful comments, as well as Sophie Mahseredjian for programming assistance. This work was supported by the Belgian Federal Government (P.A.I, contract), the Canada Research Chair Program (Chair in Econometrics, Universite de Montréal), the Institut de Finance mathématique de Montréal (IFM2), the Canadian Network of Centres of Excellence [program on *Mathematics of*

Information Technology and Complex Systems, (MITACS)], the Communauté françaíse de Belgique (A.R.C. contract), the Intercollegiate Center for Management Sciences (Bruxelles), the Fonds de recherche sur la société et la culture (Québec), the Fonds de recherche sur la nature et les technologies (Québec), the Natural Sciences and Engineering Research Council of Canada, the Social Sciences and Humanities Research Council of Canada, the Canada Council for the Arts (Killam Fellowship), and the Alexander-von-Humboldt Foundation (Germany).

Appendix A. Proofs

Proof of Proposition 1. Let $Z_{kt} = X_t X_{t+k}, t = 1, \ldots, n-k$, and let $\mathrm{sgn}(x)$ be the sign function: $\mathrm{sgn}(x) = -1$ if $x<0, 0$ if $x = 0$, and 1 if $x>0$. Then we can write

$$r_k = D_k(|X|) \sum_{t=1}^{n-k} w_{kt} S_{kt} = D_k(|X|) R_k, \tag{24}$$

where $S_{kt} = \mathrm{sgn}(Z_{kt})$, $t = 1, \ldots, n-k, R_k = \sum_{t=1}^{n-k} w_{kt} S_{kt}$, and $\sum w_{kt}^2 = 1$. When $Z_{k1} = \cdots = Z_{k,n-k} = 0$, we have $r_k = D_k(|X|) = 0$, so that $\mathsf{P}[r_k \geqslant y \,|\, |X|] = 0$, a.s., and the result holds trivially. We now suppose that $Z_{kt} \neq 0$ for at least one t. Let $A_k(|X|) = \{t : |X_t| \neq 0, 1 \leqslant t \leqslant n-k\}$ and $B_k(|X|) = \{t : |X_t X_{t+k}| \neq 0, 1 \leqslant t \leqslant n-k\}$. Clearly, $t \in B_k(|X|)$ if and only if $t \in A_k(|X|)$ and $t+k \in A_k(|X|)$, hence

$$R_k = \sum_{t=1}^{n-k} w_{kt} S_{kt} = \sum_{t \in B_k(|X|)} w_{kt} S_{kt}. \tag{25}$$

By the independence of $X_1, \ldots, X_n$ and by the symmetry assumption, the variables in the set $\{\mathrm{sgn}(X_t) : t \in A_k(|X|)\}$ are independent conditional on $|X|$, with $\mathsf{P}[\mathrm{sgn}(X_t) = -1 \,|\, |X|] = \mathsf{P}[\mathrm{sgn}(X_t) = 1 \,|\, |X|] = 0.5$. Further, since $\mathrm{sgn}(Z_{kt}) = \mathrm{sgn}(X_t)\mathrm{sgn}(X_{t+k})$, it is easy to see that the variables in the set $\{S_{kt} : t \in B_k(|X|)\}$ are independent conditional on $|X|$ with

$$\mathsf{P}[S_{kt} = -1 \,|\, |X|] = \mathsf{P}[S_{kt} = 1 \,|\, |X|] = 0.5; \tag{26}$$

see Dufour (1981). It is clear from (24)–(26) that the conditional distribution of r_k given $|X|$ is symmetric about zero. Further, using Markov's inequality and observing that $\cosh(w_{kt} z) = \cosh(0) = 1$ for $t \notin B_k(|X|)$, we have

$$\mathsf{P}[R_k \geqslant y \,|\, |X|] \leqslant \mathsf{E}[\exp(z R_k \,|\, |X|)] / \exp(zy) = \prod_{t=1}^{n-k} \cosh(w_{kt} z) / \exp(zy) \tag{27}$$

for all $z \geqslant 0$ and for all y. Consequently, for all $y>0$,

$$\begin{aligned}
\mathsf{P}[R_k \geqslant y \,|\, |X|] &\leqslant \inf_{z \geqslant 0} \left\{ \exp(-zy) \prod_{t=1}^{n-k} \cosh(w_{kt} z) \right\} \leqslant \exp(-y^2) \prod_{t=1}^{n-k} \cosh(w_{kt} y) \\
&= \exp(-y^2) \prod_{t \in B_k(|X|)} \cosh(w_{kt} y) \leqslant \exp(-y^2) \left\{ \cosh(y/\sqrt{n_k^*}) \right\}^{n_k^*} \\
&< \exp(-y^2) \left\{ \exp[(y/\sqrt{n_k^*})^2/2] \right\}^{n_k^*} = \exp(-y^2/2)
\end{aligned}$$

where the second inequality is obtained by taking $z = y$ in (27), the third one follows from Corollary 1 and Example 2 of Eaton (1970), and the last one is obtained by noting that $\cosh(x) < \exp(x^2/2)$ for $x > 0$ (Edelman, 1986). Inequality (4) follows from (27) on observing that $r_k = D_k(|X|)R_k$. □

Proof of Proposition 2. When $X_t X_{t+k} = 0$, for $t = 1, \ldots, n-k$, we have $r_k = 0$ and (10) clearly holds. When $X_t X_{t+k} \neq 0$ for some t, the result follows from (24) to (26), and then by applying Proposition 1 from Dufour and Hallin (1993) to R_k in (25). □

Proof of Proposition 3. When $X_t X_{t+k} = 0$, for $t = 1, \ldots, n-k$, we have $r_k = 0$, and (13)–(14) clearly hold. Otherwise, the result follows from (24) to (26), and Proposition 2 in Dufour and Hallin (1992). □

Proof of Proposition 4. The result is immediate from (24) to (26) and Proposition 3 from Dufour and Hallin (1992). □

Appendix B. Conditional moments of the autocorrelations

The conditional moments $\mathsf{E}(r_k^p \mid |X|)$ in (13) can be computed by noting that

$$\mathsf{E}(r_k^p \mid |X|) = D_k(|X|)^p \mathsf{E}(R_k^p \mid |X|), \tag{28}$$

where, provided $D_k(|X|) \neq 0$ (otherwise, $r_k = 0$), $\mathsf{E}(R_k^2 \mid |X|) = 1$ and, for $p = 4, 6, \ldots, 12$, $\mathsf{E}(R_k^p \mid |X|)$ is given by the following formulae: setting $W_{kp} = \sum_{t=1}^{n-k} w_{kt}^p$,

$$\mathsf{E}(R_k^4 \mid |X|) = 3 - 2W_{k4}, \tag{29}$$

$$\mathsf{E}(R_k^6 \mid |X|) = 15 - 30W_{k4} + 16W_{k6}, \tag{30}$$

$$\mathsf{E}(R_k^8 \mid |X|) = 105 - 420W_{k4} + 140W_{k4}^2 + 448W_{k6} - 272W_{k8}, \tag{31}$$

$$\begin{aligned}\mathsf{E}(R_k^{10} \mid |X|) = {} & 945 - 6300W_{k4} + 6300W_{k4}^2 + 10080W_{k6} - 6720W_{k6}W_{k4} \\ & - 12240W_{k8} + 7936W_{k,10},\end{aligned} \tag{32}$$

$$\begin{aligned}\mathsf{E}(R_k^{12} \mid |X|) = {} & 10395 - 103950W_{k4} + 207900W_{k4}^2 - 46200W_{k4}^3 + 221760W_{k6} \\ & - 443520W_{k6}W_{k4} + 118272W_{k6}^2 - 403920W_{k8} \\ & + 269280W_{k8}W_{k4} + 523776W_{k,10} - 353792W_{k,12}.\end{aligned} \tag{33}$$

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